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Inflation Futures and a Riskless Real Interest Rate

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Abstract

This paper investigates the possibilities of creating a term structure of \textit{ex ante} realizable risk free real interest rates by forming portfolios of inflation futures and discount bonds. We analyze three alternative models determining the number of futures contracts necessary to hedge a given investment. Using our theoretical results, we use data for the first 15 trading months of the inflation futures contracts to map out real term structures and intertemporal changes of the real interest rate. We find indications of a non-constant \textit{ex ante} one-year real interest rate.
Inflation Futures and a Riskless Real Interest Rate

The proposal to introduce trading futures contracts on economic indices meets a real economic need. If adopted, these futures contracts on indices will improve the efficiency of markets and increase their competitiveness.

As an example, thousands of enterprises have to enter into wage contracts that will affect their costs for many months to come. Since the future rate of rises in general prices cannot now be known, the risks involved are appreciable. Once these futures markets exist, producers can hedge part of this intrinsic risk and therefore be in a position to concentrate on the tasks of making and marketing goods. Consumers, workers, and capital providers all stand to benefit. The stability of the credit system is enhanced.

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I. Introduction

The construction of a riskless real, inflation-adjusted, rate of return has long preoccupied financial economists. A real riskless bond responds most accurately to the question, how much more real consumption will the investor-consumer attain if he/she foregoes a unit of consumption today? This problem has become especially acute in countries suffering from high -- and, almost invariably, highly variable -- rates of inflation. In the post-World War II era, decision makers in high-inflation countries have been concerned with the severe macroeconomic and social implications of the high and volatile rise in the price level. One topic of particular concern has been the effect of inflation on private savings. Absent an effective tool for hedging the rate of inflation -- anticipated as well as unanticipated -- there existed substantial concern that the savings pool would quickly dry up.
In response to these issues, several national governments in Latin America, Europe and Asia devised the construct of index-linked bonds, whereby both coupon and principal are adjusted upwards by the rate of inflation. These indexed bonds have been used extensively in Argentina, Brazil, Finland, the United Kingdom and Israel. The existence of these bonds has doubtless played a significant role in preventing a diminution of the national savings rate.

In the United States, the Federal Government has abstained from the issuance of index-linked bonds. To the extent that inflation-hedging demands exist in the U.S. market, such bonds could conceivably be sold at a lower real interest rate than the (expected) real interest rate on nominal U.S. Government obligations. Despite such potential advantages, the absence of index-linked bonds may be attributable to the 20th century U.S. experience of extended periods of low and stable inflationary rates.

In this paper, we examine the possibility of using the newly-introduced CPI-W futures contracts¹ -- known more popularly as "inflation futures" -- to create a riskless real security for investors. There is, moreover, one important institutional innovation of inflation futures over index-linked bonds: whereas the latter rely upon the ability and willingness of governments to print money,² the former involve risk-sharing through the market mechanism. Consequently, the riskless real interest rates embodied in these futures contracts serve as efficient measures of the economy's riskless real rate of interest.

¹ The nomenclature follows from the index upon which the contracts are based: the Consumer Price Index for Urban Wage Earners and Clerical Workers.

² Huberman and Schwert (1984) note that, in the context of Israeli index-linked bonds, "there is some probability of default, either as a result of wars or because the Israeli government changes the terms of the bond contract ex post" (p. 7). Naturally, for index-linked bonds issued by corporations, the risk of default would preclude these from serving as riskless instruments.
The contribution of the current paper should not be viewed as restricted to the analysis of the (relatively inactive) CPI-W futures contracts. Rather, the paper also investigates the theoretical issues involved in the creation of a riskless real interest rate through the use of inflation futures contracts based on a specific price index and published at some lag after the measurement date. Additional theoretical examples of new or as-yet-untraded securities may be found in the analyses of supershares [Hakansson (1976)], hedge portfolios [Garman (1976)], and “Delta” securities [Breeden and Litzenberger (1978)].

The academic literature has contained extensive discussion of inflationary impact on asset prices in general, and the desirability of index-linked bonds in particular. While an exhaustive review of the inflation literature is beyond the scope of this article, a discussion of index-linked bonds can be found in Ahtiala (1967), Collier (1969), and Heizel and Babbel (1982). Fischer (1975) develops an intertemporal optimization model of the demand for securities and shows a positive demand for index-linked bonds under very weak assumptions. Liviathan and Levhari (1977) address the supply of index-linked bonds and conclude that high uncertainty about future inflation will discourage potential issuers. Hakansson (1976) advocates the “Purchasing Power Fund,” a financial intermediary that would issue contingent claims based on the real value of the market portfolio, thus permitting the construction of a real risk-free portfolio.

A number of empirical studies have been carried out using data from countries in which index-linked bonds have been actively traded. A discussion of the Brazilian experience may be found in Fishlow (1974). The Israeli case is analyzed in Huberman (1981) and in Huberman and Schwert (1984), the British situation in Wilcox (1985), and the Argentinian market in Boschen and Newman (1986).
A comprehensive theoretical analysis of inflation and interest rates in an intertemporal general equilibrium setting can be found in Breeden (1986), whereas Benninga and Protopapadakis (1983) examine the relation between the term structures of real and nominal interest rates under uncertainty. Attempts to test hypotheses regarding the variability over time of the real interest rate includes Fama (1975) and Nelson and Schwert (1977).

Another wholly different body of literature also relevant to our study is the writings on the relationship between prices of futures and forward contracts. Black (1976) was the first to address the question and his analysis was extended by Cox, Ingersoll and Ross (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981), who all pointed out that futures prices in general would differ from forward prices when interest rates are stochastic. However, simulations by Rendleman and Carabini (1979), and empirical studies by Cornell and Reinganum (1981) and Park and Chen (1985) indicate that the difference is insignificant.

The paper is now structured as follows. In Section II of the paper, we show how an investor can virtually eliminate uncertainty about inflation by investing in a portfolio of inflation futures and discount bonds. We present three different models each representing a different approach to handling the problems created by the report lag in the Consumer Price Index (CPI-W). In Section III we present some empirical results in which we utilize the models from Section II to map out term structures of *ex ante* realizable real risk-free interest rates for three specific dates. In addition, we also show some evidence of non-stationarity of real interest rates over the period August 1985 - October 1986. Finally, in Section IV we conclude with a short discussion of the limitations of our analysis and the applicability of our results.
II. Theoretical Analysis

The main purpose of this paper is to show how an investor can combine default free discount bonds and inflation futures in a portfolio to get a real risk free return on his/her investment. For institutional reasons, it is not possible to create an exact hedge against changes in the CPI-W, but we will develop three ways of determining an approximate hedge ratio; that is, the number of inflation futures contracts one needs to hedge an investment in discount bonds.³ We assume perfect divisibility of the contracts, and in the present chapter we also treat the inflation futures as if they were forward contracts, assuming that the marking-to-market effect is insignificant.⁴ Our analysis takes the view of a tax-exempt institution, such as a pension fund, investing in these contracts. Section II.6 considers an exploratory investigation into taxable aspects of this problem.

The major obstacle to creating a perfect hedge against unexpected changes in the CPI-W is the time lag between price data collection and the time of announcement of the level of the CPI-W. Prices are sampled throughout each month, so we can generally assume that they best represent the price level around the 15th of the month. The level of the CPI-W is announced between the 21st and 25th of the following month, making the time lag somewhat in excess of five weeks. The CPI-W futures contract calls for cash settlement of the futures contract on the release date of the CPI-W index. This lag period causes two different problems. First, the futures prices as of today partly reflects traders' beliefs about the inflation from the last CPI-W announcement and up until today. Since this obscures the relation

³ Appendix A contains the precise description of the contract specifications of the CPI-W futures contract.

⁴ We defer to Section II.5 a consideration of the distinction between forward and futures prices.
between future inflation and the price development of the futures contract, it reduces the
quality of the hedge. Second, there will be a five-week period at the end of the life of a
contract during which its value is not affected by inflation.

II.1 Model 1 - Naive Model

In our simplest model we assume away the time lags described above and show how that
leads to the possibility of creating a perfect hedge against unexpected inflation.

Notation:

$I_0$ amount invested in a default-free pure discount bond maturing $T$ years from now
(at time $T$).

$\theta_R$ nominal risk-free rate of interest from today to time $T$; not annualized.

$C_0$ level of CPI-W today.

$\bar{C}_T$ level of CPI-W at time $T$.

$\theta_F$ the futures price today of an inflation futures contract maturing at time $T$.

$\alpha$ number of inflation futures contracts bought today, maturing at time $T$ (decision
variable).\textsuperscript{5}
The real return on the portfolio will in general depend on the stochastic term \( \tilde{C}_T \) as follows:

\[
\nu^r_T = \frac{I_0(1 + R_T) + \alpha(\tilde{C}_T - \int_0^T F_T)}{I_0(\tilde{C}_T/C_0)} - 1
\]

\[
= \frac{I_0(1 + R_T) - \alpha \int_0^T F_T}{I_0(\tilde{C}_T/C_0)} + \frac{C_0}{I_0} - 1
\]

The numerator in eq. (1) considers the payoff of the futures contract to be identical to that of the forward contract: gains and losses are realized at maturity rather than on a marked-to-the-market basis throughout the life of the contract. Second, note that the divisor calculates the (random) real value of \( I_0 \) at time \( T \) as \( I_0(\tilde{C}_T/C_0) \). Thus, the model implicitly assumes that the last published CPI-W index, \( C_0 \), is the appropriate deflator (basis) for the calculation of the cumulative inflation rate over the interval \([0, T]\). A more substantive assumption is embodied in the inflation rate \( \tilde{C}_T/C_0 \): that calculation assumes that the CPI-W index is the appropriate index for the deflation of nominal values. As Breeden (1979) has pointed out, the CPI-W index is not the most appropriate theoretical measure of inflation. Although a wealth-invariant price index\(^5\) does not exist for arbitrary preferences, Breeden

\[5\] The CPI-W index provides for a multiplier of 1,000, so that the profit on a position of \( \alpha \) contracts is \( 1000\alpha(\tilde{C}_T - \int_0^T F_T) \). As this multiplier is simply a scale factor, we ignore it in the analysis which follows, thus assuming a multiplier of unity.

\[6\] A wealth-invariant (multi-commodity) price index is one which satisfies the conditions of a price index (same utility at different price levels) independent of the individual's wealth. Such a price index exists globally only for individuals whose indifference curves are "homothetic," i.e., they have unitary elasticities of demand for all goods. See Samuelson and Swamy (1974) for a careful analysis of price index results.
demonstrates that in a continuous-time environment the personal consumption expenditures deflator (the so-called “PCE deflator”) -- an average expenditures share-weighted price index -- is a more appropriate welfare measure of price inflation in a continuous-time economy.\footnote{In Breeden's words, "it can be shown that a feasible (but not necessarily optimal) allocation exists such that everyone in the economy has a consumption allocation that is preferable to his current allocation if and only if the percentage change in aggregate nominal expenditure exceeds the percentage change in the average budget share price index [emphasis added] ..." [Breeden (1979), pp. 288-289].}

In our current analysis, we abstract from these theory-related price index considerations and take the CPI-W as appropriate proxy for utility-based price changes.

Now, from eq. (1), we see that we can immunize $\theta r_T$ from the uncertainty of $\tilde{C}_T$ by choosing

$$a^* = \frac{I_0(1 + \theta R_T)}{\theta F_T}$$

(2)

Substituting this value of $a$ back into (1), we get the risk-free real interest rate

$$\theta r_T = (1 + \theta R_T) \frac{C_0}{\theta F_T} - 1,$$

(3)

which translates into the annualized rate

$$\theta r_T^* = \left[(1 + \theta R_T) \frac{C_0}{\theta F_T}\right]^{1/T} - 1.$$

(4)

This is a simple expression that can be used for a quick look at the term structure of the real
risk-free interest rate. If we take the last observed value of the CPI-W as $C_0$, the expression involves only readily observable data. Note that in a time of positive inflation, both $C_0$ and $\theta F_T$ will be downward biased because of the time lag, such that the biases will be partly cancelling. We also expect the approximation error to be inversely related to the time to maturity $T$.

Figure 1 now presents a graphical description of the inflation rate over two lag periods, $	ilde{\varepsilon}_1$ and $	ilde{\varepsilon}_2$, which impinge on the calculation of the real interest rate in eq. (4). We use as an example portfolios formed on February 1 with maturity corresponding to the July contract of the same year. Thus, the real interest rate calculated in eq. (4) will be unbiased so long as $\tilde{\varepsilon}_1 \approx \tilde{\varepsilon}_2$.

II.2 Model 2 - A More Sophisticated Approach

In this somewhat more sophisticated model, we do away with the no-lag assumption by substituting in some less dramatic assumptions. These mainly involve perceiving changes in the CPI-W level over short periods in time as being linear.

The underlying investment strategy in this model consists of creating a portfolio of discount bonds and two different maturities of inflation futures contracts. On the day of maturity for the discount bonds and the shorter term futures contract, we also close out our position in the longer term contract. The problem therefore is to solve for the optimal number of each of these two futures contracts.

---

8 The “longer-term” contract is the next maturing contract (separated by at least three months) after the “shorter-term” contract.
CALCULATION OF REAL RATES: MODEL 1

Figure 1

Implied Inflation

Nominal Interest

Time

--- e1 = e2 by assumption
Notation:

$I_0$ amount invested in a default-free pure discount bond maturing at time $T$.

$0R_T$ nominal risk-free rate of interest from today to time $T$; not annualized.

$C - d_1$ last computed level of the CPI-W, measured $d_1$ days ago.\(^9\)

$\tilde{C}_0$ true (unknown) current level of the CPI-W.

$\tilde{C}_{T - d_3}$ computed level of the CPI-W at time $T - d_3$, being announced at time $T$.\(^10\)

$\tilde{C}_T$ true level of the CPI-W at time $T$.

$0F_{d_2}$ current futures price of the next CPI-W contract to mature, being settled based on a CPI-W level $d_2$ days from now.\(^11\)

---

\(^9\) Thus, if time 0 is February 1, then $d_1 = 45$ days, as the latest available CPI-W index is the one published in late January and applicable to mid-December.

\(^10\) Under current provisions and practices, the Bureau of Labor Statistics announces the index in the third week of the month succeeding its measurement. If the average price level applies to the mid-point of the month, $d_3 \approx 15 + 21 = 36$.

\(^11\) The number of days $d_2$ is selected in such a fashion that the 15th of the month to which it applies (i.e., the 15th of the month preceding the settlement month) is chronologically later than time 0. Thus, if time 0 is April 1, 1986, then $d_2$ will be no earlier than the July 1986 contract which measures the CPI-W index on June 15, 1986. (If there existed a June 1986 contract -- with measurement date of May 15 -- then that contract would have served in the $0F_{d_4}$ role.)
current futures price of the CPI-W contract maturing at time $T$.

current futures price of the next CPI-W contract to mature after time $T$, maturing $d_4$ days after time $T$.\(^{12}\)

unknown (random) futures price at time $T$ of CPI-W contract maturing at time $T + d_4$.\(^{13}\)

number of futures contracts bought maturing at time $T$.

number of futures contracts bought maturing at time $T + d_4$.

real rate of return on the portfolio from now to time $T$; not annualized.

Using the above notation, we have the following exact (barring the difference between futures and forwards) relation governing the real rate of return on a portfolio with $I_0$ invested in discount bonds (T-bills) and, respectively, $\alpha$ and $\beta$ short and long term CPI-W futures contracts:

\(^{12}\) The reader will notice a notational difference between $0\tilde{F}_d$ and $0\tilde{F}_{T + d_4}$: $d_2$ refers to the measurement date of the CPI-W (mid-month preceding the settlement month), whereas $d_4$ is the maturity date. This identical notation for somewhat different purposes was maintained for parsimony of notation.

\(^{13}\) For the notation to be precisely correct, the number of days $d_4$ should be subscripted with a $T$ as it varies from 3 months to 6 months, depending on the length of time $T$. (See Appendix A, Contract Specification No. 2, "Trading Months and Hours.") For parsimony of notation and since no ambiguity results, the $T$ subscript on $d_4$ has been omitted.
\[ 0 \tilde{r}_T = \frac{I_0(1 + 0 R_T) + a(\tilde{C}_{T-d_3} - 0 F_T) + \beta(\tau \tilde{F}_{T+d_4} - 0 F_{T+d_4})}{I_0(\tilde{C}_T/\tilde{C}_0)} - 1 \] (5)

To eliminate the unknown values of the CPI-W in the denominator, we assume that the following conditional expectations hold

\[
E(\tilde{C}_0|C_{-d_1}, 0 F_{d_2}) = C_{-d_1} + \frac{d_1}{d_1 + d_2} (0 F_{d_2} - C_{-d_1}) = aC_{-d_1} + (1 - a) 0 F_{d_2} \tag{6}
\]

\[
E(\tilde{C}_T|\tilde{C}_{T-d_3}, \tau \tilde{F}_{T+d_4}) = \tilde{C}_{T-d_3} + \frac{d_3}{d_3 + d_4} (\tau \tilde{F}_{T+d_4} - \tilde{C}_{T-d_3}) \equiv b_T \tilde{F}_{T+d_4} + (1 - b) \tilde{C}_{T-d_3} \tag{7}
\]

An intuitive interpretation of eqs. (6) - (7) may be in order. Eq. (6) states that the current CPI-W level is an average -- weighted by number of days -- between the date at which the last CPI-W was measured and the next measurement date as given by the \(0F_{d_2}\) contract. Similarly, eq. (7) states that the value of the CPI-W at time \(T\) will be an analogously-valued weighted index.\(^{14}\) Using these conditional expectations as approximations for the real values, (5) can be rewritten as

\(^{14}\) Thus, eqs. (6) and (7) implicitly assume that, for the short time span implied in the time intervals \([0, d_2]\) and \([T, T+d_4]\), the futures price is an unbiased linear predictor of the rate of inflation.
\[ 0 \tilde{r}_T = \frac{I_0(1 + _0 R_T) + \alpha (_0 \tilde{C}_{T-d_3} - _0 F_T) + \beta (_0 \tilde{F}_{T+d_4} - _0 F_{T+d_4}) - I_0 \left[ \frac{b_T \tilde{F}_{T+d_4} + (1 - b) \tilde{C}_{T-d_3}}{a C_{-d_1} + (1 - a) _0 F_{d_2}} \right]}{I_0 \left[ \frac{b_T \tilde{F}_{T+d_4} + (1 - b) \tilde{C}_{T-d_3}}{a C_{-d_1} + (1 - a) _0 F_{d_2}} \right]} ] \]

Performing a first order Taylor expansion of the denominator around \( (_0 F_T, _0 F_{T+d_4}) \) we see that (8) can be written as

\[ _0 \tilde{r}_T = \frac{\tilde{N} - \tilde{\pi}}{1 + \tilde{\pi}} \left[ 1 - \frac{1}{1 + \tilde{\pi}} (\tilde{\pi} - \pi) \right], \tag{9} \]

where

\[ \tilde{N} = (1 + _0 R_T) + \frac{\alpha}{I_0} (_0 \tilde{C}_{T-d_3} - _0 F_T) + \frac{\beta}{I_0} (_0 \tilde{F}_{T+d_4} - _0 F_{T+d_4}) \tag{10} \]

\[ \tilde{\pi} = \frac{b_T \tilde{F}_{T+d_4} + (1 - b) \tilde{C}_{T-d_3}}{a C_{-d_1} + (1 - a) _0 F_{d_2}} \tag{11} \]

\[ \pi = \frac{b_0 F_{T+d_4} + (1 - b) _0 F_T}{a C_{-d_1} + (1 - a) _0 F_{d_2}} \tag{12} \]

For the time horizons we are looking at and with the variability of inflation currently expe-
rienced, we would expect \( \bar{\pi} \) to be close enough to \( \bar{\pi} \) so that we can get a good approximation by assuming

\[
1 - \frac{1}{1 + \bar{\pi}} (\bar{\pi} - \bar{\pi}) \approx 1
\]

(13)

Substituting (13) into (8) and rearranging we have

\[
0 \bar{F}_T \approx \left[ \frac{1}{I_0} - \frac{1 - b}{aC_{-d_1} + (1 - a) \sigma d_2} \right] \bar{C}_{T-d_3} + \left[ \frac{\beta}{I_0} - \frac{b}{aC_{-d_1} + (1 - a) \sigma d_2} \right] \bar{F}_{T+d_4}
\]

\[
\left[ \frac{b \sigma F_{T+d_4} + (1 - b) \sigma F_T}{aC_{-d_1} + (1 - a) \sigma d_2} \right]
\]

From (14) we see that we can hedge away all remaining risk related to the stochastic terms \( \bar{C}_{T-d_3} \) and \( \bar{F}_{T+d_4} \) by choosing the values for \( \alpha \) and \( \beta \) given by:

\[
\alpha^* = \frac{1 - b}{aC_{-d_1} + (1 - a) \sigma d_2} \cdot I_0
\]

(15)
\[ \beta^* = \frac{b}{aC_{a_1} + (1 - a) \phi_{d_2}} \cdot \phi_0 \]  \hfill (16)

Substituting \( \alpha^* \) and \( \beta^* \) into (13) we get the approximately real risk-free rate of return on the portfolio:

\[ \sigma_T = (1 + \sigma_T) \frac{aC_{a_1} + (1 - a) \phi_{d_2}}{b \phi_{T+d_4} + (1 - b) \phi_T} - 1 \]  \hfill (17)

And as an annualized rate of interest:

\[ \sigma_T^{*} = \left[ (1 + \sigma_T) \frac{aC_{a_1} + (1 - a) \phi_{d_2}}{b \phi_{T+d_4} + (1 - b) \phi_T} \right]^{1/T} - 1 \]  \hfill (18)

The accuracy of the approximations made above will to a large extent depend on how well a smooth linear function describes short-term inflation and on the predictive power of the inflation futures prices. In the same way as for Model 1, however, we will have that the relative impact of the approximation errors will diminish as the time horizon involved becomes increasingly large.

Figure 2 presents the graphical presentation of Model 2, in analogous fashion to the role performed by Figure 1 for Model 1.
CALCULATION OF REAL RATES: MODEL 2

Figure 2

- Implied Inflation (1)
- Implied Inflation (2)
- Implied Inflation (3)
- Nominal Interest

---

- $e_1$ given by (1)
- $e_2$ given by (2),(3)

Time

15-Dec 01-Feb 15-Mar 15-Jun 21-Jul 15-Sep
II.3 Model 3 - An Alternative Sophisticated Model

Our last proposed model involves a slight change in investment strategy. We are back to a portfolio consisting of discount bonds and single-maturity CPI-W futures contracts, but now we have bonds which mature in the middle of the month before the futures mature. This means that we will close out our position at the time of which the real CPI-W level is approximately equal to the level at which the futures contract is going to be settled. Therefore, if the market correctly guesses the current price level at this time, and this is reflected in the futures price without a bias, then our position will be well hedged.

For parsimony of notation, we have retained the notation of Model 2.

The exact real rate of return is given by

\[
0^\tau_{T-d_3} = \frac{I_0(1 + \alpha R_{T-d_3}) + \alpha(T-d_3 F_T - 0 F_T)}{I_0(C_{T-d_3})} - 1
\]  \hspace{1cm} (19)

We now assume that the futures price at time \(T-d_3\) is an unbiased (and good) predictor of the price level at time \(T-d_3\):

\[
E(C_{T-d_3}|T-d_3 F_T) = T-d_3 F_T
\]  \hspace{1cm} (20)

Substituting in the expected values from (20) and (6) into (19) and rearranging, we get
\[ 0r_{T-d_3} \approx \frac{(1 + 0 R_{T-d_3}) + \left[ \frac{\alpha}{I_0} - \frac{1}{aC_{-d_1} + (1 - a) 0F_{d_2}} \right] T - d_3 \tilde{F}_T - \frac{\alpha}{I_0} 0F_T}{\frac{T - d_3 \tilde{F}_T}{aC_{-d_1} + (1 - a) 0F_{d_2}}} \]  

\[ (21) \]

Upon application of a Taylor's series expansion as in Model 2, we replace \( T - d_3 \tilde{F}_T \) in the denominator with today's futures price \( 0F_T \), thereby getting

\[ 0\tilde{r}_{T-d_3} \approx \frac{(1 + 0 R_{T-d_3}) + \left[ \frac{\alpha}{I_0} - \frac{1}{aC_{-d_1} + (1 - a) 0F_{d_2}} \right] T - d_3 \tilde{F}_T - \frac{\alpha}{I_0} 0F_T}{\frac{0F_T}{aC_{-d_1} + (1 - a) 0F_{d_2}}} \]  

\[ (22) \]

From (22) we see that we can choose \( \alpha \) so as to make \( 0r_T \) deterministic. To have this, set

\[ \alpha^* = \frac{1}{aC_{-d_1} + (1 - a) 0F_{d_2}} I_0 \]  

\[ (23) \]

Substituting \( \alpha^* \) back into (22) yields the approximate real risk free rate of return on the portfolio:

\[ 0r_{T-d_3} \approx \left[ (1 + 0 R_{T-d_3}) \frac{aC_{-d_1} + (1 - a) 0F_{d_2}}{0F_T} \right] - 1 \]  

\[ (24) \]

which when annualized is equal to
\[ 0^{\text{er}}_{T-d_3} = \left[ (1 + 0 R_{T-d_3}) \frac{aC - a_1 + (1 - a) 0 F_{d_2}}{0 F_T} \right]^{1/T} - 1. \]  

(25)

The relative merit of this model clearly hinges on the unbiasedness and accuracy of the inflation futures price as a predictor of the price level.\(^{15}\)

Despite their greater sophistication, there is one disadvantage which Models 2 and 3 bear relative to Model 1: they require a closing out of a futures contract prior to maturity, thus subjecting the hedger to a form of "basis" risk. In contradistinction, Model 1 simply requires that the hedger let the futures contract expire and then receive the cash settlement on that date.

Figure 3 presents the graphical description of the approximations utilized to yield the riskless real interest rate in Model 3.

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\(^{15}\) Using CPI-W futures prices allows us to shed some light on the validity of this assumption. The table below indicates the predictive ability of the inflation futures price on the 16th of the month prior to publication of the CPI-W index. The data presents evidence of all five maturity dates of the CPI-W futures contract since its inception:

<table>
<thead>
<tr>
<th>Month</th>
<th>Futures Price</th>
<th>CPI-W</th>
<th>Proportional Error (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sep-85</td>
<td>319.85</td>
<td>320.5</td>
<td>-0.20%</td>
</tr>
<tr>
<td>Dec-85</td>
<td>322.25</td>
<td>322.6</td>
<td>-0.11%</td>
</tr>
<tr>
<td>Mar-86</td>
<td>323.90</td>
<td>321.4</td>
<td>0.77%</td>
</tr>
<tr>
<td>Jun-86</td>
<td>321.25</td>
<td>323.0</td>
<td>-0.54%</td>
</tr>
<tr>
<td>Sep-86</td>
<td>325.40</td>
<td>324.9</td>
<td>0.15%</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td>0.01%</td>
</tr>
</tbody>
</table>

While this data does not contain conclusive statistical evidence, the low average proportional error indicates that the data is consistent with the unbiasedness assumption of eq. (20).
CALCULATION OF REAL RATES: MODEL 3

Figure 3

Implied Inflation (2) [contract matures 7/21]

Nominal Interest

Implied Inflation (1)

e1  e2

15-Dec  01-Feb  15-Mar  15-Jun  21-Jul

--- e1 given by (1)  Time  --- e2 given by (2)
II.4 Comparisons of Hedge Ratios for Models I-III

A comparison of the number of CPI-W futures contracts across the three models demonstrates elements of the Models' distinctions. Letting subscripts denote model number, it may easily be verified that

$$a_{III} = a_{II} + \beta_{II}$$

Further, noting that $\alpha_I = [(1 + 0 R_T) / 0 F_T] I_0$ and $\alpha_{III} = I_0 / E(\bar{C}_0)$, we obtain

$$\frac{\alpha_I}{a_{III}} \equiv \frac{1 + 0 R_T}{1 + \pi_{T-d_3}}$$

where $\pi_{T-d_3} \equiv 0 F_T / E(\bar{C}_0)$ is the inflation rate implied by the $T$-period future contract and our date-0 estimates of the CPI-W level. Thus $\alpha_I > \alpha_{III}$ as long as the estimated real interest rate is positive. Finally, in terms of the face value of the inflation futures contracts per dollar invested in the nominally riskless asset, Table 1 presents the analytical comparison of the three models:

16 Clearly, the initiation of a futures contract is a zero-cost investment (abstracting from transaction costs). Nevertheless, it is instructive to examine the hedge ratio in terms of the dollar volume of futures contracts purchased.
Table 1. Comparison of Models' Dollar Volume Hedge Ratios

<table>
<thead>
<tr>
<th>Model</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model I</td>
<td>$(1 + 0 \ R_T) \ F_T / I_0 \ 0 F_T / I_0 = 1 + 0 \ R_T$</td>
</tr>
<tr>
<td>Model II</td>
<td>$(1 - b) \ F_T \ E(C_0) + b \ F_T + d_4 \ E(C_0) = (1 - b)(1 + \pi_T - d_3) + b(1 + \pi_T + d_4)$</td>
</tr>
<tr>
<td>Model III</td>
<td>$\ F_T / I_0 = 1 + \pi_T - d_3$</td>
</tr>
</tbody>
</table>

II.5 Forward and Futures Prices

The distinction between forward and futures contracts was thoroughly analyzed by Cox, Ingersoll and Ross (1981). They use a continuous-time, continuous-state framework to show that when both futures and forward contracts are traded we will have

\[
\text{sign} \ [G(t) - H(t)] = \text{sign} \ \{ \text{Cov} [H(u), P(u)] \}
\]

where

$G(t) - \text{is the forward price at time } t$;

$H(t) - \text{is the futures price at time } t$;

$P(t) - \text{is the price at time } t \text{ of a default free discount bond maturing at the same time as}$

the futures and forward contracts; and
\( u - \) is any time between \( t \) and the contracts' maturity date.

Assuming a positive relation between inflation and nominal interest rates, we have 
\[
\text{Cov}[H(u), P(u)] < 0 \quad \text{and, consequently,} \quad G(t) < H(t).
\]
Thus the inflation rate obtained from the futures price is greater than that implied by a forward contract. Consequently, our estimates of the real rates are downward biased since we have used the futures price as if it were a forward price. However, we appeal to the findings of Cornell and Reinganum (1981) and Park and Chen (1985), showing that the differences between the prices of two types of contracts were insignificant in the foreign currency markets.

Since foreign exchange rates will be heavily influenced by inflation and interest rates, we would not expect strong qualitative differences in the behavior of futures markets in currencies and inflation.

\section*{II.6 An Exploratory Analysis of Taxation}

As this section is only exploratory in nature -- and will indeed highlight the difficulty of including taxes in the analysis -- we consider only the simplest model. Letting \( \tau \) be the marginal tax bracket, the taxable analogue to eq. (1) becomes:

\[
\delta_T(\tau) = \frac{I_0[1 + (1 - \tau) \delta_R_T] + \alpha(1 - \tau)(\tilde{C}_T - \delta F_T)}{I_0(\tilde{C}_T/C_0)} - 1
\]  

(26)
One problem is immediately noticeable. The modeling of taxes as in eq. (26) does not mirror the provisions of the Internal Revenue Code. Eq. (26) posits that all taxes are payable at maturity; in contradistinction, zero-coupon bonds are taxed at the yield-to-maturity and futures contracts are taxed in accordance with the marking-to-the-market provisions.

To eliminate the problem of the treatment of taxable zero-coupon bonds, consider the use of tax-exempt instruments selling at a discount from face value. Let \( 1 +_0 R_T^e \) be the gross total return on the tax-exempt security over the interval \([0, T]\). In a world of homogeneously taxed individuals, the tax-exempt interest rate would satisfy

\[
[(1 +_0 R_T)^{1/T} - 1](1 - \tau) = (1 +_0 R_T^e)^{1/T} - 1
\]  

(27)

It is important to note that the above relation is distinctly different from

\[
(1 +_0 R_T^e)^{1/T} = \left[1 +_0 R_T(1 - \tau)\right]^{1/T},
\]

as, in the latter case, the impact of taxation is eliminated.\(^{17}\)

---

\(^{17}\) To see this, define

\[
y = \left[\frac{1 +_0 R_T}{1 + (1 - \tau)_0 R_T}\right]^{1/T}.
\]

To investigate the properties of \( y \geq 1 \), take logarithms of \( y \) to obtain

\[
\log y = \frac{1}{T} \log \left[\frac{1 +_0 R_T}{1 + (1 - \tau)_0 R_T}\right]
\]

Thus, \( y(T = 1) > 1 \) and
Consequently, a more realistic modeling of the Internal Revenue Code is eq. (28):

\[ o'_{\tau}(\tau) = \frac{I_0(1 + o R_T^e) + \alpha(1 - \tau)(\tilde{C}_T - o F_T)}{I_0(\tilde{C}_T/C_0)} - 1 \]  

(28)

where the futures contract is taxed as if it were a forward contract. Proceeding now as in Section II.1, we have

\[ o'_{\tau}(\tau) = \frac{I_0(1 + o R_T^e) - \alpha(1 - \tau) o F_T}{I_0(\tilde{C}_T/C_0)} + \frac{\alpha(1 - \tau) C_0}{I_0} - 1 \]  

(29)

Eliminating the uncertainty associated with \( \tilde{C}_T \) requires setting

\[ \alpha = \frac{I_0(1 + o R_T^e)}{o F_T(1 - \tau)} \]  

(30)

Upon substitution of (30) into (29),\(^\dagger\) we obtain

\[ \lim_{\tau \to \infty} \gamma = 1. \]

The above limit highlights the difficulty in this modeling of taxation: as maturity increases, the impact of taxation is eliminated. This is, however, due to the modeling of taxation as upon realization, rather than the "upon accrual" rule provided in the Code for discount bonds. To preserve the impact of taxation as provided by the Code, eq. (27) presents a more realistic rendition of the tax treatment.

\(^\dagger\) Note that

\[ \frac{\partial \alpha}{\partial \tau} = \frac{I_0(1 + o R_T^e)}{o F_T(1 - \tau)^2} > 0, \]

so that the hedged position in futures contracts is dependent on the tax rate \( \tau \).
\[ 0^r T(\tau) = \left(1 + 0 \, R_T^{T} \right) \frac{C_0}{0F_T} - 1, \]

which, when annualized, yields

\[ 0^r * T(\tau) = \left[ \left(1 + 0 \, R_T^{T} \right) \frac{C_0}{0F_T} \right]^{1/T} - 1. \quad (31) \]

Consequently, the use of tax-exempt investments and current modeling of the tax code has given rise to an after-tax term structure of real interest rates, where the nominally riskless tax-exempt rate of return has supplanted the taxable rate in the calculation of the term structure of real interest rates.

Finally, contrasting the taxable and tax-exempt real term structures yields

\[ \frac{1 + 0^r T}{1 + 0^r T(\tau)} = \left( \frac{1 + 0 \, R_T}{1 + 0 \, R_T^{T}} \right)^{1/T}, \]

which reveals that discrepancies between the two term structures are attributable solely to the differences in the nominal rates of return: the differences attributable to the tax status of the forward contracts have been eliminated through the use of a distinct tax-sensitive hedge ratio [eq. (30)].
III. Empirical Results

The purpose of this section is to illustrate the use of our models and to gather some insight into the shape and stability of term structures of \textit{ex ante} risk free real interest rates. Since our data set is very small, we do not attempt to carry out formal statistical tests -- thus the expository nature of the section.

III.1 The Term Structures of Nominal and Real Interest Rates

As the first empirical example of the use of inflation futures contracts, Figure 4 demonstrates the relationship between the nominal interest rate, inflation implied by the futures contracts and the real interest rate, as derived from the naive model.

III.2 An Intertemporal Comparison of the Real Term Structure Across Models I, II and III

In Figures 5-7 we have plotted the term structures of real interest rates that could have been locked in on August 2, 1985, January 2, and March 3, 1986, respectively. We have plotted the term structures using all three models for each date, where the interest rates are calculated from equations (4), (18), and (25). For the nominal interest rates we used the annualized yields on Treasury bills for maturities up to one year and the yields on strips (Treasury notes and bonds where the coupons have been stripped off) for longer maturities.
REAL RATES, MODEL 1: AUGUST 2, 1985

Figure 4

Nominal Rate

Implied Inflation

Real Rate

Maturity Date

Sep-85  Mar-86  Oct-86  May-87  Nov-87  Jun-88
REAL TERM STRUCTURE: AUGUST 2, 1985

Figure 5

Spot Real Interest Rate

Oct-85 Jan-86 Apr-86 Jul-86 Jan-87 Jul-87 Jan-88 Jul-88

□ Model 1 + Model 2 ◇ Model 3
Where necessary, we used linear interpolations of the interest rates, a procedure that should be relatively harmless given the smoothness of the nominal term structure in the sample period.

One result that seems to justify our approximations in the development of the models is the small differences between the three models. The typical deviations are on the order of 10-30 basis points, and they tend to decrease with the maturity, thus confirming our prior notion of a reduced impact of the approximation error as the time to maturity increases.

The declining real term structure evidenced on two of the three dates -- January 2 and March 3, 1986 (Figures 6 and 7) -- has an appealing intuitive interpretation. To the extent inflation risk is priced in the inflation futures contract, the hedger (who is long the contract) must pay a risk premium to the speculator assuming that risk. Consequently, a person purchasing a riskless real asset for a longer-term maturity receives inflation insurance over a more extended period of time. The hedger pays for that longer-term insurance by receiving a lower real interest rate for longer maturities vis-a-vis the shorter ones.

Furthermore, we note the striking dissimilarity in the shape and level of the term structures when we compare the three dates. While far from constituting definitive evidence, it does, however, cast a doubt on the stability of the real term structure of interest rates over time. This impression is reinforced when we turn to Figure 8, which shows the month-to-month time-series of the one-year real interest rate over the first 15 months for which the inflation futures were traded.\(^\text{19}\) Starting out with small changes over the first seven months, we see a

\(^\text{19}\) The calculation of the one-year real rates was based on contracts with a time to maturity as close to one year as possible, but not exceeding one year. Thus the actual time to maturity ranges from 8-1/2 to 12 months. The discrepancies between these (realizable) yields and estimates of the actual one year rates are quite small; linear interpolations on the data used to generate Figures 5-7 indicate an average error of 15 basis points with a maximum error of 44 basis points.
ONE YEAR RISKLESS REAL INTEREST RATES

Figure 8

Real Interest Rate

Date


□ Model 1 + Model 2 ◇ Model 3
dramatic jump taking place in February. This is entirely attributable to a change in the futures prices reflecting expectations of lower inflation rates, presumably caused by the tumbling oil prices. The real interest rate remained high for a couple of months, later to be brought down by a combination of renewed inflationary expectations and lower nominal interest rates.

### III.3 Taxable and Tax-Exempt Real Term Structures

Figure 9 presents a contrast of the taxable and tax-exempt real term structures of interest rates for August 2, 1985, in accordance with the methodology derived in Section II.6. As observed in that theoretical discussion, the tax-sensitive hedge ratio gives rise to a real term structure determined by the identical inflation rates impacting on the two different nominal term structures.

### III.4 Conclusions

The near-zero trading volume and extremely low open interest in the CPI-W futures contracts have constituted disappointing phenomena to economists, who, along with Paul Samuelson, see in these contracts an answer to the long-awaited question: whither a riskless real interest rate in the U.S. economy? It is our unsubstantiated belief that these low volumes are manifestations of a low rate -- and, more importantly, a low volatility -- in the rate of inflation. If, as the expression goes, inflation ever "rears its ugly head," investors may yet find inflation futures an attractive vehicle for investment, hedging and speculation. As a final
TAXABLE AND TAX-EXEMPT RATES: 8/2/85

Figure 9

Taxable Real Rates

Tax-Exempt Real Rates

Spot Real Interest Rate

Maturity Date

□ Model 1
+ Model 2
♦ Model 3
note, it is imperative to recognize that, despite the extremely low trading volume, prices do adjust as market makers change their bid-ask quotes. A dramatic manifestation of this phenomenon is the rapid reaction of inflation futures prices to the fall in oil prices in February 1986. As volume and open interest both grew dramatically, the inflation futures prices reacted rapidly, and -- with near-constant T-Bill prices -- the one-year riskless real interest rate shot up dramatically from 3.2 percent to 5.4 percent. Only subsequently, as T-Bill rates declined, did the real interest fall back down. Thus, paradoxically, the illiquid market reacted more rapidly to the new circumstances than the liquid -- but Federal Reserve-influenced -- Treasury Bill market.

We are particularly reluctant to draw any firm conclusion based on the findings above because of the low trading volume in the inflation futures market. The real rates graphed in Figures 5-8 were certainly realizable for a marginal investor buying one contract for each maturity, but the quoted futures prices can hardly be viewed as representing the "consensus expectations of the market," given the lack of trading. On the other hand, one would not expect the futures prices to get too far out of line before it attracted speculators believing themselves to be good at forecasting the future inflation rate.

Therefore, to conclude our empirical section, we trust our basic approach to yield good measures of real interest rates, and with longer time series of data and (hopefully) more active trading we think research on this market might yield valuable insights into the determinants and development of inflation rates and inflationary expectations.

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20 Trading volume exceeded 100 contracts a day, with open interest rising above 200 contracts.
Further, in periods of higher/more volatile inflation rates, the presence of the inflation futures would act to prevent erosion in the savings rate as institutions offered investors a real rate of return and hedged their position in the CPI-W futures. Such a socially valuable role might even be sanctioned by a governmental repeal of any taxation on the inflation futures contract.

IV. Concluding Remarks

This paper has considered the theoretical and empirical issues involved in the construction of a riskless real asset through a portfolio composed of the nominally riskless asset and inflation futures contracts. We think our models of the real interest rate are fairly robust in the sense that the approximation errors we make will be of second-order importance in any realistic setting. The assumption of infinite divisibility of the contracts is, however, a drastic assumption if viewed with the eyes of a private investor (the size of a contract is around $330,000). This leads us to the following suggestion for new product development for financial institutions: offer real interest rate deposits/securities on a retail basis and cover the position by holding portfolios as described in Section II. In that case the nondivisibility of the contracts would pose little problem, and in addition we would get a richer opportunity set for investors to choose from. This suggestion is closely related to our wondering why the trading volume is so low, which again is of course not very different from asking why the supply of index-linked bonds is zero when on theoretical grounds there seems to be a good case for their existence.

The only reasons we can see for the lack of interest in these contracts in the market is that the present rate of inflation is low and reasonably stable, such that investors don't
perceive inflation risk as economically significant. Whether this is going to be the case in the future also is a question which seems to be more properly posed within the framework of macroeconomics.
APPENDIX A

CONTRACT SPECIFICATIONS

Futures Contract on the CPI-W

1. Contract Specifications

All futures contracts shall be a cash value based on the reported month's CPI-W (consumer price index for wage earners and clerical workers U.S. city average, all items, as compiled and announced by the Bureau of Labor Statistics).

2. Trading Months and Hours

Futures contracts shall be traded, initially, for settlement during the months of January, April, July and October, extending over the period 12 months from the initial trading date. The contract months of January and July shall additionally be listed for the period beyond 12 but less than 36 months from the initial trading date.

On the first business day following the expiration of any January and July contract, trading shall begin in the relevant January or July contract for delivery three years hence. On the first business day following the expiration on any April or October contract, trading shall begin in the relevant April or October contract for delivery one year hence. The final settlement value of any month's contract shall be based on the
value of the CPI-W released in that month, which will reflect prices through the previous
month (e.g., the January contract will reflect the CPI through the immediately preceding
December, a value publicly released in January). Trading shall be conducted between
9:30 a.m. and 2:30 p.m. Eastern Standard Time.

3. **Trading Unit**

   The unit of trading shall be the index CPI-W (currently using the base year of 1967
equaling 100). The value of the contract will be $1,000 times the index. For the
contract expiring in April 1983, the value would have equaled 293.0 (the March 1983
value of CPI-W), or $293,000 per contract.

4. **Price Increments**

   All bids and offers shall be in multiples of .01 (one one-hundredth) of the index.
The minimum price fluctuation of the contract would be $10.

5. **Daily Price Limits**

   The maximum daily price fluctuation will be 3.00 points, or $3,000 per contract.

6. **Termination of Trading**

   Trading will stop at noon of the first business day after the announcement date of
the CPI-W by the Bureau of Labor Statistics. Any contracts remaining open will be
settled on a cash basis. The settlement price will be $1,000 times the value of the announced index.

7. **Contract Modifications**

Contract specifications shall be fixed as of the first day of trading of the contract and will conform to the Bureau of Labor Statistics' specification of CPI-W. If any federal government agency issues an order, ruling, directive or law that conflicts with these specifications, such order, ruling directive or law shall be construed to become part of the contract, and all new contract months listed for trading shall be subject to such government orders.


