Research Program in Finance
WORKING PAPER SERIES

WORKING PAPER NO. 177
EX-ANTE CHARACTERIZATION OF AN EFFICIENT PORTFOLIO
BY
RICHARD C. GRINOLD

Research Program in Finance Working Papers are preliminary in nature; their purpose is to stimulate discussion and comment. Therefore, they should not be cited or quoted in any publication without the permission of the author. Single copies of a paper may be requested from the Institute of Business and Economic Research.

Institute of Business and Economic Research
University of California
Berkeley
RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

The Research Program in Finance in the Walter A. Haas School of Business at the University of California has as its purpose the conduct and encouragement of research in finance, investments, banking, securities markets, and financial institutions. The present reprint and working paper series were established in 1971 in conjunction with a grant from the Dean Witter Foundation.

INSTITUTE OF BUSINESS AND ECONOMIC RESEARCH
Carl Shapiro, Director

The Institute of Business and Economic Research is an organized research unit at the University of California, Berkeley, whose mission is to promote research by faculty and graduate students in the fields of business and economics. The Institute carries out its mission by organizing programs and activities that enrich the research environment, administering extramural research awards, publishing working papers, and making direct grants for research.
Ex-Ante Characterization of an Efficient Portfolio

by

Richard C. Grinold*

Graduate School of Business Administration
University of California, Berkeley

July 6, 1987

Revision #1: August 3, 1987
#2: August 19, 1987
#3: September 14, 1987

* Preliminary: please do not quote

Finance Working Paper 177
Abstract

This paper develops a risk return relationship for financial assets using a no-arbitrage assumption. The no-arbitrage assumption implies the existence of a continuous linear functional that correctly values all assets. The model allows for an arbitrarily large number of assets, and thus can be adapted to include dynamic strategies and assets that can be replicated (options). The valuation function can be represented as a portfolio, and it is the beta of any asset's return with respect to this valuation portfolio that determines the expected return on the asset. The model is general in that it can handle dynamic strategies as well as buy and hold. The paper then offers ways in which we can obtain an ex-ante characterization of the valuation portfolio; it turns out that the valuation portfolio is inversely related to the growth optimal portfolio that would be obtained by maximizing the expected logarithm of final wealth.

* The author would like to thank the Berkeley Program in Finance for its support, and Hayne Leland and Mark Rubinstein for helpful comments.
I. Introduction

In [10], Roll demonstrated that the expected returns on risky assets can be related in a linear fashion to the beta of those assets with respect to any ex-ante efficient portfolio. Efficiency in Roll's case, and in this paper, is defined as minimum variance for an initial investment of one and a specified level of expected return. In particular, Roll's result implies that the capital asset pricing model is simply a statement about the ex-ante efficiency of the market portfolio. The upshot of this is that any model of asset pricing will be interesting only to the extent that it can identify an ex-ante efficient portfolio. The arbitrage pricing theory pioneered by Ross [11] can be seen as an alternative method of identifying ex-ante efficient portfolios without leaning heavily on the notion of the market portfolio. Several papers, including those of Connor [3] and Chamberlain [1] have demonstrated conditions under which one could identify a group of K factors that would linearly explain the expected returns on the risky assets. If K assets will do the job, then as Ingersoll [7] points out, these factors can be rotated so that just one is necessary. According to Roll's original result, this single remaining factor must be represented as the return on an ex-ante efficient portfolio.

This paper identifies an ex-ante efficient portfolio that is related to the implicit Arrow-Debreu prices that are used to value risky assets. We define a limit portfolio as a set of outcomes that can be approximated arbitrarily closely by actual portfolios; the limit
portfolio notion is only needed in cases where the number of assets is infinite. In fact, we identify a set of such Arrow-Debreu prices, demonstrate that there is a limit portfolio that will produce returns that are proportional to those prices, and that the limit portfolio is efficient. The limit portfolio can be characterized in additional ways. First, it is the portfolio that minimizes the expected second moment of total return. Second, with additional assumptions, it can be characterized as the inverse of the growth optimal strategy that an investor with logarithmic utility would choose. If the growth optimal strategy's total return in a state of the world is 1.20, then the limit portfolio's return will be 1/1.20.

This paper is related to and motivated by the work of Chamberlain and Rothschild [2] (C&R), and Harrison and Kreps [8] (H&K). C&R present a model of an asset market in a Hilbert space setting and show that a relatively simple no arbitrage condition implies the existence of two continuous linear functionals of asset returns: one that gives the expected returns, and a second that will give the initial value (cost of purchasing) that pattern of returns; we call these the expectation and valuation functionals. This paper uses the C&R framework to develop an asset pricing model in which the expectation and valuation functionals play a key role. The expectations depend on the probability measure defined on possible security market outcomes. The valuation functional defines an equivalent measure that prices the random returns. Thus we can see that the valuation functional is akin to the Radon-Nikodym of the measures. We can then turn to the work of H&K who have characterized these Radon-Nikodym derivatives in certain special cases.
Section II of the paper describes the model in detail. It is related to the C&R model in several respects. Since it is directed toward a different end, we use a less restrictive set of assumptions. Section II establishes the existence of the valuation portfolio, demonstrates its efficiency, and derives a linear relation between the expected return on any portfolio and the beta of that portfolio with respect to the valuation portfolio. In section III, we characterize the valuation portfolio in two ways: first as the portfolio with minimum second moment of total return, and second, in complete markets, as the inverse of the growth optimal portfolio. In section IV, we link the C&R framework with that of H&K in order to build a complete market in a diffusion model, and to obtain a specific characterization of the valuation portfolio.

Our notation is a cross between that of C&R and H&K; we have tried to keep the best of both. We use the qualifier (a.s.) to mean almost surely or with probability one or that the condition holds except on a set of probability zero.
II. The Model

This section follows C&R in developing the model of asset returns in which emphasis is placed on the space of outcomes rather than the assets themselves. Our treatment differs in several ways: we do not limit ourselves to a countable number of assets, for reasons that will become clear later, we need a larger set; we also do not require that the covariance matrix for any finite subset of assets or portfolios of assets be non-singular, and our no-quasi arbitrage assumption, A4 below, uses the notion of an upper bound on the amount of return that one can expect per unit of standard deviation. In addition, our notation, in an attempt to be consistent with both C&R and H&K, differs somewhat.

Consider a collection of risky assets indexed by $n \in \mathbb{N}$ over a time span from time $0$ to time $T$. Let $\Omega$ denote the space of possible outcomes, $F$ the set of possible events (a sigma algebra), and $P$ a measure on $F$ that defines the probability of those events. The value of asset $n$ at time $T$ given outcome $\omega$ in $\Omega$ is denoted $s(n,\omega)$. We will use $s(n)$ for the random return on asset $n$ or the generic $p$, $q$, and $r$ for random returns on assets or portfolios of assets. The notation for expectation, and covariance is $E[p]$, and $Cov[p,q]$. The variance of a portfolio $p$ will be denoted $Var[p] = Cov[p,p]$.

We assume, $A_1$, that one of the assets, which we will index as asset $0$, is riskless and has a payoff of $s(0,\omega) = r_f > 0$, for all $\omega$ in $\Omega$. Let $1$ be the random variable with $1(\omega) = 1$ for all $\omega$, then $s(0) = r_f 1$. 
$L^2(\Omega,F,P)$ is the space of all square integrable random variables defined on $(\Omega,F,P)$. We will assume, $A2$, that all of the random returns $s(n)$ lie in $L^2(\Omega,F,P)$. $L^2(\Omega,F,P)$ is a Hilbert space with inner product

\[(2.1) \quad (p,q) = \text{Cov}[p,q] + E[p]E[q] = E[pq]\]

and norm

\[(2.2) \quad \|p\|^2 = (p,p) = \text{Var}[p] + E[p]^2\]

Notice that $E$ is, by definition, a continuous linear functional on $L^2(\Omega,F,P)$.

A portfolio is defined as a linear combination of a finite subset of the assets. It follows that the returns on all portfolios will also lie in $L^2(\Omega,F,P)$. Let $K$ represent the finite subset, and $h(k)$ the holding in asset $k$, then the time $T$ value for the portfolio $(h,K)$ will be

\[(2.3) \quad p(\omega) = \sum_{K} h(k)s(k,\omega)\]

if outcome $\omega \in \Omega$ obtains.

Portfolios can be described in either of two ways; first by the asset holdings, $(h,K)$, and second by the random variable $p(\omega)$ that describes the possible time $T$ values of the portfolio. We will use the latter since we allow for the case in which two or more portfolios lead to the same outcome; i.e., it may happen that two portfolio holdings, say $h^1,K^1$ and $h^2,K^2$ will give rise to the same pattern of
payoffs \( p(\omega) \). Let \( G \) be the sub-space of \( L^2(\Omega,F,P) \) that consists of all returns that can be obtained by forming portfolios of the assets; \( p \) is in \( G \) if there are holdings \( (h,K) \) satisfying (2.3). If there are a finite number of assets, then \( G \) is closed, for an infinite number of assets we will have to deal with the closure of \( G \) which will be denoted as \( CG \). An element of \( CG \) is the limit of a sequence of portfolios in \( G \); elements of \( CG \) will be called limit portfolios.

The initial value of any portfolio \( p \) is denoted \( E^*[p] \). C\&R call this the cost establishing the position and use the notation \( C(p) \). We will assume, A3, that the law of one price obtains; i.e., that any two portfolios that have the same outcome would have the same cost. Thus \( E^*[p] \) is an unambiguously defined linear functional on \( G \). We will require that \( E^*[p] \) is continuous. To do this we need the quasi-arbitrage assumption A4. For any portfolio with non-zero variance we define the information ratio, \( IR[p] \), as the ratio of the expected end of period value to the standard deviation of the end of period value.

\[
(2.4) \quad IR(p) = E[p]/\{Var[p]\}^{1/2} \quad \text{if } Var[p] > 0
\]

A4. There is a finite upper bound on the information ratio of any portfolio that has initial value equal to zero.

C\&R, in corollary 1, establish A4 as a consequence of the continuity of \( E \) and \( E^* \); it seems that A4 is a more natural place to start. If we measure risk by standard deviation, then A4 says that the expected return per unit of risk in a hedge or costless portfolio is bounded. In fact, as C\&R show, the least upper bound can be interpreted
as the slope of the mean standard deviation efficient frontier.

Proposition 1: Under A1 - A4, $E^*$ is a continuous linear functional on $G$.

Proof:

Suppose $E^*$ is not continuous, then there will exist a sequence $q(m)$ with norm $q(m)$ equal to one, and $E^*[q(m)]$ becoming unbounded. Without loss of generality, we can assume that $E^*[q(m)]$ is not zero. Let $r(m)$ be $q(m)$ divided by $E^*[q(m)]$; note that $r(m)$ will have unit cost and that the norm of $r(m)$ will go to zero as $m$ goes to infinity. Let $p(m)$ be $s(0) - r(m)$; note that $p(m)$ will have zero cost. Since the norm of $r(m)$ goes to zero, it follows from (2.2) that both $\text{Var}[r(m)]$ and $E[r(m)]$ converge to 0. The riskless asset has zero covariance with all assets, so $\text{Var}[p(m)] \rightarrow 0$, and $E[p(m)] = r_f > 0$. Thus the ratio of the expected value of $p(m)$ to the standard deviation of $p(m)$ can be made arbitrarily large by moving far enough along the sequence. This violates assumption A4.

QED.

The continuity of $E$ and $E^*$ on $CG$ allows us to use the Reisz Representation Theorem (see C&R pg. 1296) in order to express $E$ and $E^*$ in terms of an inner product. By that theorem there will exist unique limit portfolios $e$ and $\rho$ in $CG$ such that

\begin{equation}
E[p] = (e, p) = \int p(\omega)e(\omega)dP(\omega)
\end{equation}

and
(2.5) \[ E^*[p] = (p, p) = \int p(\omega)p(\omega)dP(\omega) \]

The existence of a riskless asset allows us to characterize \( c \) immediately. Consider the riskless asset \( s(0, \omega) = r_f \) for all \( \omega \).

(2.6) \[ E[s(0)] = \text{Cov}[e, s(0)] + E[s(0)]E[e] = r_f. \]

Since \( s(0) \) is riskless, the covariance term is zero, thus \( E[e] \) equals 1. In fact \( e = 1 = s(0)/r_f \) for all \( \omega \), since \( s(0) \) is in \( G \), \( e \) is unique, and this value of \( e \) certainly works,

(2.7) \[ (1, p) = \text{Cov}[1, p] + E[1]E[p] = E[p]. \]

Note also, that this value of \( e \) coincides with the usual definition of expectation.

(2.8) \[ E[p] = (1, p) = \int 1(\omega)p(\omega)dP(\omega) = \int p(\omega)dP(\omega). \]

Let \( q^* = p/E^*[p] \), \( q^* \) will have value equal to 1.

We are now ready to state our first result, and will summarize assumptions to this point.

A1. There is a riskless asset with end of period value \( r_f > 0 \).
A2. All risky assets lie in \( L^2(\Omega, F, P) \).
A3. Portfolios with the same outcome sell for the same price.
A4. There is a finite upper bound on the information ratio of any portfolio that has initial value equal to zero.

Theorem: Let \( p \) be a portfolio with initial value 1. Under assumptions A1 – A4 above, the expected return on \( p \) is
given by

(2.9) \[ E[p] = r_f + b(p) \{E[q^*] - r_f\} \]

where

(2.10) \[ b(p) = \frac{\text{Cov}[p,q^*]}{\text{Var}[q^*]} \]

Proof:

Let \( H \) be the subspace of \( L^2(\Omega,F,P) \) spanned by \( e \) and \( q^* \).

By the projection theorem (see C&R p. 1286) we can write \( p \) as

\( p = p^0 + p^1 \) where \( p^0 \) is in \( H \) and \( p^1 \) is orthogonal to \( H \). Thus

(2.11) \[ E[p] = E[p^0] + (1,p^1) = E[p^0] \]

and

(2.12) \[ E^*[p] = E^*[p^0] + (\rho,p^1) = E^*[p^0] = 1. \]

For value and expectations we can ignore the component \( p^1 \) of \( p \) that is orthogonal to \( H \). \( H \) contains \( p^0 \), so we can write \( p^0 \) as a linear combination of \( s(0) \) and \( q^* \).

(2.13) \[ p^0 = d(p)s(0) + b(p)q^*, \]

since \( s(0) \) and \( q^* \) also span \( H \).

Now \( E^*[p^0] = 1 \), and \( E^*[s(0)] \) and \( E^*[q^*] \) also equal 1, so applying \( E^* \) to (2.13) yields

(2.14) \[ d(p) = 1 - b(p) \]

and
(2.15) \[ p^0 = s(0) + b(p) \{ q^* - s(0) \}. \]

The first result follows by taking expectations in equation (2.13). For the second result note that

(2.16) \[ \text{Cov}[p,q^*] = \text{Cov}[p^0,q^*] + \text{Cov}[p^1,q^*], \]

the second term is zero by virtue of the orthogonality of \( p^1 \) to \( H \). If we substitute the expression (2.13) for \( p^0 \), we obtain

(2.17) \[ \text{Cov}[p,q^*] = b(p) \text{Var}[q^*]. \]

\text{QED.}

Although this result seems deceptively easy, recall it follows from two powerful theorems, the Reisz representation theorem, and the projection theorem, and the insightful framework set up by C&R.

The proof of the theorem also demonstrates that the efficient (minimal risk for a stipulated expected return) portfolios lie in \( H \). In particular, \( q^* \) is efficient.

In the usual mean variance analysis with a riskless asset one can obtain a similar result, see Roll [10], using the portfolio that is tangent to the mean standard deviation efficient frontier or risky assets on the line running through the return on the riskless asset. The CAPM hinges on whether that tangent portfolio is indeed the market portfolio. The APT, in fact, hinges on finding other ways to characterize that tangent portfolio. The remainder of this paper will concentrate on ways of characterizing the portfolio \( q^* \).
III. Characterization of $q^*$

Theorem 1 states that expected returns on assets and portfolios are determined by a linear relationship that involves the beta of the asset with respect to a limit portfolio whose returns $q^*(\omega)$ are proportional to those of the functional that correctly prices the returns on each asset. In this section we show how that valuation portfolio can be characterized in two ways: the first, that $q^*$ is the portfolio with minimal second moment, applies in all cases, the second applies only in cases in which there are sufficient assets to complete the market.

Proposition 2: The limit portfolio $q^*$ has minimal second moment (minimal norm) among all portfolios with initial value equal to one.

Proof:

Let $p \in CG$ have unit value, then we can write $p$ as $q^* + d$ where $E^*[d] = 0$. The norm of $p$ is

$$\|p\|^2 = (p, p) = (q^*, q^*) + 2(q^*, d) + (d, d).$$

(3.1)

Since $d$ has zero value, $(\rho, d)$ and therefore $(q^*, d)$ are equal to 0. Thus the minimum norm occurs when $p = q^*$.

QED.

Consider the empirical model based on historical returns where
both \( \Omega \) and \( N \) are finite. Let \( \Omega \) stand, temporarily, for the number of periods of returns and index them by \( \omega \). Let \( R \) be the \( N+1 \) by \( \Omega \) matrix of returns for the assets, and \( P \) the \( \Omega \) by \( \Omega \) diagonal matrix with \( 1/\Omega \) on the diagonal. In practice, \( N \) would be smaller than \( \Omega \) since we require that

\[ (3.2) \quad R\rho = 1(N+1), \]

where \( 1(N+1) \) is an \( N+1 \) element vector of ones.

In addition, since \( G = CG \) in the finite case, \( \rho \) is a portfolio. There will exist a vector \( h \) such that \( R'h = \rho \), where \( R' \) is the transpose of \( R \). Thus we can solve for \( h \) and \( \rho \).

\[ (3.3) \quad h = (RPR')^{-1}1(N+1) \]

and

\[ (3.4) \quad R'h = \rho. \]

Our second result assumes complete markets and a no free lunch assumption. Complete markets simply means that the closure of the set of possible portfolios, \( CG \), is equal to the space of all possible outcomes, \( L^2(\Omega,F,P) \). The no free lunch assumption from H&K rules out investments \( q \) with \( E[q] < 0 \), and \( P\{\omega \mid q(\omega) > 0\} > 0 \), and \( P\{\omega \mid q(\omega) = 0\} = 1 \), and \( P\{\omega \mid q(\omega) < 0\} > 0 \). The investment \( q \) has zero or negative cost, has a nonnegative outcome almost surely and a positive outcome in some significant set of situations.

When the market is complete we can think of our investment choice as a strategy \( g(\omega) \) in \( L^2(\Omega,F,P) \). In fact, we can show that the
valuation function \( \rho \) is closely related to the growth optimal strategy that one obtains by maximizing the expected logarithm of the time \( T \) investment value \( g(\omega) \). The optimal strategy in this case is called the growth optimal strategy.

The problem of finding the growth optimal strategy is to choose \( g(\omega) \) in order to

\[(3.5) \quad \text{Maximize} \quad \int \ln\{g(\omega)\} \ dP(\omega)\]

subject to \( g(\omega) \) having initial value equal to one,

\[(3.6) \quad E^*[g] = \int g(\omega) \ \rho(\omega) \ dP(\omega) = 1.\]

Our analysis of this problem requires the two additional assumptions that were described above, namely:

A5. The market is complete: i.e., \( C_G = L^2(\Omega, F, P) \).

A6. There are no free lunches.

The motivation for the following result comes from Cox [4].

Proposition 3: Under Al - A6,

i. \( \rho \) is positive (a.s.)

ii. An optimal, \( g^*(\omega) \), solution to (3.5-6) exists

iii. \( \rho(\omega) = 1/g^*(\omega) \) (a.s.)

Proof:

Notice that problem (3.5-6), or any other expected utility maximization problem, will not make sense if the valuation function \( \rho(\omega) \) is
either negative or zero on some set with positive probability. Let A be the set of \( \omega \) such that \( \rho(\omega) \) is negative, and B the complement of A. Since \((e, \rho) > 0, P[B] > 0\). Now suppose that \( P(A) > 0 \), and define \( p \) as \( p(\omega) = 1 \) for \( \omega \) in A, and a positive number \( \delta \) for \( \omega \) in B, where \( \delta \) is selected so that \( E^*[p] = 0 \). Note that \( p \) is a free lunch; this rules out \( \rho \) negative on a set of positive measure. Now consider the case in which \( \rho \) is nonnegative, and let A be the set of \( \omega \) such that \( \rho(\omega) = 0 \). If \( P(A) > 0 \), then set \( p(\omega) = 1 \) for \( \omega \) in A and zero elsewhere; \( p \) is a free lunch.

Therefore the no free lunch assumption, and the complete market assumption imply \( \rho \) is positive (a.s.).

Any portfolio \( p \) with an initial value of one can be written as

\[
(3.7) \quad p(\omega) = q(\omega)/\rho(\omega) = q(\omega) g^*(\omega)
\]

where

\[
(3.8) \quad E[q] = E[p \rho] = E^*[p] = 1.
\]

Since \( q = 1 \) will work here, we see that \( g^*(\omega) \) is feasible. It follows that the objective in (3.5) can be written as

\[
(3.9) \quad \int \ln(q(\omega)) \ dP(\omega) + \int \ln(g^*(\omega)) \ dP(\omega).
\]

By Jensen's inequality

\[
(3.10) \quad \int \ln(q(\omega)) \ dP(\omega) \leq \ln[E[q]] = 0,
\]

so \( g^*(\omega) \) is optimal.

QED.
Notice that we get the same result here that would obtain if we assumed there is a representative investor with logarithmic utility, and that re-balancing strategies are allowed. We do not get the same result if the representative investor is restricted to buy and hold investments or if the market is not complete. The first order conditions for problem (3.5-6) require that for any portfolio \( p \) with initial value of one,

\[
(3.12) \quad E[p/g^*] = r_f E[1/g^*] = \text{Cov}[p,1/g^*] + E[p]E[1/g^*].
\]

If we solve (3.12) for \( E[p] \) and rearrange somewhat using (3.7) we get (2.9).
IV. Multi-period Markets

This section follows Harrison and Kreps [8], H&K, in developing a complete model in which asset prices evolve according to a diffusion process. Thus we want to look at the single period result (2.9) in a multi-period setting; i.e., we want to recognize the implications of the ability to trade within the period on single period results. We refer readers to H&K for precise definition of the diffusion process; we give explicit results for the simple example of geometric Brownian motion that is well within the limits H&K define. Under suitable assumptions, H&K demonstrate that in the case of diffusions there will exist a unique equivalent martingale measure $P^*$, and that the valuation portfolio $\rho$ is simply the Radon-Nikodým derivative of $P^*$ with respect to $P$. H&K have an explicit formula for $\rho$ in this case. As long as we can show that the model presented here and the H&K model are consistent, then H&K’s characterization of $\rho$ must apply in our situation.

We consider a model in which there are a finite number $J+1$ of traded assets and an uncountable number of rebalancing strategies for trading those assets. For $0 \leq t \leq T$ let $F_t$ be an increasing family of sigma-algebras with $F_T = F$, and $F_0$ equal to $\{0, \Omega\}$. We assume that the value of the assets $s(j, \omega)$ follows a diffusion process and that for each $t$, $s(j, t, \omega)$ is measurable with respect to $F_t$. Let $s(t, \omega)$ be the $J+1$ element vector of asset values at time $t$, given that the eventual outcome will be $\omega$.

We define a **rebalancing strategy** as:
i. An initial \(J+1\) element holdings vector \(h(0)\), with
value equal to one:

\[
(4.1) \quad \sum_{j=0}^{J} h(j,0) = 1,
\]

ii. A rebalancing time \(t\), \(0 < t < T\).

iii. A \(J+1\) element vector of functions \(h(j,t,\omega)\), each in \(L^2(\Omega,F_t,P)\),
that satisfies the self-financing condition

\[
(4.2) \quad \sum_{j=0}^{J} h(j,t,\omega)s(j,t,\omega) = \sum_{j=0}^{J} h(j,0)s(j,t,\omega), \text{ a.s.}
\]

Thus we establish an initial position \(h(0)\), hold that until time \(t\). At

time \(t\) we trade, conditional on information received up to that time;
i.e., \(h(j,t,\omega)\) is in \(L^2(\Omega,F_t,P)\), so that the value of the portfolio
does not change. We then hold the position \(h(j,t,\omega)\) until time \(T\). Any
finite combination of (i.e., portfolio of) rebalancing strategies will
correspond to the self-financing simple strategies described in H&K. Thus
Theorem 3 in H&K applies and characterizes \(\rho\).

For example, take the simple geometric diffusion process with
stationary parameters. For \(j = 1,2,\ldots,J\)

\[
(4.2) \quad ds(j)/s(j) = a(j) \, dt + \sum_{j=1}^{J} G(j,k) \, dZ(k)
\]

where \(Z(j)\) is a simple Brownian motion. If we let \(rT = \ln r_T\), then

\[
(4.3) \quad ds(0)/s(0) = r \, dt.
\]

Let \(V = GG'\) be the variance-covariance matrix of the returns on the \(J\)
risky assets. The growth optimal strategy in this case continuously rebalances to have constant proportions \( y(j) \) in each of the risky assets. It is straightforward to show that the \( y(j) \) must satisfy the equations

\[ V y = a - r. \]  \tag{4.4} \\

Thus the payoff to the growth optimal strategy is proportional to

\[ \sum_{j=1}^{J} s(j, \omega) \gamma(j). \]  \tag{4.5} \\

So the valuation portfolio's returns will be proportional to

\[ \sum_{j=1}^{J} s(j, \omega) \gamma(j). \]  \tag{4.6} \\

It is interesting to note that the specification of \( g(\omega) \) and \( q^*(\omega) \) require the knowledge of the mean vector \( a \), and covariance matrix \( V \) of the log returns. If this theory is being used in a normative way to predict future expected returns, then we have to assume we know them in order to predict them. If, on the other hand, the theory is being used to explain the relationship between risk and return, then the theory appears to offer some insight.

One might suppose that any time the means and variances of the assets are known then one could find an efficient portfolio in the traditional way; i.e., by first characterizing the set of mean standard deviation possibilities using buy and hold strategies for the \( J \) risky assets, and then finding the tangent portfolio, i.e., the portfolio with initial value of one and the highest expected excess return (over risk free) per unit of standard deviation. Let \( p \) represent the tangent
portfolio. Portfolio \( p \) will be efficient if we are restricted to buy and hold positions. However, as Leland [9] has pointed out, it will not be efficient if re-balancing strategies are allowed. For example, suppose that the tangent portfolio \( p \) is log-normally distributed with annual expected return of 1.12 and standard deviation of 0.17 and the expected return on the risk free asset is 1.05127. This is equivalent to a mean \( \mu(p) \), of 0.10194 and standard deviation, \( \sigma(p) \), of 0.15092 for the log returns, and a continuously compounded rate of interest, \( r_f = 0.05 \). The expected excess return per unit of standard deviation for the tangent portfolio is 0.40428. In this case one can show, see Cox and Leland [6], that the limit portfolio \( r \) that keeps 50% of its funds in the tangent portfolio and 50% risk free will have a return

\[
(4.7) \quad r(w) = e^{\delta} p(w)^{1/2},
\]

where

\[
(4.8) \quad \delta = r_f/2 + G(p)^2/8 = 0.02785
\]

The expected return and standard deviation of the re-balancing strategy \( r \) are \( E[r] = 1.08509 \), and \( \text{Var}[r]^{1/2} = 0.082 \). The expected excess return per unit of standard deviation for the re-balancing strategy is 0.41245. To see that this dominates the tangent portfolio \( p \), consider a mixture of the re-balancing strategy and risk free that has the same standard deviation as the tangent portfolio. A mix of 2.0732 in the re-balancing strategy and \(-1.0732\) in risk free will do the trick. That mixture will have expected return 1.12139 and a standard deviation 0.17. The standard deviation is the same as the tangent and
the expected return is higher. Thus in any market in which intermediate trading is possible one cannot guarantee that buy and hold positions (for example in the market portfolio), will be efficient in the larger set of opportunities added by considering the possibility of intermediate trading.

As a second example of a multi-period market consider the fixed income model of Cox, Ingersoll, and Ross [5], CIR. In that model the discount bonds of all maturities represent a continuum of assets whose values are driven by a single state variable, the interest rate. We complete this market by adding re-balancing strategies of the sort described above. Note we could also have restricted attention to two assets, a consol and the short run risk free investment, along with the re-balancing strategies. This subsection assumes the reader is familiar with the CIR paper and its notation.

CIR find that the instantaneous rate of return on a discount bond of maturity \( t \) is

\[(4.9) \quad \frac{dP(t)}{P(t)} = r[1-\lambda B(t)]dt - B(t)\sigma r^{1/2}dZ\]

where

- \( r \) is the short run rate of interest
- \( dZ \) is Brownian noise
- \( \sigma \) governs the volatility of \( r \)
- \( \lambda \) is measure of market risk aversion (\( \lambda < 0 \))
- \( B(t) \) measures the interest rate elasticity of the bond price. It can be interpreted as the bond's duration.
In this framework, with a single source of uncertainty, portfolio selection is equivalent to maintaining the duration, $B$, for the portfolio. Notice a linear combination of discount bond returns where the weights sum to one will produce an expression just like (4.9) except the $B$ for the portfolio will be the weighted sum of the $B(t)$ for each asset. It is not difficult to show that the growth optimal portfolio, $g^*$, will always maintain a constant duration $B(g^*)$ where

\[(4.10) \quad B(g^*) = -\lambda/\sigma^2,\]

and that the value, $W^*[r,t]$, of the growth optimal portfolio is governed by the stochastic differential equation

\[(4.11) \quad dW^*/W^* = r[1 - \lambda B(g^*)]dT - B(g^*)\sigma r^{1/2}dZ.\]

From equation (4.11), and the initial condition of $W^*$ equal to 1, we can characterize $W^*$ and thus the valuation metric and ex-ante efficient portfolio in a model of fixed income assets.
V. Summary

This paper has used the framework of Chamberlain and Rothschild that focuses on the space of investment outcomes in a market in which there are no arbitrage opportunities in the following sense: (i) the law of one price obtains, i.e., different investments with the same outcomes sell for the same price, and (ii) there is an upper limit to the amount of expected return per unit of risk (standard deviation) that one can obtain in a hedge (zero net investment) portfolio. Under these conditions we establish the existence of a valuation portfolio such that the value of any investment can be calculated in terms of its expected return and covariance with the valuation portfolio. In addition there is a linear relationship between the expected return on any asset and its beta with respect to the valuation portfolio. This gives us a single factor APT-like explanation of expected returns with a modest set of assumptions.

The remainder of the paper deals with the ex-ante characterization of the valuation portfolio. We find that two properties give us some insight into the nature of the valuation portfolio. First, it is the portfolio that minimizes the second moment of total return among all portfolios with initial value of one. Second, in markets in which we allow re-balancing strategies that will complete the market and a no free lunch assumption, we show that the valuation portfolio is inversely related to the growth optimal policy that maximizes the expected logarithm of ending wealth. This allows us to characterize the valuation portfolio quite closely in sstock market and fixed income market applications presented in Section IV.
The paper makes three contributions. The first is using the C&R framework to model markets in which dynamic, e.g. re-balancing, strategies are included as well as the traditional buy and hold strategies; allowing puts and calls and more exotic instruments is not a problem in this framework. Thus the Roll critique of the capital asset pricing model maintains that the market portfolio will be mis-specified because it will not include the right list of assets with their correct weightings could also add that the market will not include dynamic strategies and is mis-specified on that account as well. The second contribution is the identification of the valuation portfolio as the key in establishing a risk and expected return relationship. The third contribution is in the ex-ante characterization of the valuation portfolio. This should add to the testability as well as the usefulness of the theory.

CAPM type models of the risk return relationship are not able to deal with dynamic strategy or to focus on an isolated subset of assets such as default free bonds. The APT type result presented here is able to deal with both of these difficulties as well as provide an ex-ante characterization of the factor.
References


