Research Program in Finance
WORKING PAPER SERIES

WORKING PAPER NO. 166

MULTIPLE FACTOR RISK MODELS
AND EXACT FACTOR PRICING

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Multiple Factor Risk Models and Exact Factor Pricing

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Finance Working Paper 166

February 1987

My thanks to Greg Connor for his help and encouragement and K. Ushiguga and Richard Meese for valuable contributions.
Abstract

Exact linear models of expected asset returns are the fundamental result in the CAPM and APT model of asset returns. This work is related to Connor’s important work on a unified beta pricing theory; we relax assumptions that require factor portfolios and the market portfolio to be free of any source of idiosyncratic risk. The cost of this relaxation is the requirement that idiosyncratic risks are normally distributed. These assumptions allow us to derive an exact factor pricing result directly and from a state-contingent claims framework. Tests with an actual multiple factor model indicate that the assumption of normality is quite reasonable.
1. Introduction

Connor [5] presented a multiple factor model of asset returns that unifies the CAPM and APT models' asset returns. Connor's main result is an exact factor pricing relationship that gives the expected return on each asset as a linear function of the asset's exposure to a set of important common factors. In order to establish this strong result, Connor makes two strong assumptions: first, for each common factor it is possible to build a portfolio that has only an exposure to that factor and no idiosyncratic (non-factor) risk; second, the market portfolio is completely diversified. This paper builds on Connor's foundations in several ways. First, we present assumptions that are in general weaker and lead to the same exact factor pricing result. Second, we demonstrate that these weaker assumptions stand up in a multiple factor model of asset returns that was constructed in 1981. Third, we look at this problem using a state-contingent claims approach; we find that reasonable assumptions about the structure of the state-contingent claims prices will also allow us to derive an exact factor pricing result.

Multiple factor models of asset returns were first described in the early 1970s by Rosenberg [15], Merton [11], and Sharpe [20]. An early model built by Rosenberg and Marathe is described in [24]. In 1976, Ross [16] proposed the arbitrage pricing model of asset returns. These theoretical models are all similar in structure; they differ in the goals of the model builder and the assumptions made. If the models are actually constructed, they depend on the data (macroeconomic or fundamental) used and the model building philosophy. It is always hazardous to attribute motivations to others,
but it appears there were three distinct motivations for these multiple factor pioneers. Rosenberg was interested in explaining the sources of portfolio risk and in forecasting the covariance between portfolios; Merton and Sharpe were working within a generalized CAPM framework to uncover additional systematic economic factors that could influence asset returns and that would be correctly priced in an equilibrium model. Ross built a model in which expected asset returns were approximately linear: his motivation appears to be gaining independence from strong assumptions on investor preferences and the use of the market as a central construct. Connor has pointed out that the main distinction between the CAPM and APT is one of causality. The APT is based on the notion that movements in underlying factors generate the asset payoffs. The CAPM uses portfolios of asset payoffs (e.g., on the market portfolio) to explain individual asset returns. Rosenberg's initial model anticipates this since he follows a hybrid approach of saying that there are common factors that influence the residual (net of market) return; a subsequent Rosenberg model, described in section 6, follows the APT approach. However, as Connor pointed out, this distinction between CAPM and APT models is moot since the only way we can represent factor returns is through portfolios of assets.

Exact factor pricing results appear in later versions of the APT (Connor [5], Chamberlin [2], Chen and Ingersoll [4]) although notions of the market and/or stronger assumptions about preferences creep into these later papers. A key assumption made by both Connor and Chen and Ingersoll is that the market portfolio has only common factor risk. Section 5 of this paper shows how an additional factor (idiosyncratic beta) can be added to any model in order to meet this requirement, although this device seems to contradict the spirit of the model. Connor also assumes that
factor portfolios can be built that are free of idiosyncratic risk, and Chen and Ingersoll results rest on the additional assumption that there is one individual who holds the market portfolio. The main results of this paper are based on a weaker set of assumptions, namely that the market portfolio has minimal idiosyncratic risk, and that idiosyncratic risks are normally distributed. Section 6 examines these assumptions for an actual model and finds they are quite reasonable.

Section 2 introduces the model, notation, and assumptions; we follow Connor as much as possible, and present a model of value rather than a model of returns. Section 3 establishes the exact factor pricing result. In section 4, we look at markets in which state-contingent claims prices exist and in which the claims price depends upon the value of the market portfolio. In that case, it is also possible to derive an exact linear asset pricing model and to determine that idiosyncratic risks, to the extent they are correlated with the market, will contribute to expected asset return. Section 5 presents what can only be called a trick for removing all idiosyncratic risk from the market. We introduce an additional factor and then show that the basic structure of the model is not changed by that device. Section 6 looks at a multiple factor model that was constructed in 1980-81 and checks the two critical assumptions made in this paper; first, that the idiosyncratic returns are normally distributed and, second, that the market portfolio has minimal idiosyncratic risk. Both assumptions appear eminently reasonable when checked against the actual model. Section 7 is a brief summary. An appendix contains the proof of a technical result needed in section 3.
2. The Model and Assumptions

This section describes the model and assumptions. The assumptions will be summarized at the end of the section. The notation follows Connor [4] to a large extent.

We consider a single period economy that consists of \( N \) risky assets numbered by \( n = 1, 2, \ldots, N \). The initial price per share of these assets is \( \rho(n) \) and the marketwide supply of shares is \( a(M, n) \). The end of period value per share for these assets is denoted by \( \tilde{x}(n) \) where the tilde indicates uncertainty. Final values are determined by the relation

\[
\tilde{x}(n) = c(n) + \sum_{k=1}^{K} B(n, k) \tilde{f}(k) + \tilde{i}(n),
\]

where

(i) \( c(n) \) is the expected value of \( \tilde{x}(n) \)

(ii) \( \tilde{f}(k) \) for \( k = 1, 2, \ldots, K \) are random factors that influence asset payoffs

(iii) \( B(n, k) \) for \( k = 1, 2, \ldots, K \) measure the factor exposure of asset \( n \) to the random factor \( k \)

(iv) \( \tilde{i}(n) \) is the idiosyncratic return to asset \( n \). This is the portion of asset \( n \)'s random return that is not captured by the factors.
Without any loss of generality, we can assume that \( E[\tilde{f}(k)] = 0 \) for all \( k \) and that \( E[\tilde{r}(n)] = 0 \) for all \( n \). With this convenient assumption, we can define the variance covariance matrices for \( \tilde{r}, \tilde{f} \), and \( \tilde{x} \) by:

\[
V = E[\tilde{r} \tilde{r}'],
\]

\[
F = E[\tilde{f} \tilde{f}'], \quad \text{and}
\]

\[
\Sigma = E[(\tilde{x} - c)(\tilde{x} - c)'].
\]

We assume that \( E[\tilde{r} | f] = 0 \) which implies \( \tilde{r} \) and \( \tilde{f} \) are uncorrelated. Thus we have

\[
\Sigma = B'FB + V.
\]

We can assume with little loss in generality that \( B \) has rank \( K \). This is equivalent to \( B'B \) being nonsingular. If \( B \) did not have full column rank, then it would be possible to build an equivalent model with fewer factors. Although it is not needed in what follows, we will also assume that the factor covariance matrix \( F \) is nonsingular. If not, then one factor is linearly dependent.

The matrix \( V \) is positive semidefinite; note we do not assume that \( V \) is positive definite or diagonal.
We also assume that $\Sigma$ is nonsingular. If $\Sigma$ is singular, then there exists a dependent asset whose end of period value is a linear combination of the end of period values of the other securities. We can, without loss of generality, eliminate those dependent assets from our list.

Our final assumption on the returns generating process (2.1) is that the idiosyncratic returns $\tilde{r}$ are normally distributed. There will be a discussion of this point below.

In addition to the $N$ risky assets, we assume there is a riskfree asset with initial price equal to one and whose final value is equal to $x(0)$ ($x(0) > 1$) with certainty.

There are $J$ investors indexed by $j = 1, 2, \ldots, J$. Investors are characterized by utility functions $U_j(\tilde{w}(j))$ and initial allocations of shares $e(j, n)$ for $n = 0, \ldots, N$. We write this as a vector $\{e(j, 0), e(j)\}$ where $e(j, 0)$ indicates riskfree shares.

Investor $j$ with initial allocation $e(j, n)$ can choose any allocation of shares $a(j, n)$ that satisfies the budget constraint.

\begin{equation}
(a.0) \quad a(j, 0) + \sum_{n=1}^{N} a(j, n)p(n) \leq e(j, 0) + \sum_{n=1}^{N} e(j, n)p(n).
\end{equation}

With allocation $a(j, n)$ the investor's uncertain end of period wealth will be

\begin{equation}
(2.5) \quad \tilde{w}(j) = a(j, 0)x(0) + \sum_{n=1}^{N} a(j, n)\tilde{x}(n).
\end{equation}
The investor chooses $a(k, x)$ that satisfy the budget constraint (2.4) and maximize expected utility -

$$(2.6) \quad E(\tilde{U}_j(\tilde{w}(j)))$$

where $U_j$ is a strictly increasing, strictly concave, twice differentiable utility function.

Let $a(n)$ for $n = 0, ..., N$ represent the holdings in a portfolio. The portfolio $a$ will have factor exposures.

$$\begin{align*}
(2.7) \quad b(a, k) &= \sum_{n=1}^{N} a(n) B(n, k) .
\end{align*}$$

We say a portfolio is **diversified** if it has minimum idiosyncratic variance among all other portfolios with the same factor exposures.

One of our basic assumptions is that the market portfolio $a(M, n)$ is diversified. There is no other portfolio with the same factor exposures as the market and lower variance.

The key assumptions are first the conditional independence of the factor returns $\tilde{f}$ and idiosyncratic returns $\tilde{t}$, and second, the normal distribution of the idiosyncratic returns. The conditional independence of $\tilde{f}$ and $\tilde{t}$ is stronger than assuming $\tilde{f}$ and $\tilde{t}$ are uncorrelated and not quite as strong as $\tilde{f}$ and $\tilde{t}$ independent.
The normality assumption can be motivated by an appeal to the central limit theorem. We can consider $L$ more factors in our economy so that

$$
\tilde{t}(n) = \sum_{\ell=1}^{L} d(n, \ell) \tilde{g}(\ell).
$$

If $L$ is large and no single factor has a predominant impact on $\tilde{t}(n)$, then the central limit theorem indicates that $\tilde{t}$ should be approximately normally distributed. The case is much stronger for portfolios than for assets. We can think of the idiosyncratic factors $\ell = 1, 2, \ldots, L$ as being local. They affect one company (e.g., personnel, an R&D breakthrough, a strike), two companies (e.g. a lawsuit), or one locality (bad weather). Any diversified portfolio $a$ should have a reasonably small exposure to these local factors. Since

$$
\tilde{t}(a) = \sum_{\ell=1}^{L} d(a, \ell) \tilde{g}(\ell)
$$

where

$$
d(a, \ell) = \sum_{n=1}^{N} a(n)d(n, \ell)
$$

Thus a strong case can be made for each $\tilde{t}(n)$ to be close to normally distributed. An even stronger case can be made for the normality of the idiosyncratic risk $\tilde{t}(a)$ for any reasonably diversified portfolio. It should be noted that the argument in the text pertains to portfolios and not individual assets.
The last two assumptions, i.e. that the market is diversified and that the idiosyncratic risks are normal, are discussed in greater detail along with some empirical evidence in a later section. It is quite likely that the assumptions of normality could be weakened to "elliptical"; see Owens and Rabinovitch [12]. However, the motivation for factor model is that the idiosyncratic risks are in fact remanents with no economy-wide implications.

We summarize the assumptions below.

**A1:** There are \( N \) risky assets whose uncertain future share values (including dividends) are determined by:

\[
\tilde{x} = c + B\tilde{f} + \tilde{i}
\]

where the expected value of \( \tilde{i} \) given knowledge of \( f \) is zero;

\[
E[\tilde{i} | f] = 0.
\]

**A2:** The three matrices

\[
E[(\tilde{x} - c)(\tilde{x} - c)'], \ E[\tilde{f}\tilde{f}'], \ B'B
\]

are all nonsingular.
A3: \( \tilde{r} \) is normally distributed with mean zero and covariance

\[
E[\tilde{r} \tilde{r}'] = \Sigma
\]

A4: There exists a riskless asset whose per dollar payoff is \( x(0) > 1 \).

A5: There are \( J \) risk averse investors each with a strictly increasing, strictly concave, twice differentiable utility function.

A6: There exists a competitive equilibrium.

A7: The market portfolio is diversified. It has minimal variance among all portfolios with the same factor exposures.

We can define a factor portfolio for factor \( k \) as a portfolio that has a unit exposure to factor \( k \), \( b(k) = 1 \), zero exposure to other factors, \( b(j) = 0 \) for \( j \neq k \), and minimum risk. Connor's assumption of an insurable factor economy is equivalent (see [5], Theorem 1) to assuming that the market portfolio has zero idiosyncratic risk and that each factor portfolio has zero idiosyncratic risk. This, as we shall see in a later section, is an extremely strong assumption. The cost of relaxing the requirement that the market and each factor portfolio have zero idiosyncratic risks is the assumption that the idiosyncratic risks are normally distributed.
3. Exact Linear Factor Pricing

This section establishes our first result. We proceed in three steps: first describe the properties of diversified portfolios, second show that there is an equilibrium consisting of diversified portfolios, and third establish what is equivalent to exact factor pricing, i.e., that costless portfolios with no factor exposure must have a zero expected return.

Let $\mathbf{a}(0)$, a "be a portfolio with factor exposures $b' = a'B$. Portfolio $a$ is diversified if $a$ solves

$$\text{Min} \quad \frac{h'Vh}{2}$$

(3.1) \quad h'B = b'$$

i.e., if $a$ minimizes idiosyncratic variance, $h'Vh$, among all portfolios $h$ with factor exposure $b' = h'B$.

Our first task is to characterize diversified portfolios. We can establish the following (see the appendix).

**Proposition 1:** Under A1 and A2
(i) There exists an $N$ by $K$ matrix $H$ such that the unique optimal solution to (3:1) is given by

$$h^* = Hb.$$ 

(ii) For each $b$ there is a $K$ element vector $z^*$ such that

$$Vh^* = Bz^*.$$

(iii) $a$ is a diversified portfolio if and only if there exists a $b$ such that

$$a = Hb.$$ 

The columns of $H$ are called the factor portfolios. They are the minimum variance portfolios with a unit exposure to one factor and zero to all the others. The factor portfolios form a basis for all diversified portfolios; i.e., any diversified portfolio is a linear combination of factor portfolios, and any linear combination of factor portfolios is diversified.

Our next job is to show that all investors will prefer diversified portfolios to less diversified. The motivation is similar to mean variance analysis. The idiosyncratic returns have zero mean, therefore an investor should want to reduce the idiosyncratic risk as much as possible.
Proposition 2: Investors prefer diversified Portfolios.

Proof: Consider a portfolio \([a(0), a]\) with factor exposures \(b' = a'B\), and expected value \(a(0)x(0) + a'c\). Now consider the diversified portfolio \([h(0), h]\) where \(h = Hb\) has minimum variance subject to \(h'B = b'\) and \(h(0)\) is selected so that portfolio \(h\) has the same expected return as portfolio \(a\).

\[
(3.2) \quad h(0)x(0) + h'c = a(0)x(0) + a'c.
\]

The final wealth using portfolios \([a(0), a]\) and \([h(0), h]\) can be written as

\[
\tilde{w}(h) = h(0)x(0) + h'c + b'\tilde{f} + h'\tilde{i}
\]

\[
(3.3) \quad \tilde{w}(a) = a(0)x(0) + a'c + b'\tilde{f} + h'\tilde{i} + (a - h)'\tilde{i}
\]

or

\[
\tilde{w}(a) = \tilde{w}(h) + (a - h)'\tilde{i}
\]

Notice that the covariance of \(h'\tilde{i}\) and \((a - h)\tilde{i}\) is equal to zero since from proposition 1 there exists a \(z\) such that

\[
(3.4) \quad Vh = Bz.
\]
so

\[(3:5) \quad (a - h)'Vh = (a - h)'Bz = 0.\]

Since \(h'\tilde{i}\) and \((a - h)'\tilde{i}\) are uncorrelated, and \(\tilde{i}\) is normal, then

\[(3:6) \quad E[(a - h)'\tilde{i} \mid h'\tilde{i}] = 0.\]

In addition, \(E[i\tilde{f}] = 0\). Since \(\tilde{w}(h)\) depends only on \(\tilde{f}\) and \(h'\tilde{i}\), we have

\[(3:7) \quad E[(a - h)\tilde{i} \mid w(h)] = 0.\]

From Rothchild and Stiglitz [17], any risk averse investor will determine that

\[(3:8) \quad E[U(\tilde{w}(h) + (a - h)'\tilde{i})] \leq E[U(\tilde{w}(h))].\]

Thus the diversified portfolio \(h\) is preferred by all risk averse investors.

With all investors preferring diversified portfolios, the question remains can they all afford them; i.e., is there an an equilibrium in which all the investors hold diversified portfolios. The next result shows that there is. The key assumption is that the market portfolio is itself diversified.

Proof: Let \([a(0, j), a(j)]\) be the equilibrium allocation. By proposition 2 every investor \(j\) can find a diversified portfolio \([h(0, j), h(j)]\) such that the \(h\)-portfolio is at least as good as the \(a\)-portfolio.

Let \(C[h]\) be the cost of portfolio \(h\), i.e.

\[
(3:9) \quad C[h] = h(0) + \sum_{n=1}^{N} h(n)p(n).
\]

For each \(j\) we must have

\[
(3:10) \quad C[h^{*}, j] \geq C[a^{*}, j].
\]

This follows because the \(a\)-portfolio is optimal and the \(h\)-portfolio is at least as good as the \(a\)-portfolio. If the \(h\)-portfolio was cheaper, i.e., if

\[
(3:11) \quad \delta = C[a^{*}, j] - C[h^{*}, j] > 0,
\]

then we could by increasing \(h(0, j)\) to \(h(0, j) + \delta\) find a portfolio that is strictly (recall \(x(0) > 1\)) preferred to the \(a\)-portfolio and has the same cost as the \(a\)-portfolio.

Our next step in the proof is to show that (3:10) holds with equality, i.e.
(3:12) \[ C[h(\star,j)] = C[a(\star,j)]. \]

Let's sum the relationship (3:10) over \( j \); recall the definition (3:9). Let

(3:13) \[ a(0, M) = \sum_{j=1}^{J} a(0, j) \]

\[ a(M) = \sum_{j=1}^{J} a(j) \]

\[ h(0, M) = \sum_{j=1}^{J} h(0, j) \]

\[ h(M) = \sum_{j=1}^{J} h(j) \]

\( a(0, M) \) and \( h(0, M) \) are scalars. Both \( a(M) \) and \( h(M) \) are \( N \) vectors representing the total demand for the \( a \)-portfolios and the \( h \)-portfolios.

When we sum the relationship (3:10), we obtain

(3:14) \[ h(0, M) + h(M)'\rho \geq a(0, M) + a(M)'\rho. \]

The market portfolio \([a(0, M), a(M)]\) is by assumption diversified. Thus, by proposition 1, \( a(M) \) and \( h(M) = B'a(M) \) must satisfy
(3:15) \[ a(M) = HB'\alpha(M) = H\beta(M) . \]

However, for each \( j \) the \( h \)-portfolio is diversified, so

(3:16) \[ h(j) = HB'\alpha(j) . \]

If we sum (3:16) over all \( j \), we get

(3:17) \[ h(M) = HB'\alpha(M) = H\beta(M) . \]

Equation (3:17) indicates, by proposition 1, that \( h(M) \) is the minimum variance portfolio with factor exposures \( b(M) \). However, by assumption \( a(M) \) is diversified, \( a(M) \) has factor exposures \( b(M) \), and by proposition 1 the minimum variance portfolio is unique. Thus \( h(M) = a(M) \).

We are left to deal with \( h(0, m) \) and \( a(0, M) \). However, for each \( j \)

(3:18) \[ h(0, j) = a(0, j) + \frac{[\alpha(j) - h(j)'c}{x(0)} . \]

If we sum (3:18) over \( j \), we get

(3:19) \[ h(0, M) = a(0, M) + \frac{[\alpha(M) - h(M)'c}{x(0)} . \]
Since \( a(M) = h(M) \), we must have \( h(0, M) = a(0, M) \). Thus the equation (3:14) must hold with equality, and in consequence each cost relationship (3:10) must also be an equality.

We have established that every investor can find a diversified portfolio that meets the budget constraint that is at least as good as the equilibrium allocation and that satisfies the aggregation condition that equates supply and demand. Thus the \( h \)-portfolios form an equilibrium allocation.

We have established that an equilibrium exists in which all investors hold diversified portfolios. Now, following Connor [5], consider the possibility of building a hedge portfolio with zero cost and zero factor exposure, i.e. Choose \([h(0), h]\) such that

\[
(3:20) \quad h(0) + h'p &= 0 \\
&= 0.
\]

If the hedge portfolio had positive expected value \( h(0)x(0) + h'c = 1 \), then we have a portfolio with infinite expected return and finite risk.

Connor shows that the requirement that every hedge portfolio has zero expected value is equivalent to exact factor pricing; i.e., the existence of a \( K \) element vector \( m \) such that

\[
(3:21) \quad x(0)p = c + Bm.
\]
We can establish Connor's result in a slightly different way. From Gale [6: p. 32] and a slight bit of rearranging, we note that the factor pricing equations (3:21) will have a solution if and only if the equations below which describe a hedge portfolio with positive expected return have no solution.

\[(3:22)\]
\[h(0) + h'p = 0\]

\[h'B = 0\]

\[h(0)x(0) + h'c = 1\]

Thus we can establish exact pricing (3:21) by ruling out the existence of hedge portfolios with positive expected value (i.e., solutions to (3:22)). We will do this by showing they are just too good to be true; their existence is not compatible with equilibrium. Any investor would be tempted to tilt his portfolio toward the hedge portfolio.

**Theorem 2:** The asset prices are linear functions of factor exposures.

**Proof:** (follows Connor [5]) If not, then we have a solution to (3:22). Let \([a(0), a]\) be the diversified equilibrium portfolio of a typical investor with factor exposures \(b' = a'B\) and uncertain end-of-period wealth.
\[ \hat{w} = a(0)x(0) + a'c = b\tilde{f} + a'\tilde{\iota} \]

Now consider a hedge portfolio with expected return

\[ v = h(0)x(0) + h'c. \]

We hope to prove \( v = 0 \). To do this we add a scalar multiple \( \theta \) times the hedge portfolio to \([a(0), a]\). Since the hedge portfolio has zero cost and zero factor exposures, we get a portfolio with identical cost and final value \( \hat{w} + \theta v + \theta h'\tilde{\iota} \). The expected utility of this position is

\[ G(\theta) = E\{U[\hat{w} + \theta v + \theta h'\tilde{\iota}]\}. \]

Since \([a(0), a]\) is the unique optimal and \([a(0), a] + \theta[h(0), h]\) is a possible choice due to \([h(0), h]\)'s zero cost, we must have \( G(\theta) \) taking on its unique optimum at \( \theta = 0 \). Thus

\[ \frac{dG}{d\theta} \bigg|_{\theta = 0} = E[\dot{U}[(\hat{w} + h'\tilde{\iota})] = 0 \]

However, \( \hat{w} \) depends only on \( \tilde{f} \) and \( a'\tilde{\iota} \). Since \( h'B = 0 \), and \( a \) is diversified, we have \( a'\dot{V}h = 0 \). Thus, by the normality assumption

\[ E[h'\tilde{\iota} | a'\tilde{\iota}] = 0 \]
and

\[ E[h' \tilde{\iota} | \tilde{f}] = 0. \]

Thus

\[(3:28) \quad E(E[\hat{U}\{\tilde{w}\}h' \tilde{\iota} | a' \tilde{\iota}, \tilde{f}]) = 0 = E(\hat{U}\{\tilde{w}\}h' \tilde{\iota}) \]

We are then left with

\[(3:29) \quad E[\hat{U}\{\tilde{w}\}\nu] = 0 \]

Since \( \hat{U}\{w\} \) is strictly positive, this can only occur if \( \nu = 0 \). At equilibrium all investors will resist adding the costless hedge \( h'B = 0, h(0) + h'p = 0 \) to their portfolio since it has zero expected return and positive idiosyncratic risk.

This concludes the initial result. In a later section we consider some of the assumptions and conclusions with the aid of an actual factor model for stock returns. The next section derives linear factor pricing, not from an equilibrium but from the existence of state contingent claims prices.
4. Exact Factor Pricing in a State-Claims Model

The exact factor pricing result established above used the concept of economic equilibrium as a central construct. It is possible to establish the same result in a slightly more general context. Harrison and Kreps [7] and others have shown under rather general conditions that no arbitrage implies the existence of state claims prices. In this paper we assume the state claims prices have a particular form, i.e., we assume the ratio of the claim price for any outcome to the probability of that outcome is a well-behaved function of aggregate return on the market.

We will retain the basic structure of the model introduced in section 2. That is, A1, the normality of idiosyncratic returns A3, the risk free asset A4, and the diversification of the market A7. We drop the technical conditions A2, the assumptions about investors A5, and the competitive equilibrium A6. In their place we assume that there exist state claims prices that depend on factor returns and idiosyncratic returns in a particular way. To do this, however, requires some notation.

Let \( \pi(f, i) \) be the joint probability density function of \( \tilde{f} \) and \( \tilde{i} \). Recall that \( a(0, M), a(M) \) describe the market or aggregate portfolio, and \( b(M)' = a(M)'B \) is the factor exposure of the aggregate. For any values of \( \tilde{f} = f \) and \( \tilde{i} = i \), the market return is

\[
(4.1) \quad \tilde{y}(f, i) = \tilde{y}(f) + \tilde{y}(i)
\]
where

\[ \tilde{y}(f) = \tilde{x}(0)a(0, M) + b(M)\tilde{f} \]

\[ \tilde{y}(i) = a(M)\tilde{i}. \]

The state contingent claims prices \( v(f, i) \) are assumed to be of the form

\[ (4.2) \quad v(f, i) = \pi(f, i)\hat{U}[y(f) + y(i)]. \]

The notation \( \hat{U} \) is meant to be suggestive of the case where \( \hat{U} \) is the marginal utility of some average or composite investor. See Rubinstein [18] for a derivation of this type of claims price. It is, of course, not necessary to tell that sort of story.

For any asset \( n \), the payoff to asset \( n \) with outcome \( f, i \) is

\[ (4.3) \quad x(n, f, i) = c(n) + b(n)f + i(n) \]

where \( b(n)' \) is the \( n \)-th row of the matrix \( B \). The market value of these cash flows must be:

\[ (4.4) \quad p(n) = \int_{f} \int_{i} x(n, f, i)v(f, i)di df. \]
The claims prices will value the risk-free asset, so

\[(4:5) \quad 1 = x(0) \int \int v(f, i)di df.\]

From (4:5) we can treat \(x(0)v(f, i)\) as a probability density. Let \(E^*\) denote expectation with respect to this probability density. With this notation, we can multiply (4:4) by \(x(0)\) and rewrite it as

\[(4:6) \quad x(0)p(n) = c(n) + b(n)E^*[\tilde{i}] + E^*[\tilde{i}(n)].\]

Notice that (4:6) is close to the exact factor pricing result. If \(E^*[\tilde{i}(n)]\) equals zero, there is nothing left to do. More generally, if \(E^*[\tilde{i}(n)]\) is a linear function of \(b(n)\), then we are also finished. It is this second result, that \(E^*[\tilde{i}(n)]\) is linear in \(b(n)\), that we will establish.

Using (4:2) and the laws of conditional probability, we can rewrite \(E^*[\tilde{i}(n)]\) as

\[(4:7) \quad E^*[\tilde{i}(n)] = x(0) \int \pi(f) \left\{ \int \pi(i|f) \hat{U}[y(f) + y(i)] i(n)di \right\} df\]

Conditional on \(f\), the scalars \(y(f) + y(i)\) and \(i(n)\) are joint normal. If we apply a result of Rubinstein [18], we obtain
\( (4.8) \quad \int_i \pi(i[f] \hat{U} [y(f) + y(i)]i(n)di = E[\hat{U}(y(f) + y(i))|f] \text{Cov}[y(i), i(n)]. \)

Now invoke the restriction that the market portfolio is diversified. From proposition 1 this means there exists a \( z(M) \) so that

\[
B' a(M) = b(M)
\]

(4.9)

\[
V a(M) = B z(M)
\]

However, \( \text{Cov}[y(i), i(n)] \) is just the \( n \)-th element of the vector \( V a(M) \). So from (4.9)

(4.10)

\[
\text{Cov}[y(i), i(n)] = b(n)' z(M)
\]

With this substitution, the third term in (4.6) becomes

(4.11)

\[
E^*[\hat{f} (n)] = \int_f \int_i x(0)v(i, f)i(n)di df = b(n)' \kappa z(M)
\]

where

(4.12)

\[
\kappa = \left\{ x(0) \int_f E[\hat{U}(y(f) + y(i)|f] \pi(f)df \right\}
\]

---

1 Notice we could move from (4.7) to (4.8) without the normality assumption if \( \hat{U} \) is linear, i.e., quadratic utility.
when we combine (4:11) with (4:6) we obtain exact linear pricing with

\[(4:13) \quad x(0)p(n) = c(n) + b(n)'m\]

where

\[m = E^*[f] + \kappa z(M).\]

Notice that this pricing formula does not say that all value comes from the asset’s factor returns. The asset’s idiosyncratic returns, to the extent they are correlated with the market, also contribute value. This puts a CAPM twist in an APT martini. The assumptions of normality and a diversified market portfolio allow us to express the covariance of the market and asset idiosyncratic returns in terms of the assets factor exposures. In technical terms, idiosyncratic risk that is residual (orthogonal) to the market is not priced; idiosyncratic risk that is correlated with the market is priced in proportion to its covariance with market return.
5. Well-Diversified Market Portfolio

The model described in sections 3 and 4 requires that the aggregate (market) portfolio is diversified; i.e., it has minimal idiosyncratic risk among all portfolios with the same factor exposures. This section shows that it is possible to add an additional factor to the system and thereby insure that the market portfolio has zero idiosyncratic risk.

Let

\[ \tilde{x} = c + B\tilde{f} + \tilde{\iota} \tag{5.1} \]

be the random rule that determines future asset values, where \( E[\tilde{\iota} | f] = 0 \) for all \( f \).

Recall that \( a(M) \) is the total shares outstanding in the aggregate or market portfolio. Define \( d \) as

\[ d = \frac{Va(M)}{a(M)\'Va(M)} \tag{5.2} \]

The \( N \) element vector \( d \) gives the idiosyncratic beta of each asset.

Notice that we can rewrite (5.1) as
\[(5:3) \quad \tilde{x} = c + B \tilde{f} + d(a(M)\tilde{i} + [I - da(M)]\tilde{i}).\]

We can use (5:3) to define a \(K + 1\) factor model. Let

\[(5:4) \quad \tilde{g} = \begin{bmatrix} \tilde{f} \\ a(M)\tilde{i} \end{bmatrix}\]

be the \(K + 1\) factor returns and

\[(5:5) \quad D = [B, d], \quad \text{and} \quad Q = I - da(M)'.\]

\(D\) is the \(N\) by \(K + 1\) matrix of factor exposures.

The new idiosyncratic vector is

\[(5:6) \quad \tilde{j} = \tilde{Q}i \quad \text{and}\]

the revised model is

\[(5:7) \quad \tilde{x} = c + D\tilde{g} + \tilde{j}.\]

Note that \(\tilde{j}\) is normally distributed since \(\tilde{i}\) is normal. The variance of \(\tilde{j}\) is
\( QVQ' \). Also, since \( a(M)'d = 1 \), the idiosyncratic component of the market \( a(M)'\tilde{j} \) is identically zero, so the market has no idiosyncratic risk.

To show that the adjusted model (5:4) satisfies

\[
(5:8) \quad E[\tilde{j} | g] = 0 \quad \text{for all } g.
\]

Note that \( \tilde{j} = Q\tilde{i} \) and \( a(M)'\tilde{i} \) are jointly normal and uncorrelated since (5:2) implies

\[
(5:9) \quad \text{Cov}[Q\tilde{i}, a(M)'\tilde{i}] = QV a(M) = 0.
\]

Thus

\[
(5:10) \quad E[Q\tilde{i} | a(M)'i] = 0
\]

and (5:8) follows directly since we already have

\[
(5:11) \quad E[Q\tilde{i} | f] = 0.
\]
6. A Discussion of the Assumptions

Researchers frequently become advocates for one particular point of view. It's a natural phenomenon, and might even have some benefits since the fire of advocacy may provide the extra energy required to uncover new and interesting results. However, it is often useful and promotes humility when one takes the opposite point of view. We try to do some of that in this section. Here the question is not "Why are linear models of expected return correct?"; it is more, "Is the world consistent with the kind of assumptions we have to make in order to prove that linear models of expected return are correct?". In the world of linear factor models, assumptions are usually based on investor preferences, costless access to all assets, and the distribution of asset returns. Tests are plagued by the testing of joint hypotheses, definition of the proper set of assets, and the availability of clean detailed data. Rather than test the conclusions, we will make a brief test of the assumptions.

Robert Solow [22] said, "All theory depends on assumptions which are not quite true. That's what makes it theory." We can do some analysis of the assumptions underpinning the results in this paper to find out how true they really are, and then use some judgment to consider how seriously we should take the results. A positive answer does not guarantee the results will apply, but we can at least know how many grains of salt we have to carry along in order to support the results. After all, we have lived for years with the knowledge that asset returns are not normally distributed, that transactions costs and taxes are significant, that investors cannot continuously re-balance their portfolios, and that all investors do not have the same preferences.
This analysis is based on a multiple factor model of asset return that was built by BARRA in 1980-81. The model was built with the objective of explaining portfolio risk to the largest extent possible; in particular, to provide accurate estimates of the covariance between any two portfolios. The model was not constructed with the objective of proving that expected asset returns are linear functions of factor loadings.

The BARRA model covers over 6,000 assets and has 68 factors. It is in exactly the form specified by A1 and A2. Factor portfolios are constructed using an estimation universe of about 1,300 of the highest capitalization assets. The factor portfolios are constructed so that each factor portfolio has the minimum idiosyncratic risk among all portfolios with those exposures. What is interesting to explore with this model is whether assumption A3 (normal idiosyncratic returns) and assumption A7 (diversified market portfolio) are satisfied.

The diversification of the market was tested using the estimation universe of about 1,300 assets as a market proxy. We calculated the predicted idiosyncratic risk of the market in three different months, and we calculated the idiosyncratic risk of a portfolio that had the same factor exposures as the market and minimum idiosyncratic risk. The results are

<table>
<thead>
<tr>
<th>Date</th>
<th>Market</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan. '82</td>
<td>0.378%</td>
<td>0.322%</td>
</tr>
<tr>
<td>June '84</td>
<td>0.434%</td>
<td>0.351%</td>
</tr>
<tr>
<td>March '86</td>
<td>0.374%</td>
<td>0.324%</td>
</tr>
</tbody>
</table>
Data are monthly standard deviations.

As we can see, the market portfolio has very small idiosyncratic risk; over 99.5 percent of the variance is captured by the factors. In addition, the portfolio with the same factor exposures as the market and minimum idiosyncratic risk has a standard deviation that is about seven basis points lower than the market. From this we may sleep soundly with the assumption that the market is well diversified.

The assumption that the idiosyncratic risks are normally distributed is more difficult to defend a priori. We know that both individual asset returns and market returns exhibit a large amount of kurtosis (fat tails), so the question is really whether the extreme events that produce the fat tails are captured in the factor returns or are left to the idiosyncratic returns. To look at this point, we randomly partitioned the assets in the estimation universe into 10 groups and calculated the idiosyncratic returns to capitalization weighted portfolios made up of those assets. We introduced a selection bias by excluding assets that were not in the universe over the entire period. That bias would, one assumes, be in favor of normality since assets tend to leave the universe with an event (either takeover or bankruptcy) that has large idiosyncratic risk.

For all 10 portfolios, the idiosyncratic returns passed the normal test with flying colors. A Kolmogorov-Smirnov, a version of the Shapiro-Wilk test, and a chi-squared test based on the skewness and kurtosis of the sample data were performed; see Bickel and Doksum [1, pp.378 – 89] for details about the tests. We present results for the skewness and kurtosis tests from Jarque and Bera [9]. The test statistic is
\[ t = 164 \left( \frac{s^2}{6} + \frac{(k - 3)^2}{24} \right) \]

where \( s \) = sample skewness

\( k \) = sample kurtosis

Under the assumption of normality, this statistic has a \( \chi^2(2) \) distribution. This statistic was calculated for portfolio returns, the portion of portfolio returns that were determined by the common factors, and the portion of idiosyncratic portfolio returns. For 164 samples we reject the null hypothesis of normality at the 5 percent (1 percent) level if the test statistic exceeds 5.99 (9.21). As one can see from the table below, the idiosyncratic returns pass quite easily and that all of the portfolio returns and factor returns fail to pass. The results for Kolmogorov-Smirnov and Shapiro-Wilk tests were similar.
<table>
<thead>
<tr>
<th>Total</th>
<th>Factor</th>
<th>Idiosyncratic</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.6351</td>
<td>11.5657</td>
<td>1.2862</td>
</tr>
<tr>
<td>46.0888</td>
<td>17.0668</td>
<td>.2477</td>
</tr>
<tr>
<td>28.7505</td>
<td>34.6362</td>
<td>1.6800</td>
</tr>
<tr>
<td>18.0290</td>
<td>20.2554</td>
<td>3.3020</td>
</tr>
<tr>
<td>12.0113</td>
<td>14.3304</td>
<td>1.6193</td>
</tr>
<tr>
<td>14.7210</td>
<td>13.6375</td>
<td>3.5941</td>
</tr>
<tr>
<td>12.6551</td>
<td>18.3505</td>
<td>1.7139</td>
</tr>
<tr>
<td>25.0296</td>
<td>34.4481</td>
<td>.5558</td>
</tr>
<tr>
<td>9.0502</td>
<td>12.5213</td>
<td>1.6245</td>
</tr>
<tr>
<td>22.0269</td>
<td>22.8870</td>
<td>2.1218</td>
</tr>
</tbody>
</table>
7. Summary

In a large multiple factor model of asset returns, it is reasonable to assume that idiosyncratic returns are normally distributed. Under that assumption of normality and the additional assumption that the market portfolio has minimal idiosyncratic risk, it is possible to derive an exact factor pricing result. This result reinforces the notion that expected asset returns are influenced in a linear fashion by exposure to factors that are common to all assets. Use of this notion requires an identification of the significant common factors and an estimate of the risk premia associated with each factor. The identification problem is probably not too significant. Experience with multiple factor models indicates that any large set of factors with some independence will serve as proxies for another set. What is more difficult to determine is the stability and, preferably, fundamentals underlying the expected risk premia associated with each factor.
APPENDIX A

Proof of Proposition 1:

For any $b$, the first order conditions for (3:1) are

$$Vh - Bz = 0$$

(A:1)

$$B'h = b$$

Since $V$ is positive semi-definite, these conditions are necessary and sufficient for an optimum.

Suppose $h^*$ and $\hat{h}$ were both optimal for (3:1). Let $z^*$ and $\hat{z}$ be the corresponding solutions of (A:1). Let $\Delta h = h^* - \hat{h}$ and $\Delta z = z^* - \hat{z}$. From (A:1) we have

$$V\Delta h = B\Delta z$$

(A:2)

$$B'\Delta h = 0$$

That implies

$$\Delta h' BFB' \Delta h = 0$$

and
\( \Delta h'V\Delta h = \Delta h'B\Delta z = 0 \)

so

\( (\Delta h)\Sigma\Delta h = 0 \).

However, by assumption A2, \( \Sigma \) is positive definite. It must be that \( \Delta h = 0 \) or \( h^* = ^A\). Note also that \( z \) is unique since \( \Delta z = (B'B)^{-1}V\Delta h = 0 \).

The matrix \( B \) has \( K \) linearly independent columns. Let \( D \) be an \( N \) by \( N - K \) matrix whose columns span the \( N - K \) dimension subspace orthogonal to the \( K \)-dimensional space spanned by the columns of \( B \). Note that \( B'D = 0 \) \( D'B = 0 \), and \( D'D \) is non-singular.

Let \( h^* \) be the unique optimal solution and write

\[
(A:3) \quad h^* = Bv + Dy.
\]

Since \( B'h^* = b \), we must have \( v = (B'B)^{-1}b \). By substituting (A:3) into (A:1), we get

\[
(A:4) \quad VBv + VDy = Bz.
\]

Multiplying (A:4) by \( D' \) yields

\[
(A:5) \quad D'VBv + D'V Dy = 0.
\]
The matrix $D'VD$ is non-singular. If it was singular, then there exists a $w \neq 0$ such that $w'D'VDw = 0$. Since $D$ has full column rank, $a = Dw$ is not zero. However, note $a'B = 0$, and $a'Va = 0$ so $a'\Sigma a = 0$. This is a contradiction since we have assumed $\Sigma$ is non-singular.

Since $D'VD$ is non-singular, we can solve (A:5) for $y$.

(A:6) \[ y = - (D'VD)^{-1}D'VB(B'B)^{-1}b. \]

Thus

(A:7) \[ h^* = By + Dy = Hb. \]

where

(A:8) \[ H = [I - D(D'VD)^{-1}D'V]B(B'B)^{-1} \]

is of rank $K$. Note that

(A:9) \[ B'H = I, \quad D'VH = 0 \]

and that $h = Hb$ implies

(A:10) \[ h = HB'h. \]
Finally, if \( V \) has an inverse, then

\[
A:11 \quad H = V^{-1} B (B' V^{-1} B)^{-1}.
\]
REFERENCES


