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New Cross-Sectional Regression Tests of Beta Pricing Models

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ABSTRACT

This paper develops two-pass cross-sectional regression tests for beta pricing models using an approximate factor model of returns. We develop two procedures: one applies to individual asset returns and the other uses portfolio-grouped returns. We show how our equilibrium APT based model also can be used to test the CAPM. We apply the techniques to monthly return data and relate our findings to existing empirical results. We find that our procedures substantially improve the accuracy of two-pass cross-sectional regression estimates. Our results corroborate the findings of Chan and Chen (1987) and Chan, Chen and Hsieh (1986) that the size effect (but not the January effect) is caused by statistical inadequacies in standard cross-sectional regression tests of the CAPM.
1. Introduction

This paper develops two-pass cross-sectional regression tests for beta pricing models assuming that asset returns obey an approximate factor model. We develop tests for individual asset returns and for portfolio-grouped returns. We show how the tests can be applied to both the APT and CAPM. We apply the tests to monthly return data for the period 1953-1982. Our procedures seem to substantially outperform the now standard two-pass cross-sectional regression procedure of Fama and MacBeth (1973). We find that the size effect discovered in past tests of the CAPM seems to be due to statistical inadequacies in existing cross-sectional regression procedures. Our tests, using an approximate factor model rather than the diagonal market model of standard tests, do not reject the model in favor of a size-based alternative. However, our procedure is not able to explain the January seasonality in returns or the relationship between this seasonality and size. We relate these findings to recent conflicting evidence on the size effect in the CAPM and in the APT.

Our two tests have close parallels in the existing literature. Our basic innovation is that our model requires that asset returns follow an approximate factor model and uses asymptotic principal components as first-pass regressors. This differs from the statistical framework of past tests, which rely implicitly or explicitly on the diagonal market model and use a market index as first-pass regressor.

The individual asset test which we propose is analogous to one developed by Litzenberger and Ramaswamy (1979) (hereafter LR). Although the analogy is not perfect, one can view our individual asset procedure as
a multi-factor extension of the procedure suggested by LR for the diagonal market model.

The portfolio grouped procedure which we develop is very similar to one first described by Black, Jensen and Scholes (hereafter BJS) and later extended by Fama and MacBeth (1973) (hereafter FM).

In section 2, we review some of the relevant results from LR, BJS and FM as a way of introducing our procedures.

Section 3 describes the statistical model we use and derives the individual asset test. We show that a two-pass cross-sectional regression test can be applied to individual assets given that asset returns follow an approximate factor model and that the analyst uses the errors-in-variables correction that we propose.

In section 4 we develop a portfolio grouped procedure which extends the FM procedure to the case of an approximate factor model.

The tests of sections 3 and 4 can be directly applied to the APT. The CAPM can also be tested given that it is treated as special case of the equilibrium APT. The CAPM imposes a linear restriction across the multiple factor risk premia of the equilibrium APT. Except for this linear restriction, the two models are not testably distinct within our framework. This extension of the APT-based test to the CAPM is developed in Section 5. We also develop a new test of the equilibrium APT.

Section 6 presents our empirical results from applying the techniques to the APT and CAPM. The two models are tested against the particular alternative that firm size affects expected return. The CAPM is tested as a cross-factor restriction on the multi-factor equilibrium APT. The evidence indicates that the the APT does not mis-price small firms on
average over the full calendar year, although there is unexplained seasonal
(January) variation in asset risk premia and this seasonality is related to
size. We cannot reject the CAPM as a nested hypothesis on the equilibrium
APT, but this may reflect low power of our test to distinguish between
them.

Section 7 summarizes our findings.

2. Review of Some Previous Results

This section provides a brief review of some earlier results, and
particularly of some results from LR, BJS and FM. The purpose of this
brief review is to lay an intuitive foundation for the analytical
derivation of the test statistics in sections 3 and 4. We hope to show how
our tests can be viewed as a multi-factor extensions of the
Litzenberger-Ramawamy and Fama-MacBeth tests.

Let \( \tilde{r}_1, \ldots, \tilde{r}_n \) denote the returns to \( n \) securities, \( \tilde{r}_q \) denote the return
to the market portfolio \( \tilde{r}_q \), and \( r_0 \) denote the risk-free return. In this
section, following BJS (p. 91) we assume that asset returns follow the
"diagonal market model" with homoskedastic non-market returns:

\[
\tilde{r}_i - r_0 = a_i + \beta_i (\tilde{r}_q - r_0) + \tilde{\varepsilon}_i \quad (1)
\]

\[
\text{E}[\tilde{\varepsilon}_i / r_q] = 0 \quad \text{E}[\tilde{\varepsilon}_i \tilde{\varepsilon}_j] = 0 \quad j \neq i \quad (2)
\]

\[
\text{E}[\tilde{\varepsilon}_i^2] = \sigma^2 \quad i = 1, 2, \ldots, n. \quad (3)
\]

Second, we assume that the CAPM holds:

\[
\text{E}[\tilde{r}_i / r_q] = r_0 + \beta_i (r_q - r_0) \quad i = 1, \ldots, n. \quad (4)
\]

Third, we impose two conditions which are used in deriving cross-sectional
asymptotic results:

\[
limit \frac{1}{n} \beta' \beta = \theta > 0, \quad n \to \infty. \quad (5)
\]

\[
\tilde{\varepsilon}_i, \tilde{\varepsilon}_j \text{ multivariate normal, } i, j = 1, \ldots, n. \quad (6)
\]
If one could observe $\beta$ without error, then one could test (4) using a variety of cross-sectional regression procedures. Let $R$ denote an $n \times T$ matrix of observed returns and $\beta$ the $n$-vector of true betas. Consider the ordinary least squares regression of $R_{t}$ on $\beta$ and a unit vector:

$$R_{t} = \gamma_{0t} + \gamma_{1t} \beta + \varepsilon_{t}$$

(7)

It is easy to show that given (1) - (6) we have $\lim_{n \to \infty} \gamma_{0t} = r_{0t}$ and $\lim_{n \to \infty} \gamma_{1t} = (r_{qt} - r_{0t})$, where $\lim$ denotes the limit in probability.

Following Shanken (1983), we will call this limiting property $n$-consistency.

Given that one has $n$-consistent estimates of $\gamma_{0t}, \gamma_{1t}$, one can test the CAPM against a variety of alternatives. For example, one can test whether the time-series mean of $\gamma_{1t}$ is positive and/or whether the time-series mean of $\gamma_{0}$ equals the time-series mean of the observable risk-free (Treasury bill) return. Or one can add an additional cross-sectional variable (such as firm size) to (7) and see whether it receives a zero coefficient. See Fama (1976) for a detailed analysis of these types of tests and others using this procedure.

The problem in practice is that $\beta$ is not observable and must be estimated from time-series returns. Let $R_{0}$ denote a $T$-vector of observations of the risk-free or zero-beta return. Consider the time-series ordinary least squares regressions:

$$R_{i} - R_{0} = \hat{\alpha}_{i} + \hat{\beta}_{i} (R_{q} - R_{0}) + \hat{\varepsilon}_{i}, \quad i = 1, 2, \ldots, n,$$

giving a vector of estimated betas $\hat{\beta}$. One can use the estimated betas from these regressions in place of the true betas in the cross-sectional ordinary least squares regression:

$$R_{t} = \hat{\gamma}_{0t} + \hat{\gamma}_{1t} \hat{\beta} + \hat{\varepsilon}_{t}$$

(8)
where \( R_t \) is the returns from a time period not used in the estimation of \( \hat{\beta} \).

Unfortunately, the estimates from (8) are biased and inconsistent. For the simple model given by (1) - (6), BJS derive the asymptotic \((n \to \infty)\) bias. Let \( \sigma_q \) denote the sample variance of \(( r_q - r_0 )\). BJS show (p 92) that:

\[
\lim_{n \to \infty} \hat{\gamma}_{it} = (r_q t - r_{0t}) (1 + \sigma/(\Phi \sigma_q)).
\]  

(9)

A similar expression can be derived for \( \hat{\gamma}_{0t} \); for simplicity of exposition we will concentrate on \( \hat{\gamma}_{it} \).

LR (pp. 178-181) use (9) to develop a corrected estimation procedure. They note that, although the second-pass estimates from (9) are biased, the form of the bias is known. Equation (9) not only shows that there is an asymptotic bias, it also suggests a correction for this bias. Suppose that we know \( \sigma \) and \( \Phi \). Multiplying the estimate from (8) by \((1 + \sigma/(\Phi \sigma_q))\) will give an \( n \)-consistent estimate. Or suppose that \( \sigma \) and \( \Phi \) can be \( n \)-consistently estimated. Then replacing \( \sigma, \Phi \) with \( \hat{\sigma}, \hat{\Phi} \) gives the corrected estimate:

\[
\hat{\gamma}^*_{it} = (1 + \hat{\sigma}/(\hat{\Phi} \sigma_q)) \hat{\gamma}_{it}.
\]  

(10)

Since the plim of a product is the product of plims we have the desired \( n \)-consistency result:

\[
\lim_{n \to \infty} \hat{\gamma}^*_{it} = (r_q t - r_{0t}).
\]

A similar correction can be used for \( \hat{\gamma}_{0t} \); it will be skipped for sake of brevity. LR show that (3) is not necessary for a generalized version of this correction, but they do require (1) and (2).

Fama and MacBeth (1973) deal with the errors-in-variables problem by grouping assets into portfolios and using the portfolio betas and portfolio
returns in their cross-sectional regressions. Recall from the above that the errors-in-variables problem arises because asset betas are estimated with error. The beta of a portfolio is the weighted average of the betas of the assets in the portfolio. BJS (1972) were the first to note that if asset returns obey the diagonal market model then the estimation errors in the asset betas are uncorrelated and equally-weighted portfolio betas are measured with asymptotically (n→∞) zero error. FM apply this insight, along with a more sophisticated procedure than BJS for calculating t-statistics of the estimated parameters.

Let the assets be ordered according to an instrument which is correlated with true beta. Let $R_q$ denote an $m$-vector of portfolio returns each with $(n/m)$ assets:

$$R^q_j = \frac{(n/m)}{n} \sum_{i=(n/m)(j-1)+1}^{(n/m)j} R_i$$

Let $\beta^q$ denote the $m$-vector of true betas of these portfolios and $\hat{\beta}^q$ denote the betas estimated by time series ordinary least squares. Under the assumptions of a diagonal market model, holding $m$ and $T$ fixed, $\lim_{n→∞} \hat{\beta}^q = \beta^q$. Hence, with enough assets in each portfolio and under the assumption of a diagonal market model, the errors-in-variables problem diminishes toward zero.

The problem with both the LR and FM procedure is they require too much structure on security returns. In particular, the diagonal market model, (1) - (6), is very restrictive. It requires that there is no correlation between asset returns except as they co-vary with the market return.

We replace (1) - (3), (5), and (6) with a much weaker model called an approximate factor model. The one-beta market model (1) is replaced with a
multi-beta factor model. The zero correlation condition $E[s_i s_j] = 0$ is replaced by a condition that the correlations are "weak" (the covariance matrix has bounded eigenvalues). The homoskedasticity condition (3) is replaced with the condition that the variances have an upper bound. Condition (5) is written in a multi-beta form. Condition (6) is replaced with the much weaker condition that the idiosyncratic risks obey the weak law of large numbers. Using this model, we describe multi-factor versions of the two tests. Since an approximate factor model provides a better description of asset returns than (1) - (6), the resulting estimates should be more reliable than those given by the LR or FM procedures.

3. A Two-Pass Cross-Sectional Regression Procedure for Individual Asset Returns

This section presents the statistical model used throughout the remainder of the paper, and the estimation procedure for individual asset returns. First we describe the assumptions which replace (1) - (6). Next, we show that the results from the cross-sectional regression are n-consistent when true betas are used. Next, we describe the asymptotic bias when the true betas are replaced in the cross-sectional regression by time-series estimated betas. Last, we describe a correction to the second-pass cross-sectional regressions which eliminates the bias and inconsistency induced by errors-in-variables in the first-pass beta estimates.

The model assumes that stock returns obey an approximate factor model. This means that the return on each asset is a linear combination of a few economy-wide pervasive factors plus an asset-specific variate. In a strict
factor model, the asset-specific variates are uncorrelated across assets.

In an approximate factor model (a generalization of the strict factor model) the asset-specific variates are weakly correlated and there are many assets.

Let \( \tilde{f}_1, \ldots, \tilde{f}_K \) denote a set of mean-zero uncorrelated random variates with unit variances. Let \( \| \| \) denote the Euclidian norm for matrices. The necessary assumptions on the sequence of returns are as follows.

\[
\tilde{r}_i = c_1 + B_{11} \tilde{f}_1 + \ldots + B_{1K} \tilde{f}_K + \tilde{e}_i \quad i=1,\ldots,n \quad (11)
\]

\[
E[\tilde{r} \mid f] = 0 \quad E[\tilde{r} \tilde{r}^\prime \mid f] = \Sigma \quad (12)
\]

\[
\lim_{n \to \infty} \| \overline{Y}^n \| < c \quad (13)
\]

\[
\lim_{n \to \infty} ((1/n) \overline{y}^n \overline{y}^n)^{-1} = \Lambda \quad (14)
\]

\[
\lim_{n \to \infty} (1/n) \tilde{y} \tilde{y}^\prime = \sigma^2 \text{ for some finite } \sigma^2. \quad (15)
\]

\[
\tilde{r}_q = c_q + B_{11} \tilde{f}_1 + \ldots + B_{K} \tilde{f}_K. \quad (16)
\]

Equation (11) says that returns obey an approximate factor model and (12) says that the asset-specific variates are conditionally mean zero given the factors and have a conditionally constant covariance matrix.

These equation replace (1) and (2). Equation (13) is a much weaker version of (3). Equation (14) is the direct matrix generalization of (5). Equation (15) says that the weak law of large numbers applies to \( \tilde{\varepsilon} \); multivariate normality and (13) are sufficient but not necessary for this
condition. See Chamberlain and Rothschild (1983), and Connor and Korajczyk (1986, 1987b) for a discussion and interpretation of these conditions in the context of an approximate factor model.

Equation (16) says that the market portfolio is well-diversified so that asset-specific risk disappears from its return. This is a standard assumption of the equilibrium version of the APT.

In addition to (11) - (16) we will assume that the exact form of the APT holds true:

$$E[r_i] = r_0 + B_{1i} \pi_1 + \ldots + B_{ki} \pi_k.$$  \hspace{1cm} (17)

In a one-period competitive equilibrium model, Connor (1984) shows that (17) follows from (11) - (16). In a multiperiod competitive equilibrium model, Connor and Korajczyk (1987a) develop a version of (17) in which $r_0$ is a zero-beta return different from the one-period risk-free return. The test in this paper assumes the static (risk-free) version as a null hypothesis and tests this against the zero-beta version as an alternative.

Let $R$ denote an $n \times T$ matrix of realized excess returns on the $n$ securities for $T$ time periods with independent and identical distributions across time. In matrix notation we can write these returns, using (11) and (17) as:

$$R = B \pi + \varepsilon$$  \hspace{1cm} (18)

where $F$ is the $k \times T$ matrix of the realizations of $f_1 \pi_1, \ldots, f_k \pi_k$ and $\varepsilon$ is the $n \times T$ matrix of idiosyncratic returns. Let $R$, $F$ and $\varepsilon$ denote the sample deviations (minus the sample means) for $R$, $F$ and $\varepsilon$. Using (18) it is easy to show that we have:

$$R = BF + \varepsilon$$  \hspace{1cm} (19)
It will also be convenient to have a representation for the sample of raw returns. Let \( R^* \) denote the matrix of raw returns, let \( B^* = (e \mid B) \) where \( e \) is an \( n \)-vector of ones, and let \( F^* = (R_0 \mid F) \) where \( R_0 \) is the \( T \)-vector of risk-free or zero-beta returns. Then we can write:
\[
R^* = B^* F^* + \varepsilon.
\]

First, suppose that \( B \) is observable without error. Consider the cross-sectional ordinary least squares regression of \( R_{i,t} \) on \( B_i^* \). We assume that \( B_i^* \) obeys (14) for some matrix \( \Lambda^* \). Denote the \( k+1 \)-vector of estimated coefficients by \( \hat{\gamma}_{i,t} \). These are \( n \)-consistent estimates of the riskless return and \( F_{i,t} \).

**Theorem 1:** Given (10) - (14) then \( \lim_{n \to \infty} \hat{\gamma}_{i,t} = (R_0e', F_{i,t}) \).

Next suppose that \( B \) is not observable but that \( F \) is observable. Suppose that we estimate \( B_i \) by time-series ordinary least squares of \( R_{i,t} \) on \( F_i \). Let \( \hat{B} \) denote the matrix of estimated coefficients. From least-squares theory we know that \( \hat{B} = B + eF'(FF')^{-1} \). Suppose that we use \( \hat{B} \) in place of \( B \) in the cross-sectional regressions. Let \( \hat{B}^* = (e \mid \hat{B}) = B^* + [0^n \mid eF'(FF')^{-1}] \) where \( 0^n \) denotes an \( n \)-vector of zeros. Let \( \hat{\gamma}_{i,t} \) denote the \( k+1 \)-vector of regression coefficients (where \( t \) is a period not used in the estimation of \( \hat{B} \)). These estimates are biased and inconsistent. The next theorem describes the asymptotic bias.

**Theorem 2:** Given (11) - (17) then \( \lim_{n \to \infty} \hat{\gamma}_{i,t} = (\Lambda^* + 2\sigma)^{-1} \Lambda^* (R_0e', F_{i,t}) \), where
The last step is to correct for the asymptotic bias. Let \( \hat{\sigma} \) denote an n-consistent estimate of \( \sigma \). Define the new estimates:

\[
\hat{\gamma}_t^* = ((1/n)\{\hat{B}^\ast B^\ast\} - \hat{\sigma} \hat{\sigma})^{-1}(1/n)\hat{B}^\ast R_{it}.
\] (20)

These new estimates are n-consistent, as is shown next.

**Corollary 1:** Given (11) - (17), then \( \varlimsup_{n \to \infty} \hat{\gamma}_t^* = (R_{0t}, F_{it}) \).

In order to apply Corollary 1 we need to observe the pervasive factors \( F \). Connor and Korajczyk (1986) describe a set of econometric techniques for obtaining n-consistent estimates of \( F \). They call their factor estimation procedure asymptotic principal components. It is similar to standard principal components except that it relies on asymptotic results as the number of cross-sections becomes large.

We wish to identify \( F \) (or at least a \( k \times k \) nonsingular linear transformation of \( F \)). Let \( \Omega = \frac{1}{n} RR' \), the \( T \times T \) cross-product matrix of excess returns. The asymptotic principal components procedure uses the dominant eigenvectors of \( \Omega \) as estimates of the factors \( F \). As \( n \) approaches infinity, this provides asymptotically exact estimates of \( F \). See Connor and Korajczyk (1986) for a more detailed presentation.

Let \( G^n \) denote the first \( k \) eigenvectors of \( \Omega^n \). Connor and Korajczyk (1986a) show that \( G^n \) is asymptotically equal to \( F \), except for a rotational indeterminacy. That is, there exists a nonsingular \( k \times k \) matrices \( L \) and a
sequence of $k \times T$ matrices $\phi^n$ such that $G^n = LF + \phi^n$ and $\lim_{n \to \infty} \phi^n = 0$. Hence, the eigenvectors $G^n$ approach the true factors except for the rotational indeterminacy represented by $L$. Let $G^n$ and $F$ denote the de-meaned versions of $G^n$ and $F$.

Remark 1: Given (11) - (17) there exists a nonsingular $k \times k$ matrix $L$ such that $\lim_{n \to \infty} G^n = LF$.

Let $H$ denote the rotated version of the factors: $H = LF$. Now it is simple to combine Remark 1 with Corollary 2 and produce $n$-consistent estimates. Redefine $\hat{\gamma}^*_t$ as in (20) except using $G$ in place of $F$ (for simplicity of notation we will write $G$ rather than $G^n$).

Corollary 2: Given (11) - (17), $\lim_{n \to \infty} \hat{\gamma}^*_t = (r_0, H, s)_t$.

Corollary 2 is sufficient if we wish to use the $k$-vector of estimated risk premia and/or the estimated zero-beta returns to test the model. Often, we also wish to test the model against specific alternatives such as the hypothesis that firm size has an effect on expected returns. Suppose that we have an $n$-vector $s$ (such as the log of firm size) and an alternative hypothesis as follows:

$$E[\tilde{r}] = B(f + n) + \delta s,$$

where $\delta$ is a scalar, constant through time. We can test this by performing the same corrected cross-sectional regression (20) using the expanded beta matrix $B^{**} = [B^* | s]$. As before we need that $B^{**}$ obeys (14) for some $A^{**}$. 
Let \( \hat{B} \) denote the time series ordinary least squares estimates from regressing \( G \) on \( R \) and define \( \hat{B}^{**} = [\hat{B}^{**} \mid s] \). Define:

\[
\hat{\nu}_t = ((1/n)\hat{B}^{**} \hat{B}^{**'} - Z^\top (1/n)\hat{B}^{**} R_{t,t})^{-1} (1/n)\hat{B}^{**'} R_{t,t},
\]

where

\[
Z = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]

Corollary 3: Given (11) - (16) and (19) then \( \lim_{n \to \infty} \hat{\nu}_t = (\hat{\nu}_t, H_t, s) \).

4. Portfolio Grouping Under an Approximate Factor Structure

Rather than using an errors-in-variables correction, most existing cross-sectional regression tests of beta pricing models employ portfolio grouping. For existing tests of the CAPM, this requires that the diagonal market model holds. Note that the diagonal market model is a special case of our approximate factor model (11) - (16) where \( k=1 \) and \( V \) is diagonal. In this section we generalize the standard portfolio grouping procedure to the more general case of an approximate factor model. This provides a portfolio grouping procedure which can be used directly for estimating and testing the APT. As discussed in the next section, it can also be employed for a new cross-sectional regression test of the CAPM which does not require the diagonal market model.

Let the assets in \( \mathbb{R}^n \) be ordered according to an instrument (such as firm size) which is cross-sectionally correlated with each of the factor betas (see equation (21) below). Let \( m \) denote a fixed finite number and divide the \( n \) asset returns into \( m \) equally-weighted portfolios containing
\[ R_{jt}^g = (m/n) \sum_{i=j(n/m-1)+1}^{j(n/m)} R_{it} \quad j=1, \ldots, m; \quad t=1, \ldots, T. \]

Since the instrument ordering the assets is cross-sectionally correlated with each of the factor betas, we assume that the matrix of portfolio betas is nonsingular and linearly independent of the unit vector.

\[ \lim_{n \to \infty} \{e^g B^g\} = \Delta \text{ a nonsingular matrix,} \quad (21) \]

where \( e \) is an \( m \)-vector of ones and \( B^g \) is the \( mxk \)-matrix of portfolio betas.

Let \( \hat{B}^g \) denote the \( mxk \)-matrix of times series ordinary least squares estimates of the portfolio betas. Let \( \hat{\gamma}^g \) denote the \( k+1 \)-vector of cross-sectional estimates from regressing time \( t \) portfolio returns on \( \hat{B}^g \) and an \( m \)-vector of ones.

\[ \hat{\gamma}^g = R^g G'(GG')^{-1} \]

\[ \gamma_t^g = (\{e, \hat{B}^g\}' [e, \hat{B}^g])^{-1} [e, \hat{B}^g] R^g_t \quad t=1, \ldots, T. \]

The next theorem makes clear that an approximate factor model together with (21) is sufficient for the \( n \)-consistency of portfolio grouped two-pass cross-sectional regression.

**Theorem 3:** Given (11) - (17) and (21), then \( \lim_{n \to \infty} \gamma_t^g = (r_{0t}, H_{0t}). \)

The error-correction procedure of section 3 and the portfolio grouped procedure of this section both require an approximate factor model for \( n \)-consistency. Of course in finite samples one technique or the other may prove more reliable; we compare them empirically in section 6.
Theorem 3 can be extended to the case where additional independent variables are included in the cross-sectional regression as long as these variates are sufficiently correlated with the indexing instrument used for grouping (i.e., (21) must hold). This highlights one advantage of the individual asset error-correction procedure of section 3 -- since there is no portfolio grouping it does not require the observation of an instrument which is correlated with all of the cross-sectional variables.

5. Tests of Factor Risk Premia in the CAPM and APT

The estimation procedures of section 3 and 4 apply most naturally to the APT. The econometric assumptions used to justify the procedures are nearly identical to the theoretical assumptions used to derive the equilibrium version of the APT. In this section we discuss a new way to adapt these procedures to the estimation and testing of the CAPM. We also develop a new test statistic for the equilibrium version of the APT by relying on the special nature of the market portfolio in that model.

As noted by Shanken (1985a, 1987) the CAPM and equilibrium version of the APT are closely related pricing models. The CAPM prices assets according to their covariance with the market portfolio. The equilibrium APT prices assets according to their covariances with a set of factors and requires that the market portfolio is a linear combination of these factors. It turns out that given (11) - (17) the CAPM is simply an econometrically stronger version of the equilibrium APT. To see this combine (11) and (16) and define $\beta_1 = \text{cov}(f_1, r_q)/\text{var}(r_q), \ldots, \beta_k = \text{cov}(f_k, r_q)/\text{var}(r_q)$. Equation (3) then becomes equivalent to the intersection of (17) and:

$$\frac{\pi_1}{\beta_1} = \frac{\pi_2}{\beta_2} = \ldots = \frac{\pi_k}{\beta_k}. \quad (22)$$
This is proven next.

**Theorem 4:** Given (11) - (17) then (4) holds if and only if (22) holds.

Hence, under our assumptions, the CAPM imposes a linear restriction on the equilibrium APT. Of course, the general version of the CAPM is not a special case of the equilibrium APT since it does not require (11) - (16). However, given (11) - (16), the models are nested. We can test the CAPM by estimating the APT and testing whether the estimated multi-factor risk premia obey (22).

Fama and MacBeth rely on the diagonal market model for their estimation and testing of the CAPM. Theorem 4 gives a new way to estimate and test the CAPM. Rather than applying cross-sectional regression to the market model, we apply it to our approximate factor model. This produces estimates of the factor portfolio excess returns. Next, we use these estimates for a test of the CAPM by examining if they conform to (22). This procedure is in the spirit of Fama-MacBeth but avoids their undesirable implicit assumption of a diagonal market model. The results from this test appear in the next section.

The close relationship between the CAPM and equilibrium APT described in Theorem 4 also can be utilized to improve the test of the equilibrium APT. One of the problems with cross-sectional regression tests of the APT is that the estimated factors are arbitrary up to a linear transformation. This renders the tests of individual factor risk premia problematic, since the means of the realized risk premia depend on the arbitrary rotation of the factors. A conventional solution (e.g., Chen (1981)) is to test the
k-vector hypothesis that at least one of the risk premia is non-zero. This solves the problem at considerable cost in terms of lost statistical power. An alternative solution is to use the fact that in the equilibrium APT the market portfolio is a linear combination of the factors. It is easy to show that the the total risk premia (the sum of the factors betas times the factor risk premia) on the market portfolio should be positive. By multiplying the factor risk premia times the factor betas of market portfolio, one derives the contribution of each factor risk premia to the total risk premia of the market portfolio.

\[ \pi_j^* = \beta_j \pi_j \quad (23) \]

where \( \beta_j \) are given by (16). The sum of these risk premia equals the risk premium on the market portfolio which should be positive:

\[ \pi_i^* + \ldots + \pi_k^* = \pi_q > 0. \quad (24) \]

Equation (24) gives a more efficient test of the equilibrium APT than the usual k-vector hypothesis \( \pi_j \neq 0 \) for some \( j \). The scaling given by (23) also has intuitive content since it measures each risk premia in terms of its contribution to the total risk premia of the market portfolio.

We describe the results of these new tests in the next section.

6. **Empirical Application**

Given past findings, an obvious alternative hypothesis to either the CAPM or APT is that the market value of equity ("size") affects asset expected return. Recently there have been conflicting findings on the size effect. Our results offer some new evidence which may help to resolve these conflicting findings.
Chan and Chen (1987) argue that the size anomaly in the CAPM (except for the January seasonality) is a statistical illusion caused by the incomplete elimination of errors-in-variables in the standard Fama-McBeth two-pass cross-sectional regression estimates. Chan and Chen seek to eliminate the errors-in-variables bias more fully by using very long time-series samples of portfolio returns. They note that this will eliminate the errors-in-variables asymptotically ($T \to \infty$) since the portfolio betas will be measured with error approaching zero as $T$ approaches infinity. Their empirical results indicate that this completely eliminates any unexplained size-based return premium. In this paper we explore Chan and Chen's claim with a very different errors-in-variables correction. Rather than relying on large time-series and the FM procedure, we improve the efficiency of the procedure for a fixed time-series sample with asymptotic principal components techniques and a multifactor return model. Because our approach is very different and yet has the same objective, it provides a useful corroboration, or lack thereof, to the Chan and Chen findings.

We also seek to reconcile conflicting results on the APT. Chan, Chen and Hsieh (1986) argue that replacing the CAPM with the APT eliminates the size effect. This contradicts the findings of Connor and Korajczyk (1987) and Lehmann and Modest (1986) who argue that the size effect is reduced but not eliminated using the APT in place of the CAPM. The estimation techniques differ in these three papers. Connor and Korajczyk and Lehmann and Modest use factor analytic procedures to estimate the factor portfolio returns and a multivariate procedure to estimate the size effect. Chan, Chen and Hsieh begin with a set of macrovariables as factors and then apply
two-pass cross-sectional regression on size-based portfolios to estimate the factor portfolio returns and the size effect.

This paper combines the methodologies of Connor and Korajczyk and Chan Chen and Hsieh. Like Connor and Korajczyk, we begin with a set of asymptotic principal components as factors. Unlike Connor and Korajczyk, we then apply two-pass cross-sectional regression to these factors to re-estimate them, both with and without a size variable. We find that, as in Chan, Chen and Hsieh, this two-pass cross-sectional regression procedure eliminates the size effect. It is the two-pass cross-sectional regression procedure, not the choice of factors, which separates Chan, Chen and Hsieh from Connor and Korajczyk and Lehmann and Modest.

We interpret our evidence as indicating that two-pass cross-sectional regression, when efficiently applied to the APT, eliminates the size effect because it more adequately deals with the errors-in-variables problem. However, it is possible that two-pass cross-sectional regression eliminates the size effect because, for some reason, it is less powerful against a size-based alternative than the aforementioned tests of Connor and Korajczyk and Lehmann and Modest.

This section presents empirical tests using monthly data for the period 1953 - 1982 and the full cross-section of NYSE equity-returns with no missing observations within the relevant subperiod. We divide the data into six five-year blocks in order to better approximate the assumption that asset betas are constant; the 53-57, 58-62, 63-67, 68-72, 73-77, 78-82 blocks contain 955, 947, 979, 1037, 1295, 1229 continuously traded assets, respectively. We test the APT against the specific alternative that firm size affects expected returns. We test whether the cross-sectionally
estimated risk premia on the market portfolio is positive and whether on average it equals the observed time-series risk premia on the value-weighted index. We test whether the cross-sectionally estimated zero beta rate is positive and whether on average it equals the observed monthly Treasury Bill rate.

For each five-year block, we estimate $\hat{\beta}_t$ by regressing $R$ on $F$. Then we cross-sectionally regress $R^*_t$ on a unit vector and $\hat{\beta}_t$ to get $\gamma_{0t}, \gamma_{1t}$. Note that this regression uses the error-in-variables correction or portfolio grouped returns, not ordinary least squares estimates on individual assets. Repeating for each $t$ gives a time-series of estimates $(\gamma_{0t}, \gamma_{1t})_{t=1,T}$. The statistical theory assumes that the $t^{th}$ cross-sectional regression does not use a beta calculated using the $t^{th}$ period return observation. We could have achieved this by deleting the $t^{th}$ observation, calculating betas with the remaining five-year sample, and then performing the cross-sectional regressions on the $t^{th}$ period returns. Incorporating this modification would have increased the computational requirements to an impracticable level.

The estimated risk premia are scaled by the market betas of the factor portfolios, as given by (23). The equilibrium APT predicts that the sum of the $k$ factor risk premia is positive and equals the realized risk premia on the market portfolio. The static version of the APT predicts that $\gamma_{0t}$ equals the return on a one-period riskless asset, which we proxy with the return on one-month Treasury Bills.

The same procedure is repeated with an $n$-vector of the natural logarithm of equity market value for each firm included in the cross-sectional regressions. Market value is measured at the end of the
November 4 preceeding each five year period. This gives a time-series of 
k+2 = 7 estimated coefficients, \( (\gamma_0, \gamma_1, \ldots, \gamma_k) \). As before, the factor 
risk premia are scaled using the market betas of the factor portfolios.

We also estimate by replacing the estimated zero-beta return with the 
observed Treasury Bill returns. That is, we perform the cross-sectional 
regressions using excess returns and without an estimated intercept.

We also estimate using the approaches of Chan, Chen and Hsieh and Fama 
and MacBeth. For both these models, we use five-year blocks and twenty 
equally weighted portfolios indexed by size. We use the macrovariates 
suggested in Connor and Korajczyk (1987b): the excess return to low-grade 
bonds, the excess return to long-term corporate bonds, the shock to 
industrial production, the shock to unemployment, and the shock to 
inflation. See Connor and Korajczyk (1987b) for an analysis of these 
macrovariates in the 1968-1982 period. We estimated the same variates in 
the 1953-1967 period; our results for this earlier period are available in 
an unpublished appendix.

Tables 1 categorizes the various models we analyzed. Taking into 
account all of the combinations (with and without size, Treasury Bill 
versus zero beta versions, etc.) there are a total of sixteen models. 
Table 2 summarizes the key results from all of the models.

The APT outperforms the CAPM when tested against a size-based 
alternative hypothesis. Of the six implementations of the APT only one, 
the zero-beta model estimated against individual assets, shows a 
significant size effect. Both of the CAPM models have a significant size 
effect, a well known finding.
The riskless asset version of the APT performs as well or better than the zero-beta version. There is no significant difference between the mean Treasury Bill return and the estimated mean zero-beta return. Adding an estimable parameter reduces the efficiency of estimation if that parameter is observable. It may be that using the Treasury Bill return as the riskless asset return improves the efficiency of estimation of the other coefficients. Choosing the riskless asset version eliminates the one version of the APT which has a significant size effect.

Our procedures do little to explain the January seasonality in returns and the relationship between this seasonality and size. It may be that beta pricing models can explain long-run expected returns but not the short-run predictable variations in returns (January effects, day-of-the-month effects, weekend effects).

Tables 3 describes the results of the test of the CAPM as a cross-factor premia restriction on the equilibrium APT. We use the $T^2$ test to test the linear restriction given by equation (22); see Shanken (1985b, p. 330) for a complete description of this test. We aggregate the six subperiod results by converting the $F$ statistics into equivalent Normal statistics (see Shanken (1985b, p. 339-340).

The results of Table 3 test the CAPM as a restricted version of the equilibrium APT. In this form, in which the econometric strengths of the equilibrium APT are inherited by the CAPM, we cannot reject the CAPM. This argues, in line with the work of Chan and Chen, that it is the statistical inadequacies of standard tests of the CAPM which lead to its rejection against a size-based alternative.
7. Summary

This paper develops a new approach to two-pass cross-sectional regression estimation of beta pricing models. A key difficulty with two-pass cross-sectional regression estimation is that the betas estimated in the first step are used as independent variables in the second step, giving rise to an errors-in-variables problem. Litzenberger and Ramaswamy develop an errors-in-variables correction for estimating the CAPM under the assumption of a diagonal market model; Fama and MacBeth use portfolio grouping to eliminate the errors-in-variables, also under the assumption of a diagonal market model. This paper develops new versions of these two procedures under the much weaker assumption of an approximate factor model for returns by using the asymptotic principal components theory of Connor and Korajczyk (1986). We apply the procedure to the APT. We also develop an extension which allows us to test the CAPM as a special case of the equilibrium version of the APT.

Recently there have been some conflicting findings on the size effect in the CAPM and the APT. Chan and Chen (1987) find that when long time series of size-grouped portfolios are used, the size effect (but not the January seasonality) disappears from the CAPM. This conflicts with a wealth of studies on the size effect in the CAPM (e.g., Banz (1981), Reinganum (1981), Brown, Kleidon and Marsh (1983)). Chen, Chen and Hsieh (1986) find that the size effect disappears in their two-pass cross-sectional regression estimates of the APT. This conflicts with Connor and Korajczyk (1987b) and Lehmann and Modest (1986) who find that the size effect is reduced but not eliminated by replacing the CAPM with the APT. All the APT studies show some kind of unexplained January seasonality in factor premia and/or size premia.
Our approach seems to corroborate the findings of Chan and Chen and Chan, Chen and Hsieh. Firm size is closely correlated with equity betas. Any errors-in-variables in measuring betas will give rise to a spurious size effect. Our procedure provides a more reliable algorithm for eliminating errors-in-variables than is used in previous studies. We find that the size effect disappears. We can therefore claim that previous findings on the size effect are due to inadequate treatment of the errors-in-variables problem in two-pass cross-sectional regression procedures.

The January effect, which is closely linked to the size effect, is not eliminated in our estimates. Although there is no size effect on average over time, there is an unexplained size effect and/or a mis-estimation of the market risk premium in January. We only succeed in adequately explaining the per-calendar-year relative rates of return of large versus small firms. The January effect remains a mystery.
Footnotes

1. For technical reasons, assuming the diagonal market model describes asset returns implies that the market portfolio return is not a linear combination of the returns of the observed assets. This implicit assumption is not a problem for the model as we will use it.

2. We follow Shanken's (1983) reformulation of the Litzenberger-Ramaswamy test more closely than the original maximum likelihood procedure of Litzenberger-Ramaswamy.

3. The proof that \( \pi_q > 0 \) goes as follows. Let \( U(\cdot) \) denote the one period utility function of the representative investor, who holds the market portfolio in competitive equilibrium. Let \( W \) denote his time 0 wealth. Expected returns must obey the following first-order condition in order to induce the representative investor to hold the market portfolio:

\[
E[U'(W_q) r_q] - E[U'(W) r_q] = 0,
\]

which implies:

\[
E[U'(W r_q)] \pi_q = - \text{Cov}[U'(W r_q), r_q].
\]

Since \( U'(\cdot) > 0 \) we have \( E[U'(W r_q)] > 0 \). Since \( U''(\cdot) < 0 \) we have \( \text{Cov}[U'(W r_q), r_q] < 0 \) and the result follows.

4. We use November, rather than December, measures of size because we believe that it is less likely to contain the effects of tax-loss selling and other end-of-the-year effects which may overstate the size effect.
Appendix

This appendix contains the proofs of the theorems.

Proof of Theorem 1: From the definition of ordinary least squares estimates (see Thiel p.112),
\[ \hat{\gamma}_t = (B^*B^*)^{-1}B^*R_t = F_t + (B^*B^*)^{-1}B^*\varepsilon_t. \]

We must show that \( \lim_{n \to \infty} (B^*B^*)^{-1}B^*\varepsilon_t = 0 \). Multiplying and dividing by \( \frac{1}{n} \) gives:

\[ \lim_{n \to \infty} (1/n)(B^*B^*)^{-1}(1/n)B^*\varepsilon_t = \lim_{n \to \infty} (1/n)(B^*B^*)^{-1}\lim_{n \to \infty} (1/n)B^*\varepsilon_t \]

\[ = \Lambda^* \lim_{n \to \infty} (1/n)(B^*\varepsilon_t). \]

We will prove that \( \lim_{n \to \infty} (1/n)(B^*\varepsilon_t) \) is zero by showing that the cross-product matrix converges to a zero matrix and noting that mean-square convergence implies convergence in probability. Taking the norm of the cross-product matrix gives:

\[ \| \text{E}((1/n)(B^*\varepsilon_t))(1/n)(B^*\varepsilon_t)' \| = \| (1/n)\| (1/n)B^*V'N'V \| \]

\[ \leq (1/n)\| (1/n)(B^*B^*)\| \| V^\|\|. \] Since all three terms are bounded above and \( (1/n) \to 0 \) we have the result.

\[ \text{B.E.D.} \]

Proof of Theorem 2: From the definition of ordinary least squares estimates:

\[ \hat{\gamma}_t = (\hat{B}^*\hat{B}^*)^{-1}\hat{B}^*R_t = ((1/n)\hat{B}^*\hat{B}^*)^{-1}(1/n)\hat{B}^*R_t. \]

where we have multiplied and divided by \( (1/n) \). First, we will show that

\[ \lim_{n \to \infty} (1/n)\hat{B}^*\hat{B}^*R_t = (1/n)B^*\varepsilon_t = \Lambda^*(r_{\text{out}}, F_t). \]

Then we will show that \( \lim_{n \to \infty} ((1/n)\hat{B}^*\hat{B}^*)^{-1} = (\Lambda^* + \sigma I)^{-1} \). Combining these two results gives the conclusion.
Using the definition of $\hat{B}^*$ gives

$$\frac{1}{n}\hat{B}^* R_{,t} = \frac{1}{n}\hat{B}^* R_{,t} + 0 + \left(FF'\right)^{-1}F\left(1/n\right)\varepsilon' R_{,t} . \tag{25}$$

As shown in Theorem 1 above, plim $\frac{1}{n}\hat{B}^* R_{,t} = \Lambda^\ast\left(r_{0t}^\ast F_{,t}\right)$. For the other nonzero term in (25) we have:

$$\text{plim } \left(FF'\right)^{-1}F\left(1/n\right)\varepsilon' R_{,t} = \left(FF'\right)^{-1}F \text{ plim } \left(1/n\right)\varepsilon' R_{,t} = \left(FF'\right)^{-1}F \left(\text{plim } \left(1/n\right)\varepsilon' B_{,t} F_{,t} + \left(1/n\right)\varepsilon' \varepsilon_{,t}\right) = 0 ,$$

where we use the fact that $\varepsilon$ and $\varepsilon_{,t}$ are from different time periods. This proves that $\text{plim } \hat{B}^* R_{,t} = \Lambda^\ast\left(r_{0t}^\ast F_{,t}\right)$.

Again using the definition of $\hat{B}^*$, and expanding $\left(1/n\right)\hat{B}^* \hat{B}^*$ gives:

$$\left(1/n\right)\hat{B}^* \hat{B}^* \left(1/n\right) = \left(1/n\right)B^\ast B^\ast + Z^n \right)^{-1} , \text{ where} \quad Z^n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} .$$

Since, as noted in the proof of Theorem 1, $\text{plim } \left(1/n\right)\varepsilon' \varepsilon = \sigma I_{K'}$, it is easy to see that $\text{plim } Z^n = Z$, and this completes the proof.

Q.E.D.

Proof of Corollary 1: We have from Theorem 2 that $\text{plim } \left(1/n\right)(\hat{B}^* \hat{B}^*) = \Lambda^\ast + \sigma Z$. Since the inverse of a matrix is a continuous function of its elements, and the plim of a continuous function is the function of the plim, it follows that $\text{plim } \left(\left(1/n\right)\hat{B}^* \hat{B}^* - \sigma Z\right)^{-1} = \Lambda^\ast \left(r_{0t}^\ast F_{,t}\right)^{-1}$. Combining this with $\text{plim } \left(1/n\right)\hat{B}^* R_{,t} = \Lambda^\ast\left(r_{0t}^\ast F_{,t}\right)$ gives the result.

Q.E.D.

Proof of Remark 1: It is immediate from Connor and Korajczyk (1986,
Theorem 6: That \( \lim_{n \to \infty} G^n = LF \). We must show that this implies \( \lim_{n \to \infty} G^n = LF \).

From the definition of a de-meaned series: \( G^n = G^n(I_T - e'e') \) and \( F = F(I_T - e'e') \) where \( I_T \) is the \( T \times T \) identity matrix and \( e \) denotes a \( T \)-vector of ones. Note that \( \lim_{n \to \infty} G^n = \lim_{n \to \infty} (G^n(I_T - e'e')) = LF(I_T - e'e') = LF \).

G.E.D.

Proof of Corollary 2: For notational convenience we let \( L = I_k \), the \( k \times k \) identity matrix. It is straightforward to extend the result to the case where \( L \) is any nonsingular \( k \times k \)-matrix. Note that \( \gamma_t^* \) (from equation (20)) is a continuous function of \( F \). Let \( \gamma_t^* \) denote \( \gamma_t^* \) except that we replace \( F \) by \( G \). From Corollary 1 we have \( G^n = F + \psi^n \) where \( \lim_{n \to \infty} \psi^n = 0 \). Since the plim of a continuous function is the function of the plim we have \( \lim_{n \to \infty} \gamma_t^* = \lim_{n \to \infty} \gamma_t^* \).

G.E.D.

Proof of Corollary 3: The proof is an identical repetition of the proofs of Theorem 2 and Corollaries 1 and 2. Simply replace \( B^* = [e|\hat{B}] \) with \( B^{**} = [e|\hat{B}|s] \) and proceed exactly as above.

G.E.D.

Before proving Theorem 3 we need the following.

Lemma 1: Given (11) - (17), \( \lim_{n \to \infty} s_j = 0 \) for \( j=1,\ldots,m \).
Proof of Lemma 1: Let $V_1$ denote the $(n/m) \times (n/m)$ covariance matrix of the first $(n/m)$ idiosyncratic returns. Note that $\|V_1\| \leq \|V^n\|$ for all $n$, since the maximum eigenvalue of a submatrix is less than or equal to the maximum eigenvalue of the whole matrix. From the definition of $s^2_1$ we have
\[ E(e^2_1) \leq (n/n)^2 e\cdot V^n e \] where $e$ denotes an $(n/m)$-vector of ones. Note that $(n/n)^2 e\cdot V^n e \leq (n/n) \| V_1 \| e \rightarrow 0$ as $n \rightarrow \infty$. Hence we have $\lim_{n \rightarrow \infty} E(e^2_1) = 0$ which implies that $\plim_{n \rightarrow \infty} s^2_1 = 0$ since mean-square convergence implies plim convergence. The same steps can be repeated for $s^2_j, j = 2, \ldots, m$.

Q.E.D.

Proof of Theorem 3: Recall the equations defining $\hat{\gamma}^g$ and $\gamma^g$ from the text:
\[ \hat{\gamma}^g = R^g F (F F')^{-1} \]  \hspace{1cm} \text{(26)}
\[ \gamma^g_t = (e, \hat{\gamma}^g) (e, \hat{\gamma}^g)_t^{-1} R^g \]  \hspace{1cm} \text{for } t = 1, \ldots, T. \hspace{1cm} \text{(27)}

First we will show that $\plim_{n \rightarrow \infty} \hat{\gamma}^g = \gamma^g$. Rewriting (26) using $R^g = B^g F + \varepsilon^g$ gives:
\[ \hat{\gamma}^g = (B^g F + \varepsilon^g) F (F F')^{-1} \].

From Lemma 1 we have that $\plim_{n \rightarrow \infty} \varepsilon^g = 0$ and so $\plim_{n \rightarrow \infty} \hat{\gamma}^g = B^g (F F') (F F')^{-1} = B^g$. Next, we will show that $\plim_{n \rightarrow \infty} \gamma^g_t = (r_{0t}^g, F_t)$. Note that we can express $R^g_t$ as $R^g_t = e r_{0t}^g + B^g F_t + \varepsilon^g_t = (e, B^g) (r_{0t}^g, F_t) + \varepsilon^g_t$. From (26) and (27) we have:
\[ \plim_{n \rightarrow \infty} \gamma^g_t = \plim_{n \rightarrow \infty} ((e, \hat{\gamma}^g) (e, \hat{\gamma}^g)_t^{-1} (e, B^g) (r_{0t}^g, F_t) + \varepsilon^g_t) \].
Using $\plim_{n \rightarrow \infty} \hat{\gamma}^g = \gamma^g$ and (21):
\[ = (r_{0t}^g, F_t) + ((e, B^g) (e, B^g)_t^{-1} (e, B^g) \varepsilon^g_t \]
Proof of Theorem 4: First we show that (4) implies (20). Let $r_{0j}$,
$j=1,\ldots,k$, denote the return to a portfolio with a unit loading on the $j$th
factor and a zero loading on the other factors. From the definition of
$\gamma_1,\ldots,\gamma_k$ we have $E[r_{0j} - r_0] = \gamma_j$. Given that (4) holds true, we also
have $E[r_{0j} - r_0] = \beta_j \pi_q$ where $\pi_q$ is the expected excess return on the
market portfolio. Combining these two relationships gives (20).

Now we show that (20) implies (4). Let $r_i$ denote the return on an
arbitrary asset $i$. From (16) we have $E[r_i - r_0] = \sum_{j=1}^{n} B_{ij} \gamma_j$. Using (20)
this becomes $E[r_i - r_0] = \pi^* \sum_{j=1}^{n} B_{ij} \left( \frac{\text{cov}(r_q, f_j)}{\text{var}(r_q)} \right)$, where $\pi^* = \gamma_j / \beta_j$.
From the definition of the CAPM beta and using (16) we have $\beta_i = \sum_{j=1}^{n} B_{ij} \frac{\text{cov}(r_i, f_j)}{\text{var}(r_q)}$. Combining expressions we
have $E[r_i - r_0] = \pi^* \beta_i$ which implies that the CAPM holds true for every
asset.

Q.E.D.
Bibliography


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Arbitrage Pricing Theory" Northwestern University working paper.

-------- and ------ (1987b) "Risk and Return in an Equilibrium APT," Northwestern University working paper.


Table 1
A SUMMARY OF THE MODELS

<table>
<thead>
<tr>
<th>Theoretical Pricing Model:</th>
<th>APT</th>
<th>CAPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>First-Pass Regressors:</td>
<td>Statistical Factors</td>
<td>Macroeconomic Variates</td>
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<tr>
<td>Error-Correction Technique:</td>
<td>Portfolio Grouping</td>
<td>Individual Asset Error Correction</td>
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<td>Risk-free version without size</td>
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<td>FIT</td>
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<tr>
<td>Risk-free version with size</td>
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<td>FITS</td>
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<td>FIZ</td>
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<tr>
<td>Zero-beta version with size</td>
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<td>FIZS</td>
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</tbody>
</table>

Acronym Codes

- **F** = statistical factors used as first-pass regressors
- **M** = macrovariates used as first-pass regressors
- **V** = value-weighted index used as first-pass regressor
- **G** = portfolio groups used in second-pass regressions
- **I** = individual assets used in second-pass regressions
- **T** = Treasury Bill return assumed as the risk-free return
- **Z** = zero-beta return estimated in the second-pass regressions
- **S** = size variable included in the second-pass regressions
### Table 2

**A SUMMARY OF THE 30-YEAR RESULTS**

<table>
<thead>
<tr>
<th>Acronym</th>
<th>FGT</th>
<th>FIT</th>
<th>MGT</th>
<th>VGT</th>
<th>FGT$^T$</th>
<th>FIT$^T$</th>
<th>MGT$^T$</th>
<th>VGT$^T$</th>
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<td>$.448 \times 10^{-2}$</td>
<td>$.700 \times 10^{-2}$</td>
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<td>(2.11)</td>
<td>(2.96)</td>
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<th>MGZ$^T$</th>
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Table 3
Test of the CAPM as a Restriction on the Equilibrium APT

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