AN INTERTEMPORAL EQUILIBRIUM
BETA PRICING MODEL

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An Intertemporal Equilibrium Beta Pricing Model

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ABSTRACT

This paper develops an intertemporal, discrete-time, competitive equilibrium version of the Arbitrage Pricing Theory (APT) and explores the econometric implications of this model under various restrictions on investor preferences and the dynamic behavior of equity cash flows. We describe conditions under which the standard econometric specification used for estimating and testing the APT can be fully justified by an economic model. We relate this intertemporal version of the APT to the static APT and to Merton's Intertemporal Capital Asset Pricing Model (ICAPM).
1. Introduction

This paper develops an intertemporal, discrete-time, competitive equilibrium version of Ross's Arbitrage Pricing Theory (APT) and discusses the econometric content of this model under various restrictions on investor preferences and on the multivariate stochastic process determining equity cash flows. We discuss the testable distinctions between this model, the static APT, and Merton's Intertemporal Capital Asset Pricing Model (ICAPM).

We show that it is straightforward to extend the static equilibrium APT to an intertemporal environment if one ignores problems of estimability. If one constrains the pricing model to be estimable with standard time series techniques (which require, in particular, that asset betas are constant over time and, in some cases, that returns are intertemporally homoscedastic) the modeling problem is more difficult and the necessary assumptions stricter. This may argue that time-varying beta estimators and/or heteroscedasticity consistent estimators are more appropriate than the techniques used in most published empirical work.

Our intertemporal APT share similarities to Merton's ICAPM. The two models are different, but they may not be empirically distinguishable. We also describe the testable difference between our intertemporal and earlier static versions of the APT.

Section 2 proves a general version of the intertemporal equilibrium APT. The key distinction from the static APT is that corporate dividends, rather than equity returns, are assumed to obey an approximate factor model. The factor model on equity returns, as well as the APT restriction on asset expected returns, is derived endogenously. This model is similar to the intertemporal extension of the capital asset pricing model (CAPM) developed by Bossaerts and Green (1987).
Section 3 develops two special cases of the model in section 2 which have more empirical content. In the first version, the representative investor has constant absolute risk aversion and corporate profits are given by an approximate factor model and are independently distributed through time. In the second, the representative investor has constant relative risk aversion and the proportional changes (rather than levels) of corporate profits are independent through time. The advantages and disadvantages of the two models are discussed.

Section 4 describes the econometric distinctions between three beta pricing models: the static APT, the intertemporal APT, and Merton's ICAPM. Section 5 summarizes the paper.

2. An Intertemporal Competitive Equilibrium Version of the APT

In this section we develop an intertemporal extension of the competitive equilibrium version the APT presented in Connor (1984). The original Arbitrage Pricing Theory is a static asset pricing theory. This means that one views the pricing equilibrium as occurring only once, followed by a terminal realization of investor wealth. Of course, the model is usually tested by relying on time-series data -- that is, the observation of the repeated price setting process which occurs in security markets. The time-series data must have various "stationarity" properties to render the model estimable. In a path-breaking work, Lucas (1978) suggests an integrated approach in which the statistical model (a time-stationary relationship) is derived endogenously in an infinite-horizon asset pricing model. This approach to asset pricing econometrics was extended by Prescott and Mehra who named the model a recursive competitive equilibrium (RCE) model. The RCE model has been influential, but by no means
universally accepted, in the finance literature. In this paper we apply the RCE approach to the APT. We use a model which is similar to the RCE model that Bossaerts and Green (1987) develop in their intertemporal extension of the CAPM.

We believe our RCE model is a useful complement to recent empirical studies which develop statistical tests with time-varying betas; time-varying risk premia; and/or conditional heteroscedasticity [e.g., Bollerslev, Engle, and Woolridge (1985); French, Schwert, and Stambaugh (1986); Ferson, Kandel, and Stambaugh (1987); and Gennette and Marsh (1987)]. All of these statistical phenomena are intertemporal in an essential way and so cannot be properly explored within a static model such as the CAPM or static APT. In its general form our model does not provide a closed-form solution for heteroscedasticity or time-variation in risk parameters but it does provide a solid foundation for exploring these properties. We examine some special cases which provide closed form solutions.

The economy consists of an infinite number of identical investors. The investors live forever and have additively separable, time-independent von Neumann-Morgenstern utility functions for the one good produced each period. The price of this good is normalized to 1 each period. Since all investors are identical, we act as if there is a single "representative" investor with time-separable utility function \( u(\cdot) \) and discount factor \( \rho \).

There exists a countable infinity of corporations in the economy, all of which are purely equity financed. The per-share dividends of firm \( i \) at time \( t \) will be denoted \( X_{it} \) where \( - \) denotes a random variable. The \( \mathbb{R}^\infty \)-vector of corporate dividends as time \( t \) follows an approximate factor model:
\[ \bar{x}_t = cf_{0t} + B \bar{z}_t + \bar{\xi}_t, \] (1)

where \( c \) is an \( \mathbb{R}^\infty \)-vector of expected dividends, \( B \) is an \( \mathbb{R}^\infty \times k \)-matrix of asset betas, \( f_{0t} \) is a scalar known at time \( t - 1 \), \( \bar{z}_t \) is a mean-zero, \( k \)-vector stochastic process, and \( \bar{\xi}_t \) is a mean-zero \( \mathbb{R}^\infty \)-vector stochastic process. In this section we will impose very weak assumptions on the stochastic processes \( \bar{z} \) and \( \bar{\xi} \).

In the simplest version of the model, \( f_{0t} = 1 \) for all \( t \) so that the expected dividends to the equities are constant through time. However, we also allow \( f_{0t} \) to be time trend (deterministic growth) or a stochastic process (random growth in expected dividends). Note that \( f_{0t} \) is known at time \( t - 1 \) and so we can speak of \( cf_{0t} \) as expected dividends for time \( t \) given the information available at time \( t - 1 \).

A portfolio, \( \alpha_{t-1} \), is a linear function from the random vector of asset payoffs to a random payoff. The space of portfolios is restricted to those linear functions with finite mean-square payoff and we use the mean-square payoff to define a norm on the space of portfolios.\(^1\) The market portfolio, \( q \), is the per-capita supply of the assets. Since no new shares are issued or redeemed in the economy, \( q \) is constant through time.

We summarize the model with a set of assumptions.

A1: There exists an infinite-lived representative investor with risk-averse von Neumann-Morgenstern utility function \( u(\cdot) \) and time discount factor \( \rho \).

A2: There exists a countable infinity of purely equity financed firms with net per-share dividends given by (1).

A3: \[ \mathbb{E}_{t-1}[\bar{\xi}_t | f_t] = 0. \]

A4: \[ \mathbb{E}_{t-1}[(q'\bar{\xi}_t)^2] = 0. \]

A5: \[ \lim_{t \to \infty} \mathbb{E}_{t-1}[\rho^S u'(C_{t+S})(\bar{x}_{it+s} + \bar{p}_{it+s})] = 0, \text{ for all } i=1,2,\ldots \]
The first and second assumptions were described above. Note that the second assumption applies to dividends paid out rather than total profits. Reinvestment of profits is permitted in the model as long as the cash paid out follows the assumed factor model. We do not explicitly model the reinvestment decision and so treat dividends as exogenous.

Assumption A3 states that the idiosyncratic risks are "increasing risk" in the sense of Rothschild-Stiglitz (1970). This guarantees that a risk-averse investor holding a well-diversified portfolio will not want to hold any ε risk (see Connor (1984) for more discussion on this point). A4 requires that the market portfolio be well-diversified. This is a key assumption behind the static competitive equilibrium version of the APT. A5 is a transversality condition which guarantees that no asset has infinite value to the representative investor.

Let \( p_t \) denote an \( \mathbb{R}^n \)-vector of asset prices at time \( t \). The following is an intertemporal extension of Connor (1984), Theorem 3.

Theorem 1: Given A1 - A5 there exists a unique sequence of competitive equilibrium prices given by:

\[
p_t = \pi_0 t^c + B \pi_t
\]

\[
\pi_0 = \mathbb{E}_t \left[ \sum_{s=1}^{\infty} \rho^s u'(q'x_{t+s}) f_{0t+s} / u'(q'x_t) \right] 
\]

\[
\pi_t = \mathbb{E}_t \left[ \sum_{s=1}^{\infty} \rho^s u'(q'x_{t+s}) f_{jt+s} / u'(q'x_t) \right] 
\]

(All proofs appear in the appendix.) We have proven that prices are beta
linear: the price of each asset is a linear multiple of its expected payoff, \( c_i \), and its sensitivities to the \( k \) factors, \( B_i \). This is the basic prediction of the APT. Note, however, that the "factor prices" \( \pi_{ot} \), \( \pi_t \) need not be constant through time. The APT pricing relationship holds cross-sectionally at each point in time but is not necessarily constant through time. This makes the model difficult to estimate, as will become apparent.

In order to render (2) compatible with time series return data, we must rewrite the relationship in terms of rates of return. Define the \( \mathbb{R}^k \)-vector \( r_t = (p_t + x_t)/p_{t-1} \), where the division is performed element-wise. In order to perform this division we must be sure that \( p_{it} \neq 0 \) for all \( i \) and \( t \), but this is not a substantive problem.

Let \( \alpha_{m1} \) denote a well-diversified portfolio with \( \alpha_{m1} c = 0 \), \( \alpha_{m1} B = (1, 0, \ldots, 0) \). We will call this the first factor portfolio since it mimics the first factor. Similarly define \( \alpha_{m2}, \ldots, \alpha_{mk} \) to mimic the factors \( 2, \ldots k \). Under fairly weak conditions these portfolios will have nonzero prices. Define \( r_{mlt}, \ldots, r_{mkt} \) as the returns to these portfolios. Define the zero-beta portfolio as a well-diversified portfolio such that \( \alpha_0 c = 1 \), \( \alpha_0 B = (0, \ldots, 0) \), and the return on this portfolio \( r_{0t} \) as the zero-beta return. The beta pricing result of Theorem 1 can be restated in rate-of-return form using these portfolios.

**Corollary 1:** Given A1 - A5 then

\[
\bar{r}_t - \bar{r}_{0t} = \beta_{1t-1}(\bar{r}_{mlt} - \bar{r}_{0t}) + \ldots + \beta_{kt-1}(\bar{r}_{mkt} - \bar{r}_{0t}) + \epsilon_t^* \tag{3}
\]

\[
\beta_j^t = (\beta_{1jt}, \beta_{2jt}, \ldots)
\]

\[
\iota^t = (1, 1, 1, \ldots)
\]
\[
\beta_{ijt} = B_{ij} \pi_{jt} / (c_{i0t} + B_{il} \pi_{lt} + \ldots + B_{ik} \pi_{kt}) \quad j=1, \ldots, k, \quad i=1, 2, \ldots
\]

\[
\epsilon_{it}^* = \tilde{\epsilon}_{it} / (c_{i0t-1} + B_{il} \pi_{lt-1} + \ldots + B_{ik} \pi_{kt-1}) \quad i=1, 2, \ldots
\]

There are two key distinctions between our asset pricing relationship, (3), and the static APT. First, the beta coefficients have a time subscript. In theory the static model does not consider the time dimension and so makes no statement about the presence or absence of time-variation in betas. In statistical implementations, where the model must be estimated with time series data, most analysts implicitly assume that betas are time-constant. This implicit assumption may not hold, though, when the time-dimension is included explicitly as we have done.

The second distinction from the static model is new and interesting. The static APT implies that the one-period risk-free return is the appropriate "zero-beta" return. In our intertemporal model, even if this asset exists, it does not give the appropriate return. The zero-beta return in our multiperiod model is the return on a long-run "riskless" asset. The asset satisfying (3) has a one-period-ahead cash flow (but not capital gain) that is nonrandom. The capital gain on the security is random, however. For simplicity, consider the case in which \( f_{0t} \) is constant and equal to 1. Then the zero beta portfolio is a pure consol paying one unit of consumption each period.

This feature of our intertemporal APT arises from the multiperiod cash flow assumption imposed on equities. If (as in the static APT) one models corporate equities as one-period assets, paying a single, liquidating dividend and then immediately expiring, the appropriate zero-beta security is a one-period bond with no random capital gain. If equities are modeled as multiperiod sources of cash flow, then the zero-beta asset is a multiperiod
asset. There is a similar definition of the zero-beta security in an earlier paper by Rubinstein (1981).

3. Two Special Cases

The two pricing models presented in this section are special cases of the model in Theorem 1. In each, the process determining equity cash flows and the preferences of the representative investor are restricted. This gives rise to more easily estimated and tested models.

The first model is simple. We assume that the representative investor has constant absolute risk aversion and equity cash flows are independently and identically distributed through time.

A6: The representative investor’s utility function is \( u(C_t) = (1/\gamma)e^{-\gamma C_t} \).

A7: The stochastic process \( f_t \) is independently and identically distributed through time and \( f_{0t} = 1 \) for all \( t \).

Corollary 2: Given A1 - A7 then there exists a competitive equilibrium such that:

\[
F_{it} - F_{0t} = \beta_{i1}(r_{m1t} - F_{0t}) + \ldots + \beta_{ik}(r_{mkt} - F_{0t}) + \epsilon^*_t,
\]

where \( \beta_{ij} \) is constant through time for all \( i \) and \( j \).

Note that \( \beta \) has no time subscript. Also, the risk premia associated with the factor-mimicking portfolios are time varying. That is,

\[
\beta_{ij} \tilde{E}_{t-1}(r_{mjt} - F_{0t}) = \phi_{ij} u'(q'_{xt-1})
\]

where \( \phi_{ij} \) is independent of time, and positive. The result is intuitive. When
current output \( q'x_{t-1} \) is high, marginal utility is low and hence the representative agent requires less compensation for bearing factor \( j \) type risk. This result is consistent with some of the empirical literature on time-variation in risk premia. Note that \( p_{t-1} \) is a function of \( q'x_{t-1} \) through \( \pi_{t-1} \) and \( \pi_t \). High current output leads to high prices of the assets. Thus the model predicts an inverse relation between risk premia and current prices. Keim and Stambaugh (1986) find such an inverse relation in bond and stock returns.

Thus, the model is one with constant betas and time variation in risk premia. An additional implication of the model is that the idiosyncratic variability of asset returns depends on \( [u'(q'x_{t-1})]^2 \). That is, if

\[
E_{t-1}[\varepsilon_t \varepsilon_t' | \bar{f}_t] = V_t
\]

then

\[
E_{t-1}[\varepsilon_t \varepsilon_t' | \bar{f}_t] \alpha_u [u'(q'x_{t-1})]^2 V_t.
\]

Thus, even if the errors in (1) are temporally homoscedastic, \( (V_t - V, \text{for all } t) \) the errors in (3) are conditionally heteroscedastic.

Corollary 2 provides an econometrically estimable version of the APT but it has the unattractive feature that corporate cash flows are intertemporally independent in their levels. An increase in corporate dividends this period has no effect on expected future dividends. A more attractive assumption is intertemporal independence in the rates of change of corporate dividends. The next model leads to i.i.d. rates of change in corporate dividends and conditional homoscedasticity of idiosyncratic returns.

We let the dividends to firms follow an i.i.d. approximate factor model scaled by last period’s per capita dividends. That is:
\[ x_t = (c + B\phi_t + \epsilon_t) (q'x_{t-1}). \] (4)

For convenience we assume that \( q'c = 1 \) so that the average growth rate in per capita consumption is zero. We assume that \( q'B\phi_t \) is bounded above minus 1, so that per capita dividends are never negative. Note that per-capita dividends are intertemporally independent in proportional changes. We also assume that the representative investor has constant relative risk aversion.

A8: Corporate dividends are given by (4).

A9: The utility function of the representative investor is
\[ U(C_t) = (C_t^{1-\gamma} - 1)/(1 - \gamma) \] with \( \gamma \geq 0 \).

A10: \( E_{t-1}[\tilde{\epsilon}_t|\tilde{f}_t] = 0 \).

Corollary 3: Given A1, A3-A5, A8-A10 there exists a unique sequence of competitive equilibrium returns given by
\[ r_{it} - r_{0t} = \beta_{i1}(r_{mkt} - r_{0t}) + \ldots + \beta_{ik}(r_{mkt} - r_{0t}) + \epsilon^x_{it+1} \]
where \( \beta_{ij} \) are constant through time for all \( i \) and \( j \) and \( \epsilon^x_t \) is homoscedastic, conditional on \( \tilde{f}_t \).

As in the previous case we have time constant betas but time variation in risk premia. That is,
\[ \beta_{ij}E_{t-1}(r_{mjt} - r_{0t}) = \psi_{ij}/q'x_{t-1}. \]
Again, risk premia are inversely related to current output and asset prices.

4. A Comparison Of Three Beta Pricing Models

In this section we describe the testable content of the intertemporal APT
developed above, and compare it to the static APT and the ICAPM. We argue that, in principle, these three models are testably distinct but that in practice it may be hard to distinguish them empirically. We speculate that theoretical or econometric considerations may provide better criteria than empirical tests for choosing between them.

Let \( r_{ft} \) denote the return on a one-period risk-free asset. The static equilibrium APT implies that asset returns have the form:

\[
R_{it} = r_{ft} + \beta_{i1} (r_{mt} - r_{ft}) + \epsilon_{it},
\]

(5)

where \( \beta_{i*} = (\beta_{i1}, \ldots, \beta_{ik}) \) and \( r_{mt} = (r_{mlt}, \ldots, r_{mkt})' \). The only difference between (3) and (5) is that \( r_{0t} \) is replaced by \( r_{ft} \). Subtracting \( r_{ft} \) from both sides of (5) to give an equation for asset excess returns, and also expressing the mimicking portfolio returns as excess returns gives:

\[
R_{it} = \beta_{i*} R_{mt} + \epsilon_{it},
\]

(6)

where \( R_{it} = r_{it} - r_{ft} \) and \( R_{mt} = r_{mt} - r_{ft} \). This key equation of the static APT is testable by a variety of procedures, see for example Connor and Korajczyk (1987).

Suppose we can observe the zero-beta security of the intertemporal APT (for example, Theorem 2 posits that a default-free real consol gives the zero-beta return). Then we can define asset excess returns \( R_{mt} = r_{mt} - r_{0t} \) and mimic equation (6) exactly. In this way we can test the model with the same procedures used for the static APT. There are two differences between this implementation of the intertemporal APT and the static APT. First, the zero-beta asset has a different interpretation and, second, the time-constancy of the parameters is derived endogenously.
A problem with this approach is that the zero-beta return of the intertemporal model may be difficult to identify. An alternative is to define excess returns relative to the one-period risk-free return, as in the static model, but treat the difference between the intertemporal zero-beta return and one-period risk-free return as an additional factor. Let $R_{it} = r_{it} - r_{ft}$ and we can rewrite (6) as:

$$R_{it} = (r_{0t} - r_{ft}) + \beta_i^* (r_{mt} - r_{0t}) + \epsilon_{it}$$

(7)

Let $\beta_i^* = (1, \beta_i)$ and $R_{mt}^* = ((r_{0t} - r_{st}), R_{mt})$ and we can write (7) as:

$$R_{it} = \beta_i^* R_{mt}^* + \epsilon_{it}$$

(8)

identical to the estimable equation (6) except that the number of factors has increased from $k$ to $k+1$. In this way, we add a "term structure" factor to the model and produce the same equation as the static APT.

Equation (8) is not completely identical to (6) since the beta matrix has a "unit beta" constraint: one of the columns of the matrix is a vector of ones. The matrix is not singular, but the expanded matrix $(1, \beta^*)$ is singular. We can test for the difference between the static and intertemporal versions of the APT by testing this unit-beta restriction.

If we test (6) or (8) using factor analytic methods as for example in Connor and Korajczyk (1987), then we can not identify the particular column of $B^*$ which has the unit beta constraint since the model is subject to a rotational indeterminacy as follows. Let $L$ denote a $k \times k$ nonsingular matrix. Factor analytic methods cannot distinguish between the model $BR_t^*$ and the model $(BL)(L^{-1}R_t^*)$. Allowing for this indeterminacy, the restriction placed upon $B^*$ is that there exists a $k$-vector $\lambda$ such that
\[ B^* \lambda = \mu. \] (9)

Given an estimate of \( B^* \) this can be tested as a nonlinear restriction on the matrix of estimates. The restriction is non-linear because \( \lambda \) is not known. See Connor and Korajczyk (1987) for an implementation of this test.

If the intertemporal zero-beta return is not observable, the static and intertemporal versions of the APT are only testably distinct to the extent that the constraint (9) is violated. This still leaves a key advantage to the intertemporal version that it is consistent with the use of time series observations, even if it is not testably much different from the static APT.

Merton's ICAPM provides an alternative to either version of the APT. We will describe a simple discrete-time analog of Merton's continuous-time model which captures the essential features of the Merton model. Let \( J_t \) denote the derived utility of the representative investor at time \( t \) and let \( J_{W_t} \) denote his marginal derived utility given a change in time \( t \) wealth:

\[ J_t = E_t \left[ \sum_{s=0}^{\infty} \beta^s u(C_{t+s}) \right] \]

\[ J_{W_t} = \partial J_t / \partial W_t. \]

Assume that we can express \( J_{W_t} \) as a linear combination of \( k \) uncorrelated state variables: \( J_{W_t} = z_{1t} + \ldots + z_{kt} \). From a first-order condition of competitive equilibrium at time \( t-1 \) we have that

\[ u'(C_{t-1}) = E_{t-1}[J_{W_t} \gamma_{it}] \] (10)

for every asset return \( \gamma_{it} \). Let \( \gamma_{mjt} \) \((j = 1, \ldots, k)\) denote a set of portfolio
returns such that $r_{mjt}$ is perfectly negatively correlated with $z_{jt}$. Then it is easy to show\(^3\) that we can write (10) as:

\[
\bar{r}_{it} - \bar{r}_{ft} = \beta_{1lt-1}(\bar{r}_{mlt} - \bar{r}_{ft}) + \ldots + \beta_{ikt-1}(\bar{r}_{mkt} - \bar{r}_{ft}) + \bar{\varepsilon}_{it} \tag{11}
\]

where $\beta_{ijt-1} = \text{cov}_{t-1}(r_{it}, r_{mjt}) / \text{var}_{t-1}(r_{mjt})$ and $E[\varepsilon_{it}] = 0$.

Note the similarity between (11) and the APT equation (3). The basic difference between the models lies in the definition of the portfolio returns on the right-hand side of the equation for asset returns. In the APT, these portfolios mimic the pervasive factors in the covariance matrix of asset returns. The portfolio returns can be identified by factor analytic methods applied to the cross-section of asset returns. In the ICAPM, the portfolio returns mimic the state variables which index the representative investor's wealth function. The econometrician must go outside of the basic model to identify the set of state variables, and then construct mimicking portfolios for these, hopefully observable, state variables.

The ICAPM and APT are distinguishable in principal, but perhaps not in practice. There is certain to be an overlap, if not an identity, between the portfolios found by factor analytic decomposition of asset returns and those found by exogenous choice of state variables.

The three models, static APT, intertemporal APT, and ICAPM, are in principal testably distinct. However, it may prove difficult in practice to convincingly reject two of these in favor of the remaining one.\(^4\) It may be that the set of state variable portfolio returns of the ICAPM will approximately equal the factor mimicking portfolios of the APT -- in which case the APT and ICAPM are empirically indistinguishable. Given the similarity of
the predictions of the three models, it may be preferable to decide their relative merits based on theoretical or econometric criteria rather than on empirical fit.

5. Summary

This paper applies the recursive competitive equilibrium approach of Lucas (1978) and Prescott and Mehra (1980) to the Arbitrage Pricing Theory. This approach requires that the theoretical asset pricing model be consistent with the time-series statistical techniques typically used to estimate asset pricing models. Our model is a simple one that can incorporate some of the complex time-series return patterns uncovered in recent empirical asset pricing studies. Although we do not fully explain these time-series patterns, we believe that our approach is a useful first step toward explaining them within an equilibrium beta pricing model. For example, the models do not produce seasonalities in asset returns. We believe nonetheless that the RCE approach is useful first step toward explaining these phenomena within an equilibrium model.

This paper was motivated by the desire to apply the recursive competitive equilibrium approach to the APT. In fact the results also give new and different predictions which make the model testably distinct from the static APT. In the static APT a one-period riskless asset serves as the zero-beta asset. In our model the zero-beta asset is a long lived asset rather than a one-period riskless asset. This parallels a result in an earlier paper by Rubinstein (1981).

The three models we have discussed, static APT, intertemporal APT, and ICAPM are in principal testably distinct. However, we have argued that it may
be hard in practice to distinguish them based on empirical tests. Theoretical or econometric criteria may provide a better guide.

An econometric weakness of all three of these models (relative, for instance, to the CAPM) is that the ambiguity of the benchmark risk portfolios can lead to "overfitting" of the model to the data. The APT can be subject to overfitting because the pervasive factors are identified from asset return data which is also used to test the model. The ICAPM can be overfitted because the theorist is free to use whatever state variables satisfy his intuition as to "important" parameters of the representative investor's wealth function. The economist's intuition is likely to reflect, at least in part, his knowledge of the asset return data.
Appendix

This appendix gives the proofs of the theorems. The notation and assumptions are from the text.

**Proof of Theorem 1:** We must show that the representative investor will choose to hold the market portfolio given the assumed price sequence. This will be true if and only if prices at time $t$ satisfy the following first-order condition:

$$p_t u'(q'x_t) = E_t\left[ \sum_{s=1}^{\infty} \rho^s u'(q'x_{t+s})x_{t+s} \right].$$

Dividing by $u'(q'x_t)$ and using the definition of $x_{t+s}$ from (l) gives:

$$p_t = c(E_t[ \sum_{s=1}^{\infty} \rho^s u'(q'x_{t+s})f_{0t+s}] / u'(q'x_t)) +$$

$$+ B(E_t[ \sum_{s=1}^{\infty} \rho^s u'(q'x_{t+s})\xi_{t+s}] / u'(q'x_t)) +$$

$$+ E_t[ \sum_{s=1}^{\infty} \rho^s u'(q'x_{t+s})\varepsilon_{t+s}] / u'(q'x_t). \quad (A.1)$$

Note that $E_t[\xi_{t+1} | f_{t+1}] = 0$ implies $E_t[\xi_{t+s} | f_{t+s}] = 0$ which, along with A4, implies that the third additive term on the right-hand side of (A.1) is identically zero. Using the definitions of $\pi_{0t}$ and $\pi_t$ to restate (A.1) gives the result.

Q.E.D.
Proof of Corollary 1: By definition of return we have \( r_{it} = (p_{it} + x_{it})/p_{it-1} \). Using the formula for competitive equilibrium prices from Theorem 1 and the definition of \( x_{it} \) from (1) gives:

\[
r_{it} = \left[ c_i(\pi_{0t} + f_{0t}) + B_{i*}(\pi_t + f_t) + \epsilon_{it}\right]/\left(c_i\pi_{0t-1} + B_{i*}\pi_{t-1}\right).
\]

Let \( d_{it} = c_i\pi_{0t} + B_{i*}\pi_t \) for convenience of notation. Separating the additive terms in the expression above.

\[
r_{it} = \left(c_i/d_{it-1}\right)(\pi_{0t} + f_{0t}) + \left(B_{i*}/d_{it-1}\right)(\pi_t + f_t) + \ldots + \left(B_{ik}/d_{it-1}\right)(\pi_{kt} + f_{kt}) + \epsilon_{it}.
\]

(Multiplying and dividing the \( k + 1 \) terms of (A.2) by \( \pi_{0t-1}, \pi_{1t-1}, \ldots, \pi_{kt-1} \) respectively gives:

\[
r_{it} = \frac{c_i\pi_{0t-1}}{d_{it-1}} \left[ \frac{\pi_{0t} + f_{0t}}{\pi_{0t-1}} \right] + \frac{B_{i*}\pi_{1t-1}}{d_{it-1}} \left[ \frac{\pi_{1t} + f_{1t}}{\pi_{1t-1}} \right] + \ldots + \frac{B_{ik}\pi_{kt-1}}{d_{it-1}} \left[ \frac{\pi_{kt} + f_{kt}}{\pi_{kt-1}} \right] + \epsilon_{it}.
\]

Note that all these steps apply to portfolios as well as to assets. Using the definitions of \( r_{0t}, r_{mlt}, \ldots, r_{mk} \) and finding their returns from (A.3) gives:

\[
n_{0t} = \frac{\pi_{0t} + f_{0t}}{\pi_{0t-1}}, \quad r_{mlt} = \frac{\pi_{lt} + f_{lt}}{\pi_{lt-1}}, \quad \ldots, \quad r_{mk} = \frac{\pi_{kt} + f_{kt}}{\pi_{kt-1}}.
\]

Inserting these portfolio returns into the right-hand side of (A.3) and
applying the definition of $\beta_{ijt}$:

$$r_{it} = \left(\frac{c_i \pi_{0t \cdot 1}}{d_{it \cdot 1}}\right) r_{0t} + \beta_{ilt \cdot 1} r_{mlt} + \ldots + \beta_{ikt \cdot 1} r_{mkt} + \epsilon_{it}^* \quad \text{(A.4)}$$

Note that from the definition of $d_{it}$ and $\beta_{ijt}$ we have

$$c_i \pi_{0t} / d_{it} = 1 - \beta_{ilt} - \ldots - \beta_{ikt}.$$ Inserting this in (A.4) produces:

$$r_{it} - r_{0t} = \beta_{ilt \cdot 1} (r_{mlt} - r_{0t}) + \ldots + \beta_{ikt \cdot 1} (r_{mkt} - r_{0t}) + \epsilon_{it}^*.$$ Q.E.D.

**Proof of Corollary 2:** Applying Theorem 1 and Corollary 1 we have

$$r_{it} - r_{0t} = \beta_{ilt \cdot 1} (r_{mlt} - r_{0t}) + \ldots + \beta_{ikt \cdot 1} (r_{mkt} - r_{0t}) + \epsilon_{it}^*$$

where all terms are defined in Corollary 1. Note that if $\pi_{jt} / (c_i \pi_{0t} + B_i \cdot \pi_t)$ is constant through time then so is $\beta_{ij}$. From the definition of $\pi_{0t}$; $\pi_{jt}$ in Theorem 1, and A6:

$$\pi_{0t} = \mathbb{E}_t \left[ \sum_{s=1}^{\infty} \rho^s e^{-\gamma q' \tilde{x}_{t+s}} \right] / e$$

$$\pi_t = \mathbb{E}_t \left[ \sum_{s=1}^{\infty} \rho^s e^{-\gamma q' \tilde{x}_{t+s}} \tilde{z}_{t+s} \right] / e$$

Applying the assumption that $f_{0t}$, $f_t$ have time-independent distributions:

$$\pi_{0t} = \rho e^{-\gamma q' \tilde{x}_t} \mathbb{E}_t [e^{-\gamma q' \tilde{x}_{t+1}}] / (1 - \rho); \pi_t = \rho e^{-\gamma q' \tilde{x}_t} \mathbb{E}_t [e^{-\gamma q' \tilde{x}_{t+1}} \tilde{z}_{t+1}] / (1 - \rho)$$

Note that the ratio $\pi_{jt} / (c_i \pi_{0t} + B_i \cdot \pi_t)$ is time-invariant for all $i, j$.

Q.E.D.
Proof of Corollary 3: Using A9, we can write the dividend process as:

\[ x_t = c f_0^* + B f_t^* + \varepsilon_t^* \quad \text{where} \]

\[ f_0^* = (q' x_{t-1}), \quad f_t^* = f_t(q' x_{t-1}) \quad \text{and} \quad \varepsilon_t^* = \varepsilon_t(q' x_{t-1}) \quad \text{Note that using (A.5) as} \]

a version of (1) the dividend process \( x_t \) satisfies A1 - A5. We can therefore infer from Theorem 1 that there exists a unique sequence of competitive equilibrium prices given by \( p_t = c \pi_0 + B \pi_t \) where

\[ \pi_0 = E_t [ \sum_{s=1}^{\infty} \rho^s u'(q' x_{t+s}) f_0^* ] / u'(q' x_t) \]

and

\[ \pi_t = E_t [ \sum_{s=1}^{\infty} \rho^s u'(q' x_{t+s}) f_t^* ] / u'(q' x_t) \]

Using the assumed form of the utility function and the definitions of \( f_0^* \), \( f_t^* \) these become:

\[ \pi_0 = E_t [ \sum_{s=1}^{\infty} \rho^s (q' x_{t+s})^{-\gamma} (q' x_{t+s-1})^{-\gamma} ] / (q' x_t)^{-\gamma} \]

\[ \pi_t = E_t [ \sum_{s=1}^{\infty} \rho^s (q' x_{t+s})^{-\gamma} x_{t+s} (q' x_{t+s-1})^{-\gamma} ] / (q' x_t)^{-\gamma} \]

for any \( s \geq 1 \) we can write \( q' x_{t+s} = q' x_t \sum_{z=1}^{s} (1 + q' B f_{t+z}) \). Applying this to the expressions for \( \pi_0 \) and \( \pi_t \) above:

\[ \pi_0 = E_t [ \sum_{s=1}^{\infty} \rho^s (q' x_t)^{-\gamma} (1 + q' B f_{t+z})^{-\gamma} ] / (q' x_t)^{-\gamma} \]

\[ \pi_t = E_t [ \sum_{s=1}^{\infty} \rho^s (q' x_t)^{-\gamma} (1 + q' B f_{t+z})^{-\gamma} ] / (q' x_t)^{-\gamma} \]
Simplifying these expressions by collecting \((q'x_t)\) terms and using the fact that \(f_t\) is independently and identically distributed through time:

\[
\pi_{0t} = (q'x_t) \sum_{s=1}^{\infty} \rho^s \left( E[(1 + q'B\tilde{F})^{1-\gamma}] \right)^{s-1} E[(1 + q'B\tilde{F})^{-\gamma}]
\]

\[
\pi_t = (q'x_t) \sum_{s=1}^{\infty} \rho^s \left( E[(1 + q'B\tilde{F})^{1-\gamma}] \right)^{s-1} E[(1 + q'B\tilde{F})^{-\gamma}].
\]

Applying the rules for geometric series to these expressions

\[
\pi_{0t} = \frac{\rho q'x_t E[(1 + q'B\tilde{F})^{-\gamma}]}{1 - \rho E[(1 + q'B\tilde{F})^{1-\gamma}]}
\]

\[
\pi_t = \frac{\rho q'x_t E[(1 + q'B\tilde{F})^{-\gamma} \tilde{F}]}{1 - \rho E[(1 + q'B\tilde{F})^{1-\gamma}]}
\]

Note that \(\pi_{0t}\) and \(\pi_t\) have the same form as in Corollary 2 -- \((q'x_t)\) multiplied by a time-invariant expression. By the same steps as in Corollary 2 it follows that we can write returns as

\[
r_{it} - r_{0t} = \beta_{ij} (r_{m1} - r_{0t}) + \ldots + \beta_{ik} (r_{mk} - r_{0t}) + \epsilon_{it}^{**}
\]

where the \(\beta_{ij}\) terms are time invariant. Note also that

\[
\epsilon_{it} = \epsilon_{it}^{**} \left( C_i \pi_{0t-1} + B_i \pi_{t-1} \right) - (q'x_{t-1}) \epsilon_{it} / (q'x_{t-1}) \delta_i = \epsilon_{it} / \delta_i
\]

where \(\delta_i\) is a time-invariant scalar. Hence \(\epsilon_{it}\) is conditionally homoscedastic by A10.

Q.E.D.
References


Ferson, Wayne E.; Shmuel Kandel; and Robert F. Stambaugh. "Tests of Asset
Pricing with Time-Varying Expected Risk Premiums and Market Betas."

*Journal of Finance* 42 (June 1987), 201-19.


2. Actually the corollary holds for general utility functions. This can be seen by inserting \( u'(c_c) \) for \( \exp(-\gamma c_c) \) in the proof. This is shown in Bossaerts and Green (1987) and was also suggested to us by Ravi Jagannathan.

3. Substituting \( J_{it} = z_{lt} + \ldots + z_{kt} \) into (10) and using \( E[ab] = E[a]E[b] + \text{cov}(a,b) \) gives

\[
\begin{align*}
    u'(C_{t-1}) &= E_{t-1}[r_{it}]E_{t-1}[J_{it}] + \text{cov}(r_{it}, z_{lt} + \ldots + z_{kt}).
\end{align*}
\]

Rearranging terms and using \( \text{cov}_{t-1}(r_{it}, z_{jt}) = -\text{cov}_{t-1}(r_{it}, r_{jt}) \cdot \left[ \text{var}_{t-1}(z_{jt})/\text{var}_{t-1}(z_{jt}) \right]^{1/2} \) gives:

\[
    E_{t-1}[r_{it}] = \gamma_{0t-1} + \beta_1 r_{it-1} + \ldots + \beta_k r_{kt-1},
\]

where

\[
    \gamma_{0t-1} = -u'(C_{t-1})/E_{t-1}[J_{it}]
\]

and

\[
    \gamma_{jt-1} = \left[ \text{var}_{t-1}(z_{jt})/\text{var}_{t-1}(z_{jt}) \right]^{1/2}/E_{t-1}[J_{it}], \quad j = 1, \ldots, k.
\]

By substitution we get \( \gamma_{0t-1} = r_{ft} \) and \( \gamma_{jt-1} = E_{t-1}[r_{jt} - r_{ft}] \). Using these expressions and taking expectations of (11) gives the needed result that \( E_{t-1}[e_{it}] = 0 \).