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ON THE RATIONALITY OF COMMON STOCK
RETURN VOLATILITY

BY

EHUD I. RONN

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ON THE RATIONALITY OF COMMON STOCK RETURN VOLATILITY

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ABSTRACT

This paper derives the relationship between the population unconditional variance of common stock returns and the variance of expected returns conditional on a well-specified information set. As a consequence, a lower-bound is obtained for the variance of common stock returns. The sample counterpart of this bound is then empirically tested against the sample variance of returns.

The paper's main conclusion can be stated as follows: the observed volatility of real (inflation-adjusted) common stock returns is not "irrationally" large. The paper admits of this conclusion because the point estimate of the lower-bound variance derived in this model is actually larger than the point estimate of common stock return volatility. However, since these point estimates are found to have a statistically insignificant difference, equality of the two variances cannot be ruled out. Hence, "rationality" of common stock returns -- as implied by a utility-based valuation conditional on a specified information set -- cannot be rejected.
ON THE RATIONALITY OF COMMON STOCK RETURN VOLATILITY

1. Introduction

Economists have attempted to explain the volatility of security returns since trading in such instruments first began. Earlier analysts preferred the "mass psychology" [Keynes (1936)] of the speculative bubble; in contrast, later financial analysts have predominantly utilized the assumption of market rationality and efficiency.

The results of empirical tests designed to elicit the degree of market rationality and/or efficiency have been mixed: most have confirmed market efficiency; others have rejected market rationality; yet others have obtained conflicting signals depending on the subperiod under analysis. The objectives of this paper are twofold: first, to utilize the observed time series of consumption, dividends, and common stock prices to present a rational model of common stock returns; and second, to test for the "rationality" of stock return volatility. This last objective is closely related to the extensive literature dealing with variance bounds tests.

The literature on variance bounds deals with the derivation of limits on the possible variance of a particular variable of interest, e.g., stock returns. Shiller (1981a, b, c) and Leroy and Porter (1981) examine these bounds under the assumption of an intertemporally constant real discount rate. The analysis in this paper similarly begins by assuming constant real discount rates, as do the Shiller (1981) and Leroy and Porter (1981) studies. In contrast, this initial analysis is subsumed as a special instance of a more general case, one which permits discount rates to vary stochastically. Moreover, the model demonstrates that under a well-specified set of assumptions, the rationality of common stock returns volatility cannot be
ruled out. Since such volatility is admissible under a restrictive set of assumptions, it assuredly cannot be ruled out under a broader framework of analysis.

Kleidon (1985) demonstrates the errors in the methodologies of earlier variance bounds analysts. This constant discount rate analysis is generalized to allow for stochastically changing discount rates through an explicit modeling of risk-averse preferences. Moreover, the current paper uses a richer information set to derive the moments of conditional returns.

An alternative conceptual framework to deterministic discount rates is one which permits these rates to vary stochastically. The postulate of a constant relative aversion (CRRA) utility function has frequently been made to study such stochastic discount rates.

The unknown parameters of this class of utility functions are the coefficient of relative risk aversion and the rate of subjective time preference. This utility function has been extensively analyzed in both a theoretic and empirical framework.

In the analysis of stock returns' variances, LeRoy and LaCivita (1981) emphasize the importance or risk aversion (in contrast to risk neutrality) in explaining stock price volatility. Hansen and Singleton (1982) examine the time-additive CRRA utility function using maximum likelihood and so-called "instrumental variables" estimators to estimate CRRA utility parameters. Grossman and Shiller (1981) also examine such stochastic rates. In this paper the assumption of perfect foresight is relaxed in favor of a time series test designed to elicit the nature of the underlying stochastic process. Prescott and Mehra (1982) describe an economy where the growth rate in dividends is subject to an ergodic Markov chain to derive market risk premiums. In contrast, the analysis here concentrates on the variable
of interest -- the expectation of marginal utility times dividends -- and models that variable from time-series analysis. Brown and Gibbons (1983) have estimated the coefficient of relative risk aversion using a method-of-moments approach which allows for an arbitrary stationary distribution of returns. In contrast to that paper, the current analysis focuses on the variability of common stock returns and tests the "rationality" of their variability.

The model discussed in this paper derives the relationship between the population unconditional variance of common stock returns and the variance of expected returns conditional on a well-specified information set. As a consequence, a lower-bound is obtained for the variance of common stock returns. The sample counterpart of this bound is then empirically tested against the sample variance of returns.

The paper's main conclusion can be stated as follows: the observed volatility of real (inflation-adjusted) common stock returns is not "irrationally" large. The paper admits of this conclusion because the point estimate of the lower-bound variance derived in this model is actually larger than the point estimate of common stock return volatility. However, since these point estimates are found to have a statistically insignificant difference, equality of the two variances cannot be ruled out. Hence, "rationality" of common stock returns -- as implied by a utility-based valuation conditional on a specified information set -- cannot be rejected.

This paper is organized as follows. Section 2 derives the stock price model under the assumption of a constant real discount rate, as well as obtaining the resultant implications for the volatility of common stock returns. Section 3 generalizes the analysis to stochastic discount rates via the assumption of a CRRA utility function. Section 4 presents the paper's main conclusions and summary.
2. **Stock Prices under Constant Real Discount Rates**

2.1 **Derivation of Basic Model**

Under constant real discount rates,

\[ P_t = \frac{E(P_{t+1} + D_{t+1} | I_t^*)}{1 + r} \]  \hspace{1cm} (1)

where

- \( P_t \) is the real (inflation-adjusted) period-\( t \) ex-dividend price of an asset paying off \( P_{t+1} + D_{t+1} \) at time \( t+1 \),
- \( r \) is the constant discount rate,
- and where the expectation is calculated conditional on all information available at time \( t \), \( I_t^* \).

Eq. (1) can be easily extended to an infinite horizon by repeatedly substituting for future prices. The result is Eq. (2):

\[ P_t = \sum_{s=1}^{\infty} \frac{E(D_{t+s} | I_t^*)}{(1 + r)^s} \]  \hspace{1cm} (2)

Eq. (2) is a useful point from which to begin a discussion of stock
return volatilities because it relates current prices to their primitive determinants: discount factors and dividends. From (2), the real rates of return for ex-dividend prices are given by

\[ \frac{R_t}{P_{t-1}} = \frac{\sum_{s=1}^{\infty} E(D_{t+s} | I_t^s)/ (1 + r)^s}{P_{t-1}} \] (3)

From (3),

\[ \text{VAR}(R_t) = \text{VAR} \left[ \sum_{s=1}^{\infty} E \left( \frac{D_{t+s}}{P_{t-1}} | I_t^s \right) / (1 + r)^s \right] \] (4)

Both sides of Eq.(4) will now be analyzed. Consider first the LHS. When economists estimate population variances by taking the observed values of returns and computing sample variances, they are implicitly assuming the existence of moments as well as a stationary distribution for returns. Hence they posit that \( E(R_t) \) and \( \text{VAR}(R_t) \) exist and that these moments can be consistently estimated by their sample analogues.

An obvious method of rationalizing or explaining \( \text{VAR}(R_t) \) would be to estimate the righthand side of (4) and compare it to \( \text{VAR}(R_t) \). However, inspection of the RHS of Eq.(4) immediately reveals that this RHS depends on an expectation conditional on the full information set, \( I_t^s \). Making use of data actually available at time \( t \), this RHS is econometrically unobservable without complete specification of the economic environment. Furthermore, the variability of returns is functionally dependent on the volatility of (full-information) expectations: the more these expectations change, the higher the variance of returns.
While these statements are undeniably true, one would wish to make more substantive statements regarding the volatility of observed returns. The next subsection is devoted to this issue.

2.2 Dividend Dynamics Model

As observed above, more meaningful remarks about stock return volatility require additional assumptions about the behavior of the underlying stochastic variables. In considering what additional assumptions to make, it is instructive to refer to the righthand side of (4). The only random variables not within the full information set $I_t^*$ are the values of $D_{t+s}$. Thus, precise specification of $E(D_{t+s} | I_t^*)$ for all $s \geq 1$ is required for a closed-form solution of $P_t$ and $R_t$.

Up to this juncture, the analysis has focused on the full-information set $I_t^*$, the totality of all information observable at time $t$. The econometrician, however, may consider any subset of the information set. Formally, define the econometrician's information set, $I_t$, to be the past history and current value of dividends as well as last period's price of common stocks; thus $^\delta$

$$I_t \equiv \{P_{t-1}, D_{t-s}; s = 0, 1, \ldots\}$$

Box-Jenkins time-series analysis was conducted on $D_t$ and $\Delta D_t = D_t - D_{t-1}$. Empirical analysis of the autocorrelation and partial autocorrelation functions for $\Delta D_t$ revealed behavior consistent with stationarity. The modeling of the expectation $E(D_{t+s} | I_t)$ conditional on the econometrician's information set $I_t$ was based on these results.

Empirical analysis suggests that the following linear difference equation
governs the evolution over time of the conditional expectation \( E(D_{t+s} - D_{t+s-1} \mid l_t) \) for \( s \geq 1 \):

\[
E(D_{t+s} - D_{t+s-1} \mid l_t) = \delta + \phi E(D_{t+s-1} - D_{t+s-2} \mid l_t)
\]  

(5)

It is important to realize that Box-Jenkins analysis which reveals a stationary \( \Delta D_t \) process is consistent with Eq.(5) since the time series analysis relies on just the past history, not the complete information set. Thus, if the econometrician's information set were restricted to the past history of dividends -- as was essentially done in the definition of \( l_t \) -- then Eq.(5) is an entirely reasonable modeling of conditional expectations. Naturally, the rationale for Eq.(5) is quite similar to the time series-based modeling of expectations (and forecasts) used by numerous papers in finance and accounting.

Eq.(5) is stated in terms of the past history of dividends. Thus, modeled, (5) distinguishes between the econometrician's conditional expectations \( E(D_{t+s} \mid l_t) \), the full-information conditional expectation \( E(D_{t+s} \mid l_t^* ) \), and the stochastic process \( \tilde{D}_{t+s} \). (5) also explains why realized values \( D_t \) do not precisely follow the postulated linear difference equation: this follows from the existence of the "white noise" error term \( \varepsilon_t \equiv \Delta D_t - E(\Delta D_t \mid l_t) \).\(^6\)

Two further observations about Eq.(5) are in order. First, Eq.(5) is an equation which obtains for \( s = 1, 2, \ldots \), for all \( t \); that is, it involves forecasting from a fixed point \( t \) over different horizons, \( s \).

Second,

\[
\lim_{s \to \infty} E(\Delta D_{t+s} \mid l_t) = \frac{\delta}{1 - \phi}, \quad 1 \phi 1 < 1.
\]
Under this regime, \( \delta > 0 \) implies \( \lim_{s \to \infty} E(D_{t+s} | l_t) = \infty \). Thus, (5) allows the expectation of \( D_t \) to diverge as \( t \) grows arbitrarily large.

Assuming that \( E(R_t | l_t) \) exists, Eq.(3) implies \( E(R_t | l_t) = \sum_{s=1}^{\infty} \left[ \frac{1}{(1+r)} \right]^s E(D_{t+s} | l_t) / P_{t-1} \). Defining the discount factor \( f \equiv 1/(1+r) \), inserting eq. (5) into this expression and performing the summation yields

\[
E(R_t | l_t) = \frac{1}{P_{t-1}} \frac{f}{1-f} \left[ \frac{\delta}{(1-f)(1-f\phi)} + \frac{1}{1-f\phi} E(D_{t+1} | l_t) - \frac{\phi f}{1-f\phi} D_t \right]
\]

(6)

The precise technical details are provided in Appendix A.

Now, using

\[
E(D_{t+1} | l_t) = D_t + \delta + \phi(D_t - D_{t-1})
\]

from the dividend-dynamics assumption implies

\[
E(R_t | l_t) = \frac{\delta f}{(1-f)^2(1-f\phi)} + \frac{1}{P_{t-1}}
\]

\[
- \frac{\phi f}{(1-f)(1-f\phi)} D_{t-1} + \frac{f(\phi - f\phi)}{(1-f)(1-f\phi)} D_t
\]

(7)

Eq.(7) will permit the empirical comparison of \( \text{VAR}(R_t) \) with \( \text{VAR}[E(R_t | l_t)] \).

The next section derives the theoretical relationship between the (pop-
ulation) moments $\text{VAR}(R_t)$ and $\text{VAR}[E(R_t \mid I_t)]$. Then, using sample counterparts of these variances will imply conclusions about the "rationality" of common stock returns.

2.3. Stock Return Variances

This Section builds on the results of Section 2.2 to derive closed-form solutions for the variance of stock returns, thus permitting a comparison of sample variances with the model's predicted variances.

Consider once again eq. (4): $\text{VAR}(R_t) = \text{VAR} \left[ \sum_{s=1}^{\infty} E(D_{t+s}/P_{t-1} \mid I_t^s)/(1+r)^s \right]$.

Eq.(4) is derived by applying $\text{VAR}(\cdot)$ operator to (3). This procedure is admissible if $\text{VAR}(\cdot)$ exists; the assumed weak stationarity of stock returns implies the existence of $\text{VAR}(\cdot)$.

The additional assumptions of Eq.(5) permit an empirical analysis despite the existence of $I_t^s$ in (4), which would otherwise require full elaboration of the economic environment prior to such an analysis.

The relationship between $\text{VAR}(R_t)$ and $\text{VAR}[E(R_t \mid I_t)]$ may be simply obtained as follows. For any random variables $y$ and $x$, it may easily be shown that $\text{VAR}(y) = E[\text{VAR}(y \mid x)] + \text{VAR}[E(y \mid x)]$. Setting $y = R_t$, $x = I_t$ implies

$$\text{VAR}(R_t) = \text{VAR}[E(R_t \mid I_t)] + E[\text{VAR}(R_t \mid I_t)]$$

which implies

$$\text{VAR}(R_t) \geq \text{VAR}[E(R_t \mid I_t)]$$

(8)
by virtue of the non-negativity of $\text{VAR}(R_t \mid I_t)$ and the implied non-negativity of $E[\text{VAR}(R_t \mid I_t)]$.

The weak inequality in (8) provides the theoretical relationship between the population variance of returns and the population variance of the conditional expectations $E(R_t \mid I_t)$. Three remarks are in place here. First, this procedure can be generalized to any information set $I_t$; it is, however, necessary to specify what $E(R_t \mid I_t)$ is, in terms of econometrically observable data at time $t$. In the current model, $E(R_t \mid I_t)$ has been explicitly derived in eq. (7). Thus, upon estimation of the relevant parameters, the RHS of inequality (8) is well-specified. Second, as $I_t$ incorporates more information, $I_t \uparrow I_t^*$, $\text{VAR}(R_t) = \text{VAR}[E(R_t \mid I_t)]$. Finally, $\text{VAR}[E(R_t \mid I_t)]$ provides a lower-bound for the variance of stock returns, a bound that results from a rational, utility-based model. The larger this lower bound is, the larger $\text{VAR}(R_t)$ can be without being "irrational." \footnote{[7]}

2.4 Derivation of Testable Equations

The methodology proposed in this paper requires the estimation of the behavioral parameter $r$ [the constant discount rate, Eq.(1)] as well as the dividend-dynamics parameters $\delta$ and $\phi$ [in Eq.(5)].

A non-parametric estimation procedure may be employed to estimate these parameters. This procedure has the advantage of not requiring any (additional) assumptions beyond those postulated above.

Eq.(1) may be rewritten

$$\frac{1}{1 + r} E \left( \frac{P_{t+1} + D_{t+1}}{P_t} \mid I_t^* \right) = 1$$
This equation has implications not only with respect to moments conditional on complete information, but also conditional on subsets of the complete information set.

Assuming the existence of unconditional moments, the equation reduces to

$$\frac{1}{1 + r} E \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) = 1$$

(9)

Similarly, the unconditional expectations operator (assuming it exists) applied to Eq. (5) yields

$$E(D_{t+s} - D_{t+s-1}) = \delta + \phi E(D_{t+s-1} - D_{t+s-2})$$

(10)

2.5 Data

Empirical time-series analyses require the explicit specification of the empirical data involved. Real annual dividends, $D_t$, were calculated as total nominal dividends on the NYSE value-weighted index divided by the average value of the personal consumption expenditures (PCE) deflator for the appropriate year. The average annual real price level, $P_t$, was estimated as the NYSE value-weighted index (appropriately standardized) divided by the average PCE deflator for the appropriate year. These empirical data are more precisely defined and presented in Appendix B. In this manner, the price and dividend series are those that would have been obtained from a unit investment in the NYSE value-weighted index. Thus, they are the appropriate measures of prices and dividends.
2.6 Empirical Results

This section presents the comparison of $\hat{\sigma}^2(R_t)$, the estimated sample counterpart of $\text{VAR}(R_t)$, and $\hat{\sigma}^2[\hat{\text{E}}(R_t \mid I_t)]$, the estimated counterpart of $\text{VAR}[\text{E}(R_t \mid I_t)]$.

The procedure to be followed is a straightforward one.

**Step 1:** Jointly estimate $\hat{f} = 1/(1 + r)$ [in Eq.(9)] and $(\hat{\delta}, \hat{\phi})$ [in Eq.(10)] by performing nonlinear seemingly unrelated regressions of

$$
\frac{P_{t+1} + D_{t+1}}{P_t} = 1/f + \varepsilon_{t+1}
$$

$$
D_{t+2} - D_{t+1} = \delta + \phi(D_{t+1} - D_t) + \eta_{t+1}
$$

This step results in the point estimates $\hat{\mu} = (\hat{f}, \hat{\delta}, \hat{\phi})'$.

**Step 2:** For the vector $\hat{\mu}$, calculate $\hat{\text{E}}(R_t \mid I_t)$ from Eq.(7). Note that the "carat" above the $\hat{\text{E}}$ signifies the fact that this is an estimated expectation rather than the true population moment. Then, using the time-series of $\hat{\text{E}}(R_t \mid I_t)$, estimate $\hat{\sigma}^2[\hat{\text{E}}(R_t \mid I_t)]$.

**Step 3:** Calculate $\hat{\sigma}^2(R_t)$ from the time-series $R_t$ and compare $\hat{\sigma}^2(R_t)$ with $\hat{\sigma}^2[\hat{\text{E}}(R_t \mid I_t)]$.

Using the data described in Section 2.5 above, the values obtained for the estimated parameters were $(\hat{f}, \hat{\delta}, \hat{\phi})' = (0.9554, 9.82 \cdot 10^{-4}, 0.2915)$. The estimated standard errors for $\hat{\text{E}}(R_t \mid I_t)$ implied by these values, $\hat{\sigma}[\hat{\text{E}}(R_t \mid I_t)]$ equals 0.230 (23.0 percent on an annualized basis).
In contrast, the estimated standard error for returns, \( \hat{\sigma}(R_t) \), equals 0.169 (16.9 percent).

Recalling the weak inequality (8), the population relationship must satisfy \( \text{VAR}(R_t) \geq \text{VAR}[\hat{E}(R_t | l_t)] \). Clearly, however, the statistical results show that \( \hat{\sigma}[\hat{E}(R_t | l_t)] > \hat{\sigma}(R_t) \). Thus, it would appear that stock returns may not be volatile enough, i.e., they are "irrationally" stable. This conclusion is in sharp contrast to those of Shiller, LeRoy and Porter, and Prescott and Mehra.

This conclusion is, however, unwarranted at the current level of analysis. The statistics \( \hat{\sigma}^2[\hat{E}(R_t | l_t)] \) and \( \hat{\sigma}^2(R_t) \) are only point estimates of their population analogues \( \text{VAR}[\hat{E}(R_t | l_t)] \) and \( \text{VAR}(R_t) \). Any statistical procedure suffers, by definition, from the consequence that any (point) estimate of a population moment is subject to error. Consequently, any statistical results should be verified for statistical significance. The subsequent analysis is devoted to a rigorous test of the null hypothesis \( H_0: \hat{\sigma}[\hat{E}(R_t | l_t)] = \sigma(R_t) \).

As noted above, \( \hat{\sigma}[\hat{E}(R_t | l_t)] \) and \( \hat{\sigma}(R_t) \) are only point estimates of the population analogues. This section develops confidence bounds about those point estimates, such that statistically significant statements can be made.

There are two sequential questions to be addressed in this section. The first considers the null hypothesis implied by Eq.(8): \( H_0: \text{VAR}(R_t) > \text{VAR}[\hat{E}(R_t | l_t)] \). If this null hypothesis is rejected, then the next issue is this: are stock returns "irrationally stable"? This is embodied in the null hypothesis: \( H_0: \text{VAR}(R_t) < \text{VAR}[\hat{E}(R_t | l_t)] \).

The following procedure is now utilized:
Step 1: Repeat Step 1 as described in Section 2.6.1 above. This step results in point estimates $\hat{\mu} = (\hat{f}, \hat{\delta}, \hat{\phi})$ as well as the estimated variance-covariance matrix $\hat{\Sigma}$.

Step 2: From the distribution $N(\hat{\mu}, \hat{\Sigma})$, simulate a (large\textsuperscript{10}) sample of vectors $(f, \delta, \phi)^{'}$.

Step 3: For each simulated vector $(f, \delta, \phi)^{'}$, calculate $\hat{E}(R_t | l_t)$ from Eq.(7). Note that each triplet $(f, \delta, \phi)$ selected from the distribution $N(\hat{\mu}, \hat{\Sigma})$ will yield a different result of $\hat{\sigma}^2[\hat{E}(R_t | l_t)]$.

Step 4: Calculate $\hat{\sigma}^2(R_t)$ from the time-series of $R_t$ and compare $\hat{\sigma}(R_t)$ with $\hat{\sigma}[\hat{E}(R_t | l_t)]$ to determine the probability of $\sigma[\hat{E}(R_t | l_t)] \leq \sigma(R_t)$.

Prior to presenting the empirical results of this simulation procedure, it is imperative to comment on its theoretical validity. The desired theoretical construct is given by $\text{Prob} \{ \sigma(R_t) \geq \sigma[\hat{E}(R_t | l_t)] \}$. In contrast, the empirical data permits the calculation of

$$\text{Prob} \{ \sigma(R_t) \geq \sigma[\hat{E}(R_t | l_t)] | \sigma(R_t) = .169 \}.$$  

There are two differences between these two probabilities and these will now be analyzed in turn.

First, $\sigma(R_t)$ is unknown and is replaced by its unbiased empirical analogue $\hat{\sigma}(R_t)$.	extsuperscript{11} Thus, reading the value of the probability from the simulated distribution yields
\[
\Pr \{\sigma(R_t) \geq \sigma[E(R_t \mid l_t)] \mid \sigma(R_t) = 0.169\}.
\]

Now, by Bayes' law,

\[
\Pr \{\sigma(R_t) \geq \sigma[E(R_t \mid l_t)]\} = \\
\int_0^\infty \Pr \{\sigma(R_t) \geq \sigma[E(R_t \mid l_t)] \mid \sigma(R_t) = x\} f_{\sigma(R_t)}(x)dx
\]

Thus, further analysis would require specification of the prior distribution of \(\sigma(R_t)\); the empirical data can at best yield the distribution of \(\hat{\sigma}(R_t)\). As a first approximation, then, assume that the observed probability \(\Pr \{\sigma(R_t) \geq \sigma[E(R_t \mid l_t)] \mid \sigma(R_t) = 0.169\}\) is an unbiased estimate of the unconditional probability.

Further, note that the true conditional expectation \(E(R_t \mid l_t)\) cannot be calculated.\(^{12}\) Naturally, the procedure detailed above supplants \(\sigma[E(R_t \mid l_t)]\), the estimated conditional expectation, as a proxy for the true conditional expectations \(E(R_t \mid l_t)\).

The bias in using \(\hat{\sigma}(E(R_t \mid l_t))\) in lieu of \(\sigma[E(R_t \mid l_t)]\) may be determined as follows. Assuming that \(E(R_t \mid l_t)\) measures \(E(R_t \mid l_t)\) with a measurement error, \(\epsilon_t\), uncorrelated with \(l_t\),\(^{13}\)

\[
\hat{E}(R_t \mid l_t) = E(R_t \mid l_t) + \epsilon_t
\]

implies

\[
\text{VAR}[\hat{E}(R_t \mid l_t)] = \text{VAR}[E(R_t \mid l_t)] + \text{VAR}(\epsilon_t)
\]  

(11)

where equation (11) follows from the assumed existence of unconditional var-
iances.

It now follows from Eq. (11) that \( \sigma[\hat{E}(R_t | I_t)] \) is an upper-bound for \( \sigma[E(R_t | I_t)] \). The consequence of this result will be considered in the analysis of the empirical results.

Proceeding along the lines described earlier, the estimators for \( \hat{\mu} \) and \( \hat{\Sigma} \) were obtained as follows:

\[
\begin{pmatrix}
0.9354 \\
1.637 \times 10^{-4} \\
0.2915
\end{pmatrix}
\]

\[
\begin{pmatrix}
-4.277 \times 10^{-4} \\
-3.814 \times 10^{-6} & 9.427 \times 10^{-8} \\
6.072 \times 10^{-20} & -3.56 \times 10^{-6} & 1.178 \times 10^{-2}
\end{pmatrix}
\]

where the symmetrical \( \hat{\Sigma} \) is written as a lower-diagonal matrix.

The simulated empirical distribution of \( \sigma[\hat{E}(R_t | I_t)] \) presented in Figure 1 reveals that \( \text{Prob} \{ \sigma[\hat{E}(R_t | I_t)] \leq \sigma(R_t) \} = .157 \). Consequently, the null hypothesis \( H_0: \sigma(R_t) > \sigma[\hat{E}(R_t | I_t)] \) is conclusively rejected. Moreover, the hypothesis \( H_0: \sigma(R_t) < \sigma[\hat{E}(R_t | I_t)] \) cannot be rejected. The conclusion from this analysis, then, is that \( \sigma(R_t) \) is not significantly different from \( \sigma[\hat{E}(R_t | I_t)] \).

Now, since \( \sigma[\hat{E}(R_t | I_t)] > \sigma[E(R_t | I_t)] \), \( H_0: \sigma(R_t) > \sigma[\hat{E}(R_t | I_t)] \) will still be rejected, although at a lower confidence level. This assumes that \( \text{Var}(\varepsilon_t) \) is not unduly "large." In fact, the simulated distribution provides an indication of how "large" \( \text{Var}(\varepsilon_t) \) would have to be to reverse the rejection of \( H_0 \). As for \( H_0: \sigma(R_t) < \sigma[E(R_t | I_t)] \), this probability
Figure 1 - Simulated Empirical Distribution of $\sigma_{\hat{E}(R_t | I_t)}$

<table>
<thead>
<tr>
<th>MIDPOINT</th>
<th>FREQ</th>
<th>CUM. FREQ</th>
<th>PERCENT</th>
<th>CUM. PERCENT</th>
</tr>
</thead>
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<td>5</td>
<td>5</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>0.015</td>
<td>17</td>
<td>22</td>
<td>1.70</td>
<td>2.50</td>
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has an upper-bound of .157. The extent to which probability is reduced depends on the magnitude of \( \text{VAR}(\epsilon_t) \) in Eq.(11). Thus, unless \( \text{VAR}(\epsilon_t) \) is "large," the previous conclusion may be extended to state that \( \sigma(R_t) \) is not significantly larger than \( \sigma[E(R_t \mid t)] \). This conclusion implies, in turn, that stock returns are not "irrationally volatile."

This model differs significantly from previous models in at least one important sense: whereas previous models sought to examine whether stock returns variances were unduly high, the current model paradoxically encountered the need to rationalize "unduly low" volatilities. The difference in results between the current model and previous ones stems from the horizons of the various models. Previous models have typically considered a one-period return as a given in Equation (3). Conversely, the current model utilized the infinite horizon present-value relationship (2) to yield the conditional expectation (6). Note that Equation (6) has the form of an infinite summation in the \( f/(1-f) \). This term results from the expression

\[ \sum_{t=1}^{\infty} f^t. \]

The existence of the divisor \( 1-f \) results in extreme sensitivity of the conditional expectation \( E(R_t \mid I_t) \) to increases in \( f \); in particular \( \text{VAR}[E(R_t \mid I_t)] \) diverges as \( f \to 1 \). It is this particular feature which distinguishes this model from previous ones and which results in point estimates suggestive of "unduly low" stock return volatilities. However, since the variance of the conditional expectations was found to be insignificantly different from \( \text{VAR}(R_t) \), we may conclude that stock returns are not "irrationally volatile."
3. **Stock Prices under Stochastic Real Discount Rates**

3.1 **Derivation of Basic Model**

As mentioned in the introduction, the maintained hypothesis in this section is that all individuals possess the CRRA utility function

\[ U(C_0, C_1, \ldots) = \frac{1}{1-A} \sum_{t=0}^{\infty} \beta^t C^1_{t}, \quad A > 0, \quad 0 < \beta < 1 \]  \hspace{1cm} (12)

where \( C_t \) is real period-t consumption,

\( A \) is the coefficient of relative risk aversion

and \( \beta \) is the subjective rate of time preference.

The first-order necessary condition for optimality is then given by

\[ P_t = \beta E \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-A} \left( P_{t+1} + D_{t+1} \right) | I_t^* \right] \]  \hspace{1cm} (13)

where \( P_t \) is the real (inflation-adjusted) period-t ex-dividend price of an asset paying off \( \tilde{P}_{t+1} + \tilde{D}_{t+1} \) in real terms at time \( t+1 \),

and where the expectation is calculated conditional on all information available at time \( t \).

The stochastic Euler condition (2) is well-known in the financial literature. Among others, Lucas (1978) provided a discrete-time derivation, while Grossman and Shiller (1982) considered the continuous-time analogue.

Eq. (13) can be easily extended to an infinite horizon framework, either by repeatedly substituting for future prices, or by working from first principles. The result is Eq. (14):

\[ P_t = \sum_{s=1}^{\infty} \beta^s E \left[ \left( \frac{C_{t+s}}{C_t} \right)^{-A} D_{t+s} | I_t^* \right] \]  \hspace{1cm} (14)
where the expectation is calculated conditional on all information available at 
time $t$, $I_t^*$. This result is demonstrated in Appendix C.\textsuperscript{14}

Eq.(14) relates current prices to their primitive determinants: marginal
rates of substitution (discount factors) and dividends. From (14), it is easy to derive the real rates of return for ex-dividend prices

$$R_t = \frac{P_t}{P_{t-1}} = \frac{\sum_{s=1}^{\infty} \beta^s E \left[ \left( \frac{C_{t+s}}{C_t} \right)^{-A} \right] \frac{D_{t+s}}{P_{t-1}} \mid I_t^*}{1}$$ \hspace{1cm} (15)

Assuming the existence of variances, (15) implies

$$\text{VAR}(R_t) = \text{VAR} \left\{ \sum_{s=1}^{\infty} \beta^s E \left[ \left( \frac{C_{t+s}}{C_t} \right)^{-A} \frac{D_{t+s}}{P_{t-1}} \mid I_t^* \right] \right\}$$ \hspace{1cm} (16)

As in Section 2 above, in order to obtain empirically testable results, additional assumptions are required.

\subsection*{3.2 Dividend Dynamics Model}

In Section 2.2, the dividend dynamics focused on the conditional ex-
pectation $E(D_{t+s} \mid I_t^*)$ for $s \geq 1$. In Section 3, the random variables not
within $I_t$ are the values of $C_{t+s}^{-A} D_{t+s}$ for $s \geq 1$. Consequently, the econome-
trician's information set is redefined and expanded to include the past history of consumption:

$$I_t = \{P_{t-1}, D_{t-s}, C_{t-s}; \quad s = 0, 1, \ldots\}$$

Box-Jenkins time-series analysis was conducted on $C_{t}^{-A} D_{t}$ and $\Delta C_{t}^{-A} D_{t} =$
$C_t^{-A} D_t - C_{t-1}^{-A} D_{t-1}$ for various values of A between 0 and 10. Empirical analysis of the autocorrelation and partial autocorrelation functions for $\Delta C_t^{-A} D_t$ revealed behavior consistent with stationarity. Thus, $E(C_t^{-A} D_t \mid I_t)$ was modeled as

$$E(C_{t+s}^{-A} D_{t+s} - C_{t+s-1}^{-A} D_{t+s-1} \mid I_t) = \delta + \phi E(C_{t+s-1}^{-A} D_{t+s-1} - C_{t+S-2}^{-A} D_{t+s-2} \mid I_t)$$

(17)

Consequently, Eq.(5) is a special case of Eq.(17), where A has been set equal to zero.

Now, assuming once again the existence of $E(R_t \mid I_t)$, Eq.(15) implies

$$E(R_t \mid I_t) = \sum_{s=1}^{\infty} \beta^s [(C_{t+s}^{-A}/C_t)^{-A} D_{t+s} \mid I_t]/P_{t-1}.$$

Inserting (17) into this expression yields

$$E(R_t \mid I_t) = \frac{C_t^A}{P_{t-1}} \frac{\beta}{1 - \beta} \left[ \frac{\beta \gamma + \frac{1}{1 - \beta \phi}}{(1-\beta)(1-\beta \phi)} \right] \times$$

$$E(C_{t+1}^{-A} D_{t+1} \mid I_t) - \frac{\phi \beta}{1 - \beta \phi} C_t^{-A} D_t$$

(18)

The precise technical details are provided in Appendix D.

Using

$$E(C_{t+1}^{-A} D_{t+1} \mid I_t) = C_t^{-A} D_t + \delta + \phi(C_t^{-A} D_t - C_{t-1}^{-A} D_{t-1})$$
from the dividend-dynamics assumption implies

\[
\begin{align*}
E(R_t \mid l_t) &= \frac{\beta \delta}{(1 - \beta)^2 (1 - \beta \phi)} \frac{C_t^A}{P_{t-1}} \\
&\quad - \frac{\phi \beta}{(1-\beta)(1-\beta \phi)} \left( \frac{C_{t-1}}{C_t} \right)^{-A} \frac{D_{t-1}}{P_{t-1}} \\
&\quad + \frac{\beta(1 + \phi - \beta \phi)}{(1 - \beta)(1 - \beta \phi)} \frac{D_t}{P_{t-1}}
\end{align*}
\]  

(19)

Eq. (19) will permit the empirical comparison of \( \text{VAR}(R_t) \) with \( \text{VAR}[E(R_t \mid l_t)] \).

3.3 Stock Return Variances

Section 2.3 demonstrated the result that \( \text{VAR}(R_t) \geq \text{VAR}[E(R_t \mid l_t)] \). The same result obtains here, save for the fact that \( \text{VAR}[E(R_t \mid l_t)] \) is given by applying the variance operator to Eq. (19).

3.4 Derivation of Testable Equations

The parameter set in Section 3 expands on its Section 2 analogue by the addition of the relative risk aversion parameter, \( \alpha \).

In the context of the current paper, it will not be possible to determine a unique estimate of \( \alpha \). The analyses will be performed conditional on alternative (plausible) values of this parameter. Section 3.6 below implements an appropriate estimation methodology which results in a different value of \( \hat{\delta}[E(R_t \mid l_t)] \) for each value of \( \alpha \).
The non-parametric estimation procedure utilizes Eq.(13) and (17). Assuming the existence of unconditional moments, those equations reduce to

\[ \beta E \left( \left( \frac{C_{t+1}}{C_t} \right)^{-1} \frac{P_{t+1} + D_{t+1}}{P_t} \right) = 1 \]  

(20)

and

\[ E(C_t^{-A} D_{t+s} - C_{t+s-1}^{-A} D_{t+s-1}) = \delta + \]

\[ \phi E(C_{t+s-1}^{-A} D_{t+s-1} - C_{t+s-2}^{-A} D_{t+s-2}) \]  

(21)

3.5 Data

The price and dividend data were previously introduced in Section 2.5. Consumption, \( C_t \), was proxied by total annual per capita consumption expenditures on non-durables and services.

3.6 Empirical Results

Section 2.6 is now generalized to allow for stochastic real discount rates, i.e., \( A > 0 \). The methodology proposed here is such that the vector \( (\hat{\beta}, \hat{\delta}, \hat{\phi})' \) is estimated for each fixed \( A \). Namely, \( A \) is set at some value, say \( A_0 \), and then the triplet \( (\hat{\beta}, \hat{\delta}, \hat{\phi})' \) is estimated. Thus, in Step 1:

**Step 1:** Jointly estimate \( \hat{\beta} \) [in Eq.(20)] and \( (\hat{\delta}, \hat{\phi})' \) [in Eq.(21)] by non-linear seemingly unrelated regressions of

\[ \left( \frac{C_{t+1}}{C_t} \right)^{-A} \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{1}{\beta} + \epsilon_{t+1} \]
\[ C^{-A} D_{t+2} - C^{-A} D_{t+1} = \delta + \phi(C^{-A} D_{t+2} - C^{-A} D_t) + \eta_{t+1} \]

for some fixed \( A \).

Steps 2 and 3 are completed as described in Section 2.6.

As the procedure was implemented for higher \( A \) values, the locus \( \hat{\sigma}[\hat{E}(R_t \mid I_t)] \) as a function of \( A \) traced out a concave function, with a minimum at \( \hat{A} = 1.16 \).

Table 1 below presents the empirical results of the estimation procedure for \( A \) values of 0, 1 (logarithmic utility), 1.16 and 2.

<table>
<thead>
<tr>
<th>( A )</th>
<th>( \hat{\beta} )</th>
<th>( \hat{\phi} )</th>
<th>( \hat{\sigma}[\hat{E}(R_t \mid I_t)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.9354</td>
<td>.2915</td>
<td>23.0%</td>
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<tr>
<td>1</td>
<td>.9526</td>
<td>.2216</td>
<td>21.4%</td>
</tr>
<tr>
<td>1.16</td>
<td>.9554</td>
<td>.2100</td>
<td>21.3%</td>
</tr>
<tr>
<td>2</td>
<td>.9695</td>
<td>.1410</td>
<td>47.5%</td>
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</table>

The results of Section 2.6 are confirmed: it would seem that

\[ \text{VAR}[\hat{E}(R_t \mid I_t)] > \text{VAR}(R_t). \]

However, these results must be subjected to statistical analysis to determine the level of significance.

It is interesting to note the effect of an increasing \( A \) on \( \hat{\sigma}[\hat{E}(R_t \mid I_t)] \). A affects the value of \( \hat{\sigma}[\hat{E}(R_t \mid I_t)] \) both directly and indirectly. The
direct effect comes from the variable $C_t^A$ in Eq.(19); the indirect effect comes from the dependency of the parameters $\hat{\beta}$, $\hat{\delta}$, and $\hat{\phi}$ on the chosen value of $A$. The sum effect seems to be non-monotonic: at first, $\hat{\sigma}[\hat{E}(R_t \mid I_t)]$ declines; then, as $\hat{\beta}$ tends to unity, $\hat{\sigma}[\hat{E}(R_t \mid I_t)]$ grows to very large values.

While the test of statistical significance is identical to the one proposed in Section 2.6, in this section it is imperative to account for the existence of the $A$ parameter. The basic question posed all along has been this: are stock returns irrationally volatile or irrationally stable? The most conservative approach these questions is to scan over all feasible values of $A$ to ascertain whether there exists an $A$ value such that returns' variabilities can be rationally explained.

As noted above, $\hat{\sigma}[\hat{E}(R_t \mid I_t)]$ has a minimum at $\hat{A} = 1.16$. Thus, a conservative approach to hypothesis testing would dictate utilizing $\hat{A} = 1.16$. However, even this minimal value of $\hat{\sigma}[\hat{E}(R_t \mid I_t)]$ exceeds $\sigma(R_t)$. Thus, the null hypothesis $H_0$: $\sigma[\hat{E}(R_t \mid I_t)] < \sigma(R_t)$ will certainly be rejected. As for the null hypothesis $H_0$: $\sigma(R_t) < \sigma[\hat{E}(R_t \mid I_t)]$, Prob $\{\sigma[\hat{E}(R_t \mid I_t)] \leq \sigma(R_t) \mid \sigma(R_t) = .169, A \equiv 1.0\}$ serves as an upper bound on the analogous probabilities using alternative plausible values of $A$. This methodology thus provides the data with the greatest latitude to reject the null hypothesis $H_0$: $\sigma(R_t) < \sigma[\hat{E}(R_t \mid I_t)]$.

Table 2 below presents the results of the simulation analysis. The table presents the value, from the simulated distributions, of Prob $\{\sigma[\hat{E}(R_t \mid I_t)] \leq \sigma(R_t)\}$ for $A$ values of 0, 1, 1.16 and 2. In view of the numerous empirical studies which have found $A$ to lie in the interval (0,5), this interval, [0,2], is a very plausible range of values for $A$. 
Table 2 - Tests of Statistical Significance

<table>
<thead>
<tr>
<th>A</th>
<th>Prob {σ[(\hat{E}(R_t \mid l_t))] ≤ σ(R_t)}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.157</td>
</tr>
<tr>
<td>1</td>
<td>.211</td>
</tr>
<tr>
<td>1.16</td>
<td>.199</td>
</tr>
<tr>
<td>2</td>
<td>.031</td>
</tr>
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</table>

It is interesting to note, from Table 2, that the probability of \(σ[\hat{E}(R_t \mid l_t)] \leq σ(R_t)\) is not monotonic in \(σ[\hat{E}(R_t \mid l_t)]\). Namely, the probability is higher for \(A = 1\) than for \(A = 1.16\) even though \(σ[\hat{E}(R_t \mid l_t)]\) for \(A = 1\) is less than \(σ[\hat{E}(R_t \mid l_t)]\) for \(A = 1.16\). This is probably the consequence of both \(\hat{\mu}\) and \(\hat{\Sigma}\) (of the simulated distributions) changing as \(A\) changes.

Thus, it can be seen that, as was the case in Section 2.6, for the very plausible \(A\) values not exceeding 2, the data cannot significantly reject the hypothesis \(H_0: σ(R_t) = σ[\hat{E}(R_t \mid l_t)]\). The conclusion, once again, is that in the models proposed above, stock returns are not irrationally volatile.

4. Summary

This study has considered whether the volatility of inflation-adjusted common stock returns can be explained by a simple aggregate risk-neutral or risk-averse utility function.

The primary assumptions of this paper included a constant discount
rate or constant relative risk aversion utility function and a specific assumption dealing with the evolution through time of the product of (aggregate) marginal utility and dividends.

Contrary to previous models, the current model resulted in a point estimate of $\text{VAR}(R_t)$ smaller than its theoretical lower-bound $\text{VAR}[E(R_t|I_t)]$. This was found to be intimately related to discount rate factor $f = 1/(1+r)$ (under the constant discount rate case) or to the time preference parameters $\beta$ (in the CRRA utility case), and to the model's infinite-horizon valuation. Rationality of volatility was not rejected, however, since the above variances were found to be statistically indistinguishable.

In conclusion, the model contained in this study demonstrated that volatilities as high (or even higher) than observed stock return volatilities are not inconsistent with a rational aggregate utility-based valuation of stock returns.
Appendix A: Proof of Equation (6)

Define \( [s] \equiv D_{t+s} L_t \).

Now, for \( s = 1, 2, \)

\[
E([2] - [1]) = \delta + \phi([1] - D_t L_t) \\
= \delta + \phi E([1]) - \phi D_t
\]

For \( s = 2, 3 \)

\[
E([3] - [2]) = \delta + \phi(E[2] - [1]) \\
= \delta + \phi[\delta + \phi E([1]) - \phi D_t] \\
= \delta(1 + \phi) + \phi^2 E([1]) - \phi^2 D_t
\]

For \( s = 3, 4 \)

\[
E([4] - [3]) = \delta + \phi(E[3] - [2]) \\
= \delta + \phi[\delta + \phi \delta + \phi^2 E([1]) - \phi^2 D_t] \\
= \delta(1 + \phi + \phi^2) + \phi^3 E([1]) - \phi^3 D_t
\]

Thus, in general for \( s \geq 2, \)

\[
E([s] - [s - 1]) = \delta \left( \sum_{i=0}^{s-2} \phi^i \right) + \phi^{s-1} E([1]) - \phi^{s-1} D_t
\]

Further,

\[
E([s]) = \sum_{i=2}^{s} E([i] - [i - 1]) + E([1])
\]

\[
= \sum_{i=2}^{s} \left[ \delta \sum_{j=0}^{i-2} \phi^j + \phi^{i-1} E([1]) - \phi^{i-1} D_t \right] + E([1])
\]

\[
= \delta \sum_{i=2}^{s} \sum_{j=0}^{i-2} \phi^j + E([1]) \left( 1 + \sum_{i=2}^{s} \phi^{i-1} \right) - D_t \sum_{i=2}^{s} \phi^{i-1}
\]
Now, using
\[
\sum_{i=2}^{s} \phi^i = \sum_{i=2}^{s-1} \frac{s - 1}{1 - \phi} \frac{\phi}{1 - \phi} \frac{1 - \phi^s}{1 - \phi}
\]
and
\[
1 + \sum_{i=2}^{s} \phi^{i-1} = 1 + \frac{\phi(1 - \phi^s)}{1 - \phi} = \frac{1 - \phi^s}{1 - \phi}
\]
we then have
\[
E([s]) = \delta \left( \frac{s - 1}{1 - \phi} - \frac{\phi}{1 - \phi} \frac{1 - \phi^s}{1 - \phi} \right)
\]
\[
+ \frac{1 - \phi^s}{1 - \phi} E([1]) - \phi \frac{1 - \phi^s}{1 - \phi} D_t
\]
Then
\[
R_t = \frac{1}{P_{t-1}} \sum_{s=1}^{\infty} \delta_s f^s E([s])
\]
\[
= \frac{1}{P_{t-1}} \sum_{s=1}^{\infty} \delta_s \left( \frac{s - 1}{1 - \phi} - \frac{\phi}{1 - \phi} \frac{1 - \phi^s}{1 - \phi} + \frac{1 - \phi^s}{1 - \phi} E([1]) - \frac{1 - \phi^s}{1 - \phi} D_t \right)
\]
\[
= \frac{1}{P_{t-1}} \left( \frac{\delta}{1 - \phi} \left( \sum_{s=1}^{\infty} f^s - \sum_{s=1}^{\infty} f^s \phi^{s-1} \right) - \frac{1}{P_{t-1}} \frac{\phi \delta}{(1 - \phi)^2} \times \right)
\]
\[
\times \left( \sum_{s=1}^{\infty} f^s - \sum_{s=1}^{\infty} f^s \phi^{s-1} \right)
\]
+ \frac{1}{P_{t-1}} \frac{E([1])}{1 - \phi} \left( \sum_{s=1}^{\infty} f^s - \sum_{s=1}^{\infty} f^s \phi^s \right) - \frac{D_t}{P_{t-1}} \frac{\phi}{1 - \phi} \times \\

\times \left( \sum_{s=1}^{\infty} f^s - \sum_{s=1}^{\infty} f^s \phi^{s-1} \right) \\

= \frac{1}{P_{t-1}} \frac{\delta}{1 - \phi} \left[ \frac{f}{(1 - f)^2} - \frac{f}{1 - f} \right] - \frac{1}{P_{t-1}} \frac{\phi \delta}{(1 - \phi)^2} \left[ \frac{f}{1 - f} - \frac{f}{1 - f\phi} \right] \\

+ \frac{1}{P_{t-1}} \frac{E([1])}{1 - \phi} \left( \frac{f}{1 - f} - \frac{f\phi}{1 - f\phi} \right) - \frac{D_t}{P_{t-1}} \frac{\phi}{1 - \phi} \left( \frac{f}{1 - f} - \frac{f}{1 - f\phi} \right) \\

Simplifying and combining terms yields (6).
Appendix B: Calculation of Empirical Proxies for Dividends and Prices

Define

VWRETD<sub>j</sub> - total rate of return, including dividends, on value-weighted NYSE portfolio during month <i>j</i>.

VWRETX<sub>j</sub> - total rate of return, excluding dividends, on value-weighted NYSE portfolio during month <i>j</i>.

\[
\text{INDEX}_i = \begin{cases} 
1 & \text{for } i = 1 \\
\Pi_{j=2}^i (1 + \text{VWRETX}_j) & \text{for } i \geq 2 
\end{cases}
\]

B = base for standardizing price and dividend series defined below.

Then

\[
d_t = \frac{1}{B} \sum_{j=2}^{13} \text{INDEX}_i \cdot 12(t-1)+j-1 (\text{VWRETD}_{12(t-1)+j} - \text{VWRETX}_{12(t-1)+j})
\]

The nominal dividend series thus obtained excludes effects of intra-year compounding (or reinvestment) of dividends. Real dividends, D<sub>t</sub>, are then given by \( d_t/q_t \).

The remaining empirical datum is the real price series, P<sub>t</sub>. Using \text{INDEX}_i as previously defined, the base to which \( p_t \) was standardized, B, is given by

\[
B = \frac{1}{370} \sum_{i=181}^{217} \text{INDEX}_i.
\]
This gives the nominal price series $p_t$ an average value of 10 for the years 1941-43 period. While this standardization permits comparability with the Standard & Poor's price series, it in no way reduces the generality of the analysis. Indeed, $B$ is arbitrary. Using $B$ so defined, $p_t$ was given by

$$p_t = \frac{1}{13B} \sum_{j=1}^{13} \text{INDEX}_{12t-j+2}$$

Thus, $p_t$ is the average nominal price level of stocks in year $t$. The real price level, $P_t$, is then given by $P_t = p_t / q_t$. 
Appendix C: Proof of Eq.(14)

Without loss of generality, set $t = 1$.\textsuperscript{16}

Define

\[ \text{st} \in S_t = \text{set of states of the world at time } t \geq 2. \]

\[ \phi_{\text{st}} = \text{nominal price of a primitive security paying$1 at time } t \text{ iff state st occurs.} \]

Then, using

\[ \frac{\partial U}{\partial C_t} = \beta C_t^{-A}; \quad \frac{\partial U}{\partial C_{st}} = \beta^t \pi_{\text{st}} C_{C_{st}}^{-A}, \quad t \geq 2 \]

where

\[ \pi_{\text{st}} = \text{Prob(st at time } t | \bar{x}_t). \]

Under competitive equilibrium

\[ \frac{q_{\text{st}} \phi_{\text{st}}}{q_1} = \beta^{t-1} \pi_{\text{st}} C_{C_{st}}^{-A} / C_1 \]

where $q_{\text{st}}$ is the price of the consumption good in state $\text{st} \in S_t$ and $q_1$ is the analogous price at $t = 1$.

Rearranging,

\[ \phi_{\text{st}} = \beta^{t-1} \pi_{\text{st}} \frac{C_{C_{st}}^{-A}}{C_t} \frac{q_1}{q_{\text{st}}} \]

The nominal price of the complex security at time $1$, $p_1$, is given by

\[ p_1 = \sum_{t=2}^{\infty} \sum_{\text{st} \in S_t} \phi_{\text{st}} d_{\text{st}} \]

where $d_{\text{st}}$ is nominal payoff of the security in state $s$ at time $t$. 
\[ p_1 = \sum_{t=2}^{\infty} \beta^{t-1} \sum_{s \in S_t} \pi_{st} (\frac{C_{st}}{C_1})^{-A} \frac{q_1}{q_{st}} \ d_{st} \]

\[ \Rightarrow p_1 = \sum_{t=2}^{\infty} \beta^{t-1} E \left[ (\frac{C_{t-1}}{C_1})^{-A} \frac{q_1}{q_t} d_t \mid l_1^* \right] \]

Division by \( q_1 \) is permissible since \( q_1 \in l_1^* \). This yields eq. (14).
Appendix D: Proof of Eq.(18)

Eq.(18) may be proved by generalizing the procedure utilized in the proof of Eq.(6) in Appendix A.

Define \([s] = C_t^{-A}D_t^{-s\ t}\)

Then, repeating the steps in Appendix A, for \(s \geq 2\),

\[
E([s] - [s-1]) = \delta(\sum_{i=0}^{s-2} \phi^i) + \phi^{s-1} E([1]) - \phi^{s-1} C_t^{-A}D_t
\]

which then implies

\[
E([s]) = \delta \left( \frac{s - 1}{1 - \phi} - \frac{\phi}{1 - \phi} \right) \frac{1 - \phi^{s-1}}{1 - \phi} + \frac{1 - \phi^s}{1 - \phi} E([1]) - \phi \frac{1 - \phi^{s-1}}{1 - \phi} C_t^{-A}D_t
\]

Finally,

\[
R_t = \left(\frac{C_t^A}{P_t}\right) \times \sum_{s=1}^{\infty} \beta^s E([s]),
\]

which, upon simplification, yields Eq. (18).
FOOTNOTES


3. Tildes denote random variables.

4. Grossman and Shiller (1981) finesse this problem using a perfect foresight assumption: namely, they include $C_{t+1}', D_{t+1}', C_{t+2}', D_{t+2}', \ldots, D_T$ and $P_T$ (where $T$ is the terminal date) in the information set at time $t$.

5. Note that $I_t'$ lacks an asterisk, to distinguish the econometrician's information set from the full-information set $I_t^*$.

6. Note that $E(\varepsilon_t') = E[E(\varepsilon_t \mid I_t')] = E[E(\Delta D_t \mid I_t') - \hat{E}(\Delta D_t \mid I_t')] = 0$.

7. That is, eq. (8) implies that a larger $\text{VAR}[E(R_t \mid I_t)]$ can "sustain," or provide rationale for a larger $\text{VAR}(R_t)$. That greater magnitude of $\text{VAR}(R_t)$ is thus justifiable by a rational, utility-based model.

8. Note that theory relates prices to spot dividends. The data used in this paper approximate the theoretical values by using average values
of prices and dividends.

9. Any non-linear regression requires initial parameter estimates. Such initial estimates were obtained by estimating the linear two-equation system modified from that presented in the text. This linear system was obtained by replacing the first equation with \( (P_{t+1} + D_{t+1})/P_t = b + \epsilon_{t+1} \). Then, with the second equation unchanged, estimates were obtained for \( \hat{b}, \hat{\delta}, \hat{\phi} \). Consequently, the initial estimates (inserted in the non-linear system) for \( (f, \delta, \phi)' \) were \( (1/\hat{b}, \hat{\delta}, \hat{\phi}) \).

10. The sample size of the simulation was chosen to be 1,000. The IMSL subroutine GGNSM was used to simulate the random drawings.

11. Thus, any measurement error attributable to the imprecise nature of statistical estimation is assumed away.

12. \( E(R_t | l_t) \) cannot be computed without knowledge of the true parameter vector \( (f, \delta, \phi)' \).

13. This is a plausible assumption. The source of the measurement error \( \epsilon_t \) follows from the use of the estimated parameter vector \( (\hat{f}, \hat{\delta}, \hat{\phi})' \) rather than the true parameter vector \( (f, \delta, \phi)' \). Since \( (\hat{f}, \hat{\delta}, \hat{\phi}) \) was estimated from the unconditional moments, it should not be correlated with the information set \( I_t \).

14. It turns out that Eq.(9) takes full account of the impact of inflation. From the first-order conditions, it follows that
\[
\rho_1 = \sum \beta^{t-1} \mathbb{E} \left[ \left( \frac{C_t}{C_1} \right)^{-A} \frac{q_1}{q_t} d_t \mid \mid_1^x \right]
\]

where \( \rho_1 \) is the nominal price of the complex security at time 1 (no loss of generality is incurred by this "standardization" of time to unity); \( d_t \) is the (random) nominal payoff of the security at time \( t \); and \( q_t \) is the (random) price of the consumption good at time \( t \). Thus, the nominal price \( \rho_1 \) is related to real dividends at time \( t \), \( d_t/q_t \). The intuition is quite simple: we are foregoing real purchasing power in the amount of \( \rho_1 \) in return for real purchasing power in the future. Looking at the above equation it would seem that inflation interactions may be important. However, rewriting the equation,

\[
\frac{\rho_1}{q_1} = \sum \beta^{t-1} \mathbb{E} \left[ \left( \frac{C_t}{C_1} \right)^{-A} \frac{d_t}{q_t} \mid \mid_1^x \right]
\]

and, generalizing from time period 1 to arbitrary period \( t \) yields Eq.(9), since \( \rho_1 = \rho_1/q_1 \) and \( D_t = d_t/q_t \). Thus, (9) relates real prices to real dividends.

15. Alternative values of \( A \) will also result in different simulated distribution functions for \( \sigma(\hat{E}(R_t \mid I_t)) \).

16. As previously noted, this "standardization" of time is designed to achieve parsimonious notation. No unique standardization is afforded to time period \( t = 1 \).
BIBLIOGRAPHY


