THE ARBITRAGE PRICING THEORY: A STATE-PREFERENCE ANALYSIS

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ABSTRACT

In a single-period model with complete markets, linear factor pricing holds if and only if the variance of the market portfolio conditional on the factors is zero. There is no need for factors other than nonlinear functions of the market. For accurate linear pricing of all payoff patterns the factors must be rotationally equivalent to Hakansson's "supershares". In a multiperiod model, a similar set of results holds, but with consumption replacing the market payoff. The methodology of the empirical APT literature is not consistent with either model. If a single-period model is believed then there is no role for multiple dimensions of risk. If an intertemporal model is believed, then the usual analyses of contemporaneous correlations of asset returns may not capture the hedging effects.
Linear factor models have been used extensively to analyze security returns, predominantly by researchers working with the "Arbitrage Pricing Theory" (APT) paradigm. See Huberman [8] for a review of the empirical and theoretical literatures. A coherent theoretical framework is of course crucial to give direction and interpretation to empirical work, as well as to guide applications. This paper presents some new basic theoretical results on factor models and security pricing. In a single-period model with complete markets, even though the Sharpe-Lintner CAPM may not hold, the market portfolio plays such a central role in pricing that the following condition is necessary and sufficient for a vector of factors to give linear pricing: The variance of the market return conditional on the factors must be zero; that is, the factors must determine the market.

Furthermore, there is no need for factors other than (potentially nonlinear) functions of the market return. Nonmarket risk in a factor can be eliminated with virtually no loss of linear pricing accuracy. In order to get linear pricing of all assets in a complete market, the factors must be rotationally equivalent to Hakansson's [7] "supershares".

In an intertemporal model, a similar set of results holds, with consumption replacing the market return as the pivotal variable for pricing all securities.
These results are used to highlight some limitations of the existing theoretical APT literature as an interpretive foundation for APT empirical work. It is argued that the empirical work is consequently somewhat misdirected, so recommendations for redirection are made at the end of the paper.

Section I discusses single-period APT models, and introduces a complete-markets model. Six theorems are derived, which indicate that the potential role of factors is too limited to justify the usual interpretations of empirical APT studies. The theorems are extended to a multi-period setting in Section II, and related to the Connor and Korajczyk [4] intertemporal APT model. These models share much of the logic of the consumption-based asset pricing analyses of Breeden and Litzenberger [2] and Breeden [1], as well as sharing their practical limitations. Section III discusses implications of these results for empirical work. Section IV gives a summary and conclusions.

I. Single-Period Models

There are different versions of the APT in the theoretical literature. The original version, in Ross [11], is inexact in the sense that it does not determine expected returns, but rather determines a bound on deviations of expected returns from a benchmark formula. With stronger assumptions Connor [3] derived an exact version, sometimes called an "Equilibrium-APT" because of the key role of aggregate supply (the market portfolio). Our model is
closer to Connor's, so we will refer mainly to his results.

The one piece of the APT that is truly based on arbitrage is that if a security's return (or payoff) is an exact linear function (no error term) of the factors, then its expected-return premium (or current price) is a linear function of the factor premia (or factor-portfolio present values). This is just "value additivity". But the important and interesting pieces of the APT are based on equilibrium rather than arbitrage analysis. They deal with the issues of deriving conditions under which the "epsilon"s (error terms in the factor structure) are not priced (no return premium or present value); and conditions under which investors' portfolios have uniformly zero epsilons (typically the same conditions). Important questions not addressed by the APT are whether anything can be said about the factor premia, and what factor weightings are appropriate for any given investor's portfolio.

In assessing the unique contributions of the APT, it is helpful to consider the theoretical reasons why the Sharpe-Lintner CAPM may not hold. These include insufficiency of mean and variance for portfolio choice, intertemporal hedging effects, missing assets, market segmentation, taxes, and transaction costs. This section will follow Connor [3] in analyzing a model in which sufficiency of mean and variance is the only missing CAPM assumption. The limited role of factors in this setting is contrasted with the usual APT portrayal in which different factors represent different dimensions of risk. Note that in the APT literature, factors are often orthogonalized, and often identified with distinct macroeconomic variables as sources of risk.
We will make the additional simplifying assumption of market completeness. This can be relaxed to assuming the existence of options on the market portfolio, since that would give the same equilibrium prices and portfolios as with completeness. The readiness of modern security markets to create new instruments (like index options) with low transaction costs lends support to the position that the real-world equilibrium is not very different from a complete-markets equilibrium. In any case, the assumption of completeness, like those of no-transaction-costs and homogeneous beliefs, will help us focus more clearly on the more substantial issues of risk, return, and optimal portfolios.

Consider a one-period pure exchange economy of investors who trade their endowments of securities at time $t=0$ and consume only at time $t=1$. Consumption for any investor $k$ equals her wealth at time 1, which is the sum of the time-1 payoffs on the securities in her optimally chosen equilibrium portfolio. She maximizes her time-0 expectation of utility of time-1 wealth $E_0(U_k(W_k^t))$. Assume $U_k > 0$, $U_k'' < 0$, and $U_k'(0) = \infty$ for all $k$. Security payoffs are determined by the time-1 state of the economy. There are a finite number of possible states, indexed by $s$. All investors agree on the state probabilities $\{\pi_s\}$, which are all positive. The security market is complete — there is a set of $s$ securities with linearly independent payoffs across states.

It is well known that equilibrium prices in this setting are characterized by the existence of "pure security" prices $\{p_s\}$, where $p_s$ is the time-0 price of a security paying $1$ in state $s$ and $0$ in all other states.$^1$
These pure securities may or may not exist as traded securities, but any traded security's time-0 price $y^0$ will satisfy

$$y^0 = \sum_s y_s p_s$$  \hspace{1cm} (1)

where $y_s$ is the traded security's time-1 payoff in state $s$. Let $M_s$ be the dollar value of the market portfolio (aggregate wealth) in state $s$ at time 1. Then states with higher aggregate wealth will have lower pure-security prices per unit of probability (i.e. per unit of expected payoff):

For any two states $s$ and $s^*$ —

$$M_s < M_{s^*} \text{ if and only if } \frac{P_s}{\pi_s} > \frac{P_{s^*}}{\pi_{s^*}}. \hspace{1cm} (2)$$

And every investor will have the same ranking of states in order of their time-1 wealth:

$$M_s < M_{s^*} \text{ implies } w_{s^*k} < w_{s^*k} \text{ for all } k. \hspace{1cm} (3)$$

A consequence of (2) is that payoffs occurring in states with the same aggregate wealth have the same present value per unit of expected payoff, and so can be grouped together in calculation (1) as follows. Letting $m$ vary across all possible realizations of $\bar{M}$:

$$y^0 = \sum_s y_s p_s$$

$$= \sum_m \sum_{s: M_s = m} y_s p_s$$

$$= \sum_m \sum_{s: M_s = m} \left( \frac{P_s}{\pi_s} \right) (y_s \pi_s)$$

$$= \sum_m g(m) \sum_{s: M_s = m} \left( \frac{y_s \pi_s}{\text{Prob}(\bar{M} = m)} \right)$$

$$= \sum_m g(m) E(\tilde{y} | \bar{M} = m) \hspace{1cm} (4)$$
where \( g(m) = \sum_{s:M_s = m} p_s \) for any \( s \) such that \( M_s = m \).

In the case of a security with expected payoffs that are linear conditional on the market's payoffs:

\[
E(\tilde{y} | \tilde{M} = m) = a + bm
\]

then

\[
y^0 = \sum_m g(m) \ (a + bm)
= a \sum_m g(m) + b \sum_m mg(m)
= a \sum_s p_s + b \sum_s M_s p_s
= \frac{a}{1+r} + bM^0
\]

where \( r \) is the riskless rate and \( M^0 \) is the time-0 value of the market portfolio.

To translate the above result into rates of return, denote the return on the complex security by \( \tilde{R}_i = (\tilde{y}/y^0) - 1 \), and the market return by \( \tilde{R}_m = (\tilde{M}/M^0) - 1 \). Then:

\[
E(\tilde{R}_i - r | \tilde{R}_m = R_m) = \alpha_i + \beta_i (R_m - r)
\]

then \( \alpha_i = 0 \)

so that \( \mu_i - r = \beta_i (\mu_m - r) \).

This shows that equilibrium pricing will be the same as in the Sharpe-Lintner CAPM for any security with a linear conditional expectation relationship with the market, even if other securities are not conditionally linear. Dybvig and Ingersoll [5] prove a similar result in a mean-variance framework, but mean-variance analysis is not required for the above. (Also, notice that joint normality of security returns is a special case of equation (7).)
Let \( \tilde{f} \) denote a random vector \((\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_s)\), called the factors. Let \( f_s \) be the realization of the random-variable factor \( \tilde{f}_\lambda \) in state \( s \), and let \( \tilde{f}_s \) (without a tilde \( \tilde{\cdot} \)) be the vector realization of \( \tilde{f} \) in state \( s \), that is \( \tilde{f}_s = (f_{1s}, f_{2s}, \ldots, f_{ns}) \). Let \( f \) be a \( \kappa \)-vector variable, which we will use as an index, running across all possible realizations of \( \tilde{f} \).

Connor's main assumption is a linear factor structure for security payoffs. Let us adopt the simplified terminology "(time-1 payoff) \( \tilde{y} \) is linear in \( \tilde{f} \)" to mean that there exist nonrandom scalar \( a \) and vector \( b \) such that

\[
E(\tilde{y} \mid \tilde{f} = f) = a + b'f \quad \text{for all possible realizations } f \text{ of } \tilde{f}.
\]

(9)

Then Connor's assumption is that all time-1 security payoffs are linear in \( \tilde{f} \). Connor's main result is that time-0 security prices are linear with the same coefficients:

\[
y^0 = a/(1+r) + b'f^0
\]

(10)

where \( f^0 = \Sigma_s p_s f_s \) is the vector of time-0 prices of portfolios with payoffs equalling the factors \( \tilde{f} \) in every state.\(^2\) We will describe this as "\( \tilde{f} \) gives accurate linear pricing". Important among Connor's other assumptions are the conditions that the market portfolio's payoff \( \tilde{M} \) is a linear function of \( \tilde{f} \) (\( \tilde{M} = a_m + b_m' \tilde{f} \)), and that for each factor \( \tilde{f}_\lambda \) there exists a mimicking portfolio with payoffs \( \tilde{y} \) such that \( y_s = f_{s \lambda} \) for all states \( s \).

In the complete-markets context of the present paper, we can weaken the condition of \( \tilde{M} \) being a linear function of \( \tilde{f} \), and find a necessary and sufficient condition for accurate linear pricing by \( \tilde{f} \). The condition is that the market portfolio payoff \( \tilde{M} \) can be any function of the factors; that is, knowing the realization of \( \tilde{f} \) must tell you the realization of \( \tilde{M} \). This
relation between two random variables will recur often in this paper, so let us describe it by the simplified terminology "\( \tilde{T} \) refines \( \tilde{M} \)". This would mean that for any two states \( s \) and \( s^* \), if \( f_s = f_{s^*} \) then \( M_s = M_{s^*} \). It also implies that

\[
\text{Var}(\tilde{M} | \tilde{T} = f) = 0 \quad \text{for all } f. \tag{11}
\]

(Some readers may prefer to mentally substitute this last condition in place of "\( \tilde{T} \) refines \( \tilde{M} \)" in the rest of this paper.) It makes intuitive sense that \( \tilde{T} \) must determine \( \tilde{M} \) in order for conditional pricing to work, because in this economy the present value of expected payoffs is linked to the market payoff by (2).

Theorem 1, below, shows that \( \tilde{T} \) gives accurate nonlinear pricing of any payoff pattern, if and only if \( \tilde{T} \) refines \( \tilde{M} \). Theorem 2 shows that \( \tilde{T} \) gives accurate linear pricing of payoff patterns that are linear in \( \tilde{T} \), if and only if \( \tilde{T} \) refines \( \tilde{M} \), even if not all securities' payoffs are linear in \( \tilde{T} \).

**Theorem 1** (Nonlinear Conditional Pricing)

For any vector of factors \( \tilde{T} \), there is a present value function \( h(f) \) such that for any payoff pattern \( \tilde{y} \), its present value \( y^0 \) satisfies

\[
y^0 = \sum_{\tilde{T}} h(f) E(\tilde{y} | \tilde{T} = f)
\]

if and only if \( \tilde{T} \) refines \( \tilde{M} \).

[See Appendix for all proofs.]

**Theorem 2** (Linear Conditional Pricing)

\( \tilde{T} \) gives accurate linear pricing for all securities with payoffs linear in \( \tilde{T} \) if and only if \( \tilde{T} \) refines \( \tilde{M} \).
We have seen that in a complete-markets single-period model, equilibrium pricing is closely linked to the market portfolio. So what role is played by "factors" other than the market? The next section addresses this question by projecting each factor onto the market-payoff random variable. For securities that were linear conditional on the original factors, the projected factors work just as well; but the projected factors have the potential weakness of mispricing some securities that were not linear with the original factors but are linear with the projected factors.

For any vector of factors \( \tilde{f} \), define a new vector of factors \( \tilde{\phi} \) as follows:

\[
\tilde{\phi}_s = \mathbb{E}(\tilde{f}|\tilde{M} = M_s).
\]

(13)

Thus \( \tilde{\phi} \) is the pointwise projection of \( \tilde{f} \) onto \( \tilde{M} \). Note that this definition implies that \( \tilde{M} \) refines \( \tilde{\phi} \) and that \( \mathbb{E}(\tilde{f}|\tilde{\phi} = \phi) = \phi \).

**Lemma 1**

\( \tilde{\phi} \) has the same vector of present values (and thus the same vector of factor premia) as \( \tilde{f} \):

\[
\tilde{\phi}^0 = \tilde{f}^0.
\]

(14)
Lemma 2

If and only if \( \hat{f} \) refines \( \hat{\phi} \):

Any security that is linear in \( \hat{\phi} \) is also linear in \( \hat{\phi} \), and with the same coefficients:

If

\[
E(\hat{y} | \hat{\phi} = f) = a + b' \hat{f}
\]  (15)

then

\[
E(\hat{y} | \hat{\phi} = \phi) = a + b' \phi.
\]  (16)

Theorem 3

If \( \hat{f} \) refines \( \hat{M} \):

Any security that satisfies linear pricing on a vector of factors \( \hat{f} \) will satisfy linear pricing on \( \hat{\phi} \) (the projection of \( \hat{f} \) onto the market) with the same coefficients and factor premia:

\[
\text{If } E(\hat{y} | \hat{f} = f) = a + b \hat{f} \text{ for all } f \]  (17)

and \( y^0 = a/(1+r) + b \hat{f}^0 \),  (18)

then \( E(\hat{y} | \hat{\phi} = \phi) = a + b' \phi \) for all \( \phi \)  (19)

and \( y^0 = a/(1+r) + b' \phi^0 \).  (20)

To summarize, if \( \hat{f} \) refines \( \hat{M} \) then its projection \( \hat{\phi} \) gives the same conditional-linear fit for time-1 payoffs of all securities linear in \( \hat{f} \); and both \( \hat{f} \) and \( \hat{\phi} \) price these securities accurately. So if all existing securities have expected payoffs linear in \( \hat{f} \), as in Connor [3], then the projected factors \( \hat{\phi} \) work just as well. Therefore for the purposes of pricing, there is no role for factors with dimensions of risk other than that of the market portfolio. The only contribution multiple factors can make is to linearize
conditional payoffs that are nonlinear in the market; this is done by having factors that are simply nonlinear functions of the market payoff.

All the factor-pricing results in the present paper can be applied selectively to those securities with the required conditional payoff characteristics, while allowing that other securities may exist. This raises the question of whether the projected factors $\hat{\phi}$ are useful in pricing securities that are nonlinear conditional on the original factors $\hat{f}$. First we will show that there will always be payoff patterns linear in $\hat{\phi}$ but not in $\hat{f}$, unless $\hat{\phi} = \hat{f}$. Then we will present conditions under which $\hat{\phi}$ gives accurate linear pricing of those patterns.

Lemma 3

\[ \hat{\phi} = \hat{f} \]  

if and only if $\hat{M}$ refines $\hat{f}$.

Theorem 4

In the case where $\hat{f}$ refines $\hat{M}$:

There exist payoff patterns $\tilde{y}$ linear in $\hat{\phi}$ but not linear in $\hat{f}$

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\[ \tilde{f} \neq \hat{\phi}. \]  

Therefore, there can be many securities nonlinear in $\hat{f}$ that become linear in its projection $\hat{\phi}$. However, while "most" of them will be accurately priced by $\hat{\phi}$, some may not be. Applying Theorem 3 to $\hat{\phi}$, $\hat{\phi}$ will accurately price all conditionally linear securities if and only if $\hat{\phi}$ refines $\hat{M}$, a

- 11 -
condition which is not guaranteed by the fact that \( \tilde{f} \) refines \( \tilde{M} \). The following two examples illustrate this. (States are equiprobable in both.)

### Table I
\( \phi \) Refines \( \tilde{M} \)

<table>
<thead>
<tr>
<th>States→</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>s4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{M} ):</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \tilde{f} ):</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>( \phi ):</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

In Table I, \( \phi \) refines \( \tilde{M} \) — there are no two states with different market levels and the same \( \phi \), so knowing \( \phi \) tells you \( M \). So \( \phi \) correctly prices not only securities linear in \( \tilde{f} \), but all others that are linear in \( \phi \), such as:

\( \tilde{y} \): 4 4 4 6

Recognizing that state-prices are the same when market levels are the same, a simpler and more powerful factor has been obtained by projecting on \( \tilde{M} \).

### Table II
\( \phi \) Does Not Refine \( \tilde{M} \)

<table>
<thead>
<tr>
<th>States→</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>s4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{M} ):</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \tilde{f} ):</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>( \phi ):</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

In Table II, \( \phi \) does not refine \( \tilde{M} \), even though \( \tilde{f} \) did. So, as shown in the "only if" proof of Theorem 2, there are payoff patterns linear in \( \phi \) that
do not fit the corresponding linear present-value relation, such as:

\[
\tilde{y}: \quad 4 \quad 3 \quad 5 \quad 4
\]

\[E(\tilde{y} | \phi) - \phi\] but the present value \(y^0 < \phi^0\) because \(\tilde{y}\) is tilted more toward higher \(M\) values than \(\phi\) is.\(^3\) Notice that \(\tilde{y}\) is not conditionally linear in \(\tilde{\phi}\), although nonlinear pricing on \(\tilde{\phi}\) would still be accurate by Theorem 1, since \(\tilde{\phi}\) refines \(\tilde{M}\).

But in the case of Connor's model this problem does not arise. If the market portfolio is a linear function of \(\tilde{\phi}\):

\[
\tilde{M} = a + b'\tilde{\phi}
\]

then it will be the same linear function of \(\tilde{\phi}\): For any state \(s\),

\[
a + b'\phi_s = a + b'(E(\tilde{\phi} | \tilde{M} = M_s))
\]

\[
= E(a + b'\tilde{\phi} | \tilde{M} = M_s)
\]

\[
= E(\tilde{M} | \tilde{M} = M_s)
\]

\[
= M_s.
\]

Thus \(\phi\) refines \(\tilde{M}\) and, by Theorem 2, gives accurate linear pricing for all conditionally linear payoffs. Therefore if we project Connor's factors onto \(\tilde{M}\) so that each factor is a nonlinear function of the market, we lose nothing while gaining the potential of linearizing more payoff patterns, plus the simplicity of having no "diversifiable" (nonmarket) risk in the factors, which rational investors would not want to hold anyway.

It is interesting to ask what kind of factors it would take to make all securities conditionally linear in a complete market. The answer is that the factors must be rotationally equivalent to a set of "quasi-pure" securities paying off in "quasi-states", where a quasi-state is a group of states with
the groups being mutually exclusive and collectively exhausting the state space. Thus in each state, one rotated factor has the value 1 and all the others are zero. Table III shows an example of this.

Table III

Rotated Factors As "Quasi-Pure" Securities

<table>
<thead>
<tr>
<th>States→</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>s4</th>
<th>s5</th>
<th>s6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotated factors:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>rf1→</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>rf2→</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>rf3→</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>&quot;quasi-states&quot;→</td>
<td>—qs1—</td>
<td>—qs2—</td>
<td>qs3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This also implies that the $\kappa$-dimensional random vector $\bar{f}$ can only have $\kappa$ possible distinct realizations — an extremely restrictive condition.4

For simplicity let us assume at this point that there are no redundant factors — no factor can be replicated by a linear combination of the others. Thus each security’s factor loading is unique. Also let factor number 1 be a constant, $f_{is} = 1$ for all $s$; there are still $\kappa$ factors, so now only $\kappa-1$ of them are risky. The constant term "a" in a linear combination of factors becomes the first element of the vector "b".
Theorem 5
For any payoff pattern $\tilde{y}$ there exists a vector $b$ such that
\[ E(\tilde{y} | \tilde{f} = f) = b'f \] (28)
for all possible realizations $f$ of $\tilde{f}$.

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there is a nonsingular $\kappa$-by-$\kappa$ matrix $A$ such that $\tilde{\psi} = Af$ has its realization $\psi_s$ in any state $s$ being a vector of all zeroes and a single 1.

Theorem 5 says that the only way to get conditional expected payoffs of all securities to be linear in $\kappa$ factors is to partition the state space into $\kappa$ subsets, and define each factor to be a "dummy variable" for a subset. Theorem 2 said that the only way to get accurate pricing of conditionally linear payoffs is to have $\tilde{f}$ refine $\tilde{M}$, that is, to ensure that no two levels of $M$ are associated with the same factor-vector realization $f$. If we put these two results together, we see that the only way to get both linearity and accurate pricing is to use as factors Hakansson's [6, 7] "supershares". That is, partition the state space by $M$ levels, so that each subset contains all states with a given $M$ value (these subsets are called "superstates"); define $\kappa$ "quasi-pure" securities ("supershares"), each paying off $1$ in its superstate and $0$ otherwise. Any (nonsingular) rotation of these payoffs will give accurate linear pricing. Factors based on finer partitions of the state space would also give accurate linear pricing, but the refinement would increase complexity with no improvement in linearity or pricing accuracy.
Theorem 6

For every payoff pattern \( \tilde{y} \) there exists a vector \( b \) such that

\[
E(\tilde{y} | \tilde{f} = f) = b'f
\]

(29)

for all \( f \), and

\[
y^0 = b'f^0;
\]

(30)

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\( \tilde{f} \) is rotationally equivalent to Hakansson's "supershares" or to a refinement of supershares.

The results of this section show that in the single-period complete-markets model, the only useful role for factors is to provide different nonlinear functions of the market portfolio, so that securities that are nonlinear conditional on the market may be linear conditional on the factors. This contrasts sharply with most discussions of APT empirical work, which emphasize different sources and dimensions of risk. What pricing effects do we really think factors are going to capture, if not just nonlinearity conditional on the market? Remaining possibilities include intertemporal effects, missing assets, market segmentation, taxes, and transaction costs. Each of these has different implications about how empirical work should be conducted, and about how it should be applied to choosing optimal portfolios. The next section analyzes the case of intertemporal effects caused by stochastic investment opportunities.
II. Intertemporal Models

Models with more than one future consumption date allow the possibility that equilibrium pricing may not be so simply linked to the return on the market portfolio. This may make for a richer role for factors other than functions of $\hat{m}$. Connor and Korajczyk [4] ("CK") develop an "Intertemporal Equilibrium APT" based on assuming a linear factor structure for future dividends. While it is more appealing to make the dividend process exogenous than to make the security price (or return) process exogenous, this has the effect of making aggregate consumption exogenous since it equals aggregate dividends in the CK model. If we are willing to make assumptions directly about consumption and how it correlates with individual securities' dividends, we can apply some known results from the consumption-based asset pricing literature.

Following CK, consider an economy with a representative investor who at time $t$ chooses a consumption and portfolio strategy to maximize $u(C_t) + E_t \left( \sum_{r=1}^{\infty} \rho^r u(C_{t+r}) \right)$, where $u(C_t)$ is utility of time-$t$ consumption, and $\rho$ is an impatience parameter. As in the single-period model, $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $u'(0) = \infty$. For simplicity we again assume a complete market for state claims, trading at time $t$ for all future time-state combinations. Then equilibrium prices are characterized by pure security prices $p_{t+r,s}$, being the time-$t$ price, denominated in the consumption good, for receiving one unit of the good at time $t+r$ in state $s$. The consumer's first-order condition determines that

$$p_{t+r,s} = \pi_{t+r,s} \rho^r \frac{u'(C_{t+r,s})}{u'(C_t)}.$$  \hspace{1cm} (31)
In this multiperiod setting, first consider the subset of securities that happen to pay dividends only at date t+r. Then all the single-period results from Part I above have counterparts here, with \( \hat{C}_{t+r} \) now playing the role of the market portfolio's payoff \( \hat{M} \), which was equal to consumption in the single-period model. The counterparts are listed below, with "i" (for intertemporal) appended to the theorem numbers.

The time-\( t \) value \( v_t \) of a security paying dividend \( \hat{x}_{t+r} \) is

\[
v_t = \sum_x x_{t+r,s} P_{t+r,s} \tag{32}
\]

\[
= \sum_c g(c,t+r) E_t(\hat{x}_{t+r} | \hat{C}_{t+r} = c) \tag{33}
\]

where

\[
g(c,t+r) = \sum_{s:C_{t+r,s} = c} P_{t+r,s}. \tag{34}
\]

(This was shown by Breeden and Litzenberger [2].)

In the case of a security with time-(t+r) dividends linear in consumption, if

\[
E_t(\hat{x}_{t+r} | \hat{C}_{t+r} = c) = a + bc \tag{35}
\]

then

\[
v_t = aB(t,t+r) + b\gamma(t,t+r) \tag{36}
\]

where

\[
B(t,t+r) = \sum_x P_{t+r,s}. \tag{37}
\]

is a time-\( t \) unit bond price, and

\[
\gamma(t,t+r) = \text{time-}t \text{ value of } \hat{C}_{t+r} \tag{38}
\]

\[
= \sum_x C_{t+r,s} P_{t+r,s}. \tag{39}
\]

**Theorem 14** (Nonlinear Conditional Pricing)

For any vector of factors \( \hat{f} \) with realizations at time \( t+r \), there is a present value function \( h(\hat{f}) \) such that for any time-(t+r) dividend payoff pattern \( \hat{x}_{t+r} \),
its present value $v_t$ satisfies

$$v_t = \sum_{\tilde{f}} h(\tilde{f}) E_t(\tilde{x}_{t+t} | \tilde{f} = f)$$  \hspace{1cm} (40)

IF AND ONLY IF $\tilde{f}$ refines $C_{t+t}$.

Theorem 2i

$\tilde{f}$ gives accurate linear pricing for all securities paying dividends only at time $t+t$, with $\tilde{x}_{t+t}$ linear in $\tilde{f}$, if and only if $\tilde{f}$ refines $C_{t+t}$.

Here "$\tilde{x}_{t+t}$ linear in $\tilde{f}$" means there exist scalar $a$ and vector $b$ such that

$$E_t(\tilde{x}_{t+t} | \tilde{f} = f) = a + b'f$$  \hspace{1cm} (41)

for all realizations $f$ of $\tilde{f}$.

And "accurate linear pricing" means

$$v_t = aB(t, t+t) + b'F(t, t+t)$$  \hspace{1cm} (42)

where

$$F(t, t+t) = \sum_s p_{t+t,s} f_s$$  \hspace{1cm} (43)

is the vector of time-$t$ values of portfolios with time-$(t+t)$ payoffs equalling the factors $\tilde{f}$ in every state.

Parallel to the single-period case, CK assume that $C_{t+t}$ is a linear function of the factors, which guarantees that $\tilde{f}$ refines $C_{t+t}$.

Now define the projected factors $\hat{\phi}$ as

$$\hat{\phi}_s = E_t(\tilde{f} | C_{t+t} = C_{t+t,s})$$  \hspace{1cm} (44)

So $C_{t+t}$ refines $\hat{\phi}$, and $E_t(\tilde{f} | \hat{\phi} = \phi) = \phi$. 

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Lemma 1i

\( \phi \) has the same vector of time-\( t \) values as \( \tilde{\phi} \):

\[
\Phi(t, t+r) = F(t, t+r)
\]  \hspace{1cm} (45)

where

\[
\Phi(t, t+r) = \sum_s p_{t+r,s} \phi_s.
\]  \hspace{1cm} (46)

Lemma 2i

If and only if \( \tilde{f} \) refines \( \phi \):

Any time-\( (t+r) \) dividend payoff pattern linear in \( \tilde{f} \) is linear in \( \phi \), and

with the same coefficients.

To summarize the remaining intertemporal counterpart theorems: If \( \tilde{f} \) refines \( \tilde{C}_{t+r} \) then the projected factors \( \tilde{\phi} \) work just as well as \( \tilde{f} \) for linear

pricing of payoffs linear in \( \tilde{f} \); more payoff patterns will be linear in \( \tilde{\phi} \) than

in \( \tilde{f} \), except in the degenerate case where \( \tilde{\phi} = \tilde{f} \); but some of these may not be

accurately priced by \( \tilde{\phi} \); but that problem does not arise if \( \tilde{C}_{t+r} \) is a linear

function of \( \tilde{f} \), as in CK. The only way for all payoff patterns to be linear

in \( \tilde{f} \) is for \( \tilde{f} \) to be rotationally equivalent to "quasi-pure" securities on any

some partition of the state space; and the only way to have this linearity

plus accurate pricing is for \( \tilde{f} \) to be rotationally equivalent to "supershares"

on consumption, or a refinement thereof.

Turning now to securities that may pay dividends in all future periods,

we add up the present values of dividends from each future date. In the

special case of a security with dividends linear in consumption with the same

coefficients at each future date:

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If \( E_t(\tilde{x}_{t+r} | \tilde{C}_{t+r} = c) = a + bc \) for all \( r, c \) \hspace{1cm} (47) 

then \( v_t = aB(t,*) + b\gamma(t,*) \) \hspace{1cm} (48)

where \( B(t,*) = \sum_{r=1}^\infty B(t,t+r) \) = time-\( t \) price of a consol \hspace{1cm} (49)

and \( \gamma(t,*) = \sum_{r=1}^\infty \gamma(t,t+r) \) = time-\( t \) value of all future consumption. \hspace{1cm} (50)

In the case of securities with dividends linear in a time series of factors \( \tilde{f}_t \) where \( \tilde{f}_t \) refines \( \tilde{C}_t \) for each \( t \):

If \( E_t(\tilde{x}_{t+r} | \tilde{f}_{t+r} = f) = a + b'f \) for all \( r, f \) \hspace{1cm} (51) 

then \( v_t = aB(t,*) + b'F(t,*) \) \hspace{1cm} (52)

where \( F(t,*) = \sum_{r=1}^\infty F(t,t+r) \) = time-\( t \) factor values. \hspace{1cm} (53)

In the case (considered by CK) of a "zeroth" factor \( f_0 \) that is stochastic, affects dividends at time \( t+r \), but becomes known one period earlier at time \( t+r-1 \), it can be treated just like any other factor in the above valuation equations.

III. Implications for Empirical Work

The appropriate strategy for estimating and testing the APT (versus the CAPM) depends on which CAPM assumption is being violated. The violation considered in Connor [3] is that mean and variance are not sufficient for portfolio choice, or alternatively that security payoffs are not linear conditional on the market. The analysis in Section I above implies that in such a case, the only kind of factors needed is nonlinear functions of the market portfolio. Existing APT empirical work is nothing like that. It uses
multidimensional linear strategies like factor analysis or principal components analysis.

Suppose instead that the CAPM fails to hold because of a stochastic investment opportunity set (intertemporal effects). Merton's ICAPM shows that securities with returns correlated with IOS shifts will tend to have hedging premia. But standard methods focus on contemporaneous correlations, not serial correlations. There is no assurance that the serial correlations will be adequately captured in the factors rather than left in the epsilons, although one can try to make the epsilons as small as possible. This problem applies whether the theoretical model is stated in terms of Consumption-CAPM or ICAPM since the two are equivalent. So besides ensuring zero correlation of epsilons with contemporaneous factors, we should try for zero correlation of epsilons with subsequent returns.

If the problem is that we lack data on some existing capital assets, then (as in the case of intertemporal effects) the arbitrage part of the APT still works: exact replication implies exact relative pricing. But epsilons may be priced if they are correlated with returns on missing assets. Once again, finding factors that keep the epsilons small may reduce the model's inaccuracies. But notice that standard empirical methodologies concentrate on correlations among assets with data, and can not try to ensure no correlation of epsilons with the missing data.

If incomplete integration of international capital markets is causing the CAPM not to hold, then, for example, the correlation of a U.S. stock with
the Nikkei index may affect its pricing; but factor analysis on U.S. equities alone may leave much of that correlation in the epsilons.

If there are tax effects then of course the usual factor approaches are inappropriate. Differential taxation for different asset classes should be measured and allowed for directly.

Hand in hand with the above problems of pricing the epsilons is the question of whether an optimal portfolio may contain epsilon risk. Furthermore, addressing the issue of why the CAPM is not holding will lead to facing the problems of determining the equilibrium factor premia, and optimal portfolio factor loadings for specific types of investors.

IV. Summary and Conclusions

The possible roles for factors in a single-period perfect-markets model are not rich enough to provide a theoretical framework for the mainstream empirical APT research. The market portfolio is so central to such a model that the following results are shown to hold under complete markets: Linear factor pricing is exact if and only if the factors "refine" the market (i.e. knowing the factor outcomes tells you the market outcome; the variance of the market conditional on the factors is zero). If the factors refine the market then they can be projected onto the market with no loss of asset-pricing power. Thus the only "factors" needed are nonlinear functions of the market. Finally, to give accurate linear pricing of all payoff patterns, factors must
be rotationally equivalent to Hakansson's "supershares". A similar set of results holds in an intertemporal model, but with consumption replacing the market payoff.

Under each of the many possible reasons why the Sharpe-Lintner CAPM may not hold, the current empirical APT focus on contemporaneous correlations of asset returns is too narrow. For example, if Merton's ICAPM holds instead, then in principle we should be looking for serial correlations in returns.
**APPENDIX**

**Proof of Theorem 1**

"if":

Suppose \( \tilde{f} \) refines \( \tilde{M} \). Then any two states with \( f_s = f_{s*} \) will have \( M_s = M_{s*} \) and \( p_s/\pi_s = p_{s*}/\pi_{s*} \). Let

\[
h(f) = \sum_{s: \tilde{f}_s = \tilde{f}} p_s = (p_s/\pi_s) \text{Prob}(\tilde{f} = f)
\]

(A1)

for any \( s \) such that \( f_s = f \). Then any payoff pattern will satisfy

\[
y^0 = \sum_{\tilde{f}} h(f) E(\tilde{y} | \tilde{f} = f)
\]

(A2)

by the same reasoning as for equation (4).

"only if":

Suppose there is such a function \( h \). Consider any two states \( s \) and \( s* \) such that \( f_s = f_{s*} \). Applying the present value function \( h \) to the pure securities for states \( s \) and \( s* \) gives

\[
p_s = h(f_s) \pi_s / \text{Prob}(\tilde{f} = f)
\]

(A3)

and

\[
p_{s*} = h(f_{s*}) \pi_{s*} / \text{Prob}(\tilde{f} = f)
\]

(A4)

implying \( p_s/\pi_s = p_{s*}/\pi_{s*} \). Therefore by (2), \( M_s = M_{s*} \). So \( \tilde{f} \) refines \( \tilde{M} \).

Q.E.D.

**Proof of Theorem 2**

"if":

By Theorem 1 any \( \tilde{y} \) has present value

\[
y^0 = \sum_{\tilde{f}} h(f) E(\tilde{y} | \tilde{f} = f).
\]

(A5)

So any \( \tilde{y} \) that is linear conditional on \( \tilde{f} \) will have
\[ y^0 = \sum_f h(f)(a+b'f) \]
\[ = a\sum_f h(f) + b' \sum_f f h(f) \]  \hspace{1cm} (A6)
\[ = a\sum_s p_s + b' \sum_s f_s p_s \]  \hspace{1cm} (A7)
\[ = a/(1+r) + b'f^0. \]  \hspace{1cm} (A8)

"only if":

Supposing \( \hat{f} \) gives accurate linear pricing of linear payoffs, but \( \hat{f} \) does not refine \( \hat{M} \); then we will derive a contradiction. Nonrefinement implies that there are two states \( s_1 \) and \( s_2 \) such that \( f_{s_1} = f_{s_2} \) and \( M_{s_1} < M_{s_2} \). So by (2) we have
\[ p_{s_1}/\pi_{s_1} > p_{s_2}/\pi_{s_2}. \]  \hspace{1cm} (A9)

Choose any one of the factors, \( \hat{f}_\lambda \). Consider the payoff pattern \( \hat{y} \) that has realizations identical to those of \( \hat{f}_\lambda \) except in states \( s_1 \) and \( s_2 \), in which
\[ y_{s_1} = f_{s_1} + \pi_{s_1} \]  \hspace{1cm} (A10)
\[ y_{s_2} = f_{s_2} - \pi_{s_1}. \]  \hspace{1cm} (A11)

Then
\[ E(\hat{y} | \text{state } s_1 \text{ or } s_2) = \frac{\pi_{s_1}(f_{s_1} + \pi_{s_2}) + \pi_{s_2}(f_{s_2} - \pi_{s_1})}{\pi_{s_1} + \pi_{s_2}} = f_{s_1} = f_{s_2}. \]  \hspace{1cm} (A12)

so \( y \) is linear conditional on \( \hat{f} \) —
\[ E(\hat{y} | \hat{f} = f) = f_\lambda \]  \hspace{1cm} (A13)

since \( \hat{f} \) does not distinguish state \( s_1 \) from state \( s_2 \). So \( \hat{f} \) should give the present value of \( \hat{y} \) as \( y^0 = f^0_\lambda \). But
\[ y^0 = \sum_s y_s p_s \]  \hspace{1cm} (A14)
\[ = (\sum_s f_{s_1} p_s) + \pi_{s_1} p_{s_1} - \pi_{s_1} p_{s_2} \]  \hspace{1cm} (A15)
\[ > \sum_s f_{s_1} p_s \]  \hspace{1cm} (A16)
\[ \text{[because } p_{s_1}/\pi_{s_1} > p_{s_2}/\pi_{s_2}] \]  \hspace{1cm} (A17)
\[ = f_\lambda^0. \]  \hspace{1cm} (A18)
Thus we have a contradiction.

Q.E.D.

Proof of Lemma 1

By equation (4),

$$
\phi_0 = \sum_m g(m) \mathbb{E}(\tilde{\phi} | \tilde{M} = m) \tag{A19}
$$

$$
= \sum_m g(m) \mathbb{E}(\mathbb{E}(\tilde{f} | \tilde{M} = m) | \tilde{M} = m) \tag{A20}
$$

$$
= \sum_m g(m) \mathbb{E}(\tilde{f} | \tilde{M} = m) \tag{A21}
$$

$$
= f_0. \tag{A22}
$$

Proof of Lemma 2

"if".

$$
\mathbb{E}(\tilde{y} | \tilde{\phi} = \phi) = \mathbb{E}(\mathbb{E}(\tilde{y} | \tilde{f} = f \text{ and } \tilde{\phi} = \phi) | \tilde{\phi} = \phi) \tag{A23}
$$

where the outer expectation is averaging across all realizations \( f \) of \( \tilde{f} \) that are possible conditional on \( \tilde{\phi} = \phi \);

$$
= \mathbb{E}(\mathbb{E}(\tilde{y} | \tilde{f} = f) | \tilde{\phi} = \phi) \tag{A24}
$$

because \( \tilde{f} \) refines \( \tilde{\phi} \);

$$
= \mathbb{E}(a + b' \tilde{f} | \tilde{\phi} = \phi) \tag{A25}
$$

$$
= a + b'(\mathbb{E}(\tilde{f} | \tilde{\phi} = \phi)) \tag{A26}
$$

$$
= a + b' \phi. \tag{A27}
$$

"only if":

Suppose \( \tilde{f} \) does not refine \( \tilde{\phi} \). Then there are two states \( s_1 \) and \( s_2 \) such that \( f_{s_1} = f_{s_2} \) and \( \phi_{s_1} \neq \phi_{s_2} \). Consider the payoff pattern \( \tilde{y} \) that has realizations identical to those of \( \tilde{f} \) except in states \( s_1 \) and \( s_2 \), in which
\[ y_{s1} = f_{s1} + \pi_{s2} \quad (A28) \]

and

\[ y_{s2} = f_{s2} - \pi_{s1} \quad (A29) \]

\[
\tilde{E}(\tilde{y} | \text{state } s_1 \text{ or } s_2) = \frac{\pi_{s1}(f_{s1} + \pi_{s2}) + \pi_{s2}(f_{s2} - \pi_{s1})}{\pi_{s1} + \pi_{s2}} = f_{s1} - f_{s2}. \quad (A30)
\]

So \( \tilde{y} \) is linear conditional on \( \tilde{f} \)

\[ \tilde{E}(\tilde{y} | \tilde{f} = f) = f \quad (A31) \]

since \( \tilde{f} \) does not distinguish state \( s_1 \) from state \( s_2 \). But

\[
\tilde{E}(\tilde{y} | \phi = \phi_{s1}) = \tilde{E}(\tilde{f} | \phi = \phi_{s1}) + \pi_{s1}\pi_{s2}/\text{Prob}(\phi = \phi_{s1})
\]

\[ = \phi_{s1} + \pi_{s1}\pi_{s2}/\text{Prob}(\phi = \phi_{s1}) \quad (A32) \]

\[ > \phi_{s1}. \quad (A33) \]

Q.E.D.

Proof of Theorem 3

If \( \tilde{f} \) refines \( \tilde{M} \) then \( f \) refines \( \phi \) (because \( M \) refines \( \phi \)). Then apply Lemma 2 and Lemma 1.

Proof of Lemma 3

"if":

\( \tilde{M} \) refines \( \tilde{f} \) implies that the distribution of \( \tilde{f} \) conditional on any value of \( \tilde{M} \) is a singleton — zero variance. So for any state \( s \)

\[ \phi_s = \tilde{E}(\tilde{f} | \tilde{M} = M_s) = f_s. \quad (A35) \]
"only if":
\[ \tilde{\phi} = \tilde{\tilde{\phi}} \text{ means that for any state } s, \]
\[ f_s = \phi_s = E(\tilde{\tilde{f}} | \tilde{\tilde{M}} = M_s). \]  
(A36)
Consider any two states \( s \) and \( s^* \) with \( M_s = M_{s^*} \). Then
\[ f_s = E(\tilde{\tilde{f}} | \tilde{\tilde{M}} = M_s) = E(\tilde{\tilde{f}} | \tilde{\tilde{M}} = M_{s^*}) = f_{s^*}. \]  
(A37)
So \( \tilde{\tilde{M}} \) refines \( \tilde{\tilde{f}} \).
Q.E.D.

Proof of Theorem 4

"IF":
\[ \tilde{\tilde{f}} \neq \tilde{\tilde{\phi}} \text{ implies that } \tilde{\tilde{M}} \text{ does not refine } \tilde{\tilde{f}}, \text{ so there are states } s \text{ and } s^* \]
such that \( M_s = M_{s^*} \) and \( f_{\lambda s} \neq f_{\lambda s^*} \) for some factor number \( \lambda \). Consider payoff pattern \( \tilde{\tilde{y}} = \tilde{\phi}_\lambda \). Then
\[ E(\tilde{\tilde{y}} | \tilde{\phi} = \phi) = \phi_\lambda \]  
(A38)
so \( \tilde{\tilde{y}} \) is linear in \( \tilde{\phi} \). \( \tilde{\tilde{f}} \) refines \( \tilde{\tilde{M}} \) and \( \tilde{\tilde{M}} \) refines \( \tilde{\phi} \) so \( \tilde{\tilde{f}} \) refines \( \tilde{\phi} \); thus by Lemma 2, if \( \tilde{\tilde{y}} \) were linear in \( \tilde{\tilde{f}} \) then
\[ E(\tilde{\tilde{y}} | \tilde{\tilde{f}} = f) = f_\lambda. \]  
(A39)
We will derive a contradiction from this. If \( E(\tilde{\phi}_\lambda | \tilde{\tilde{f}} = f) = f_\lambda \) for all possible outcomes \( f \), then
\[ E(\tilde{\phi}_\lambda | \tilde{\tilde{f}} = f_s) = f_{\lambda s} \]  
(A40)
and
\[ E(\tilde{\phi}_\lambda | \tilde{\tilde{f}} = f_{s^*}) = f_{\lambda s^*}. \]  
(A41)
\( \tilde{\tilde{M}} \) refines \( \tilde{\phi} \) so \( \phi_{\lambda s} = \phi_{\lambda s^*} \). \( \tilde{\tilde{f}} \) refines \( \tilde{\phi} \) so the distribution of \( \tilde{\phi} \) conditional on any value of \( \tilde{\tilde{f}} \) is a singleton, so
\[ E(\tilde{\phi}_\lambda | \tilde{\tilde{f}} = f_s) = \phi_{\lambda s} \]  
(A42)
and
\[ E(\tilde{\phi}_i | \tilde{f} = f_{s^*}) = \phi_{s^*}. \] (A43)

Linking up the last five equations gives \( f_{\lambda s} = f_{\lambda s^*} \), a contradiction.

[There must be an easier proof!]

"ONLY IF":

\( \tilde{f} = \phi \) implies that any \( \tilde{y} \) linear in \( \phi \) must be linear in \( \tilde{f} \).

Q.E.D.

Proof of Theorem 5

"IF".

Assuming such an A exists, let \( \tilde{y} \) be the payoff on pure security \( s^* \).

So \( y_{s^*} = 1 \), and \( y_s = 0 \) for \( s \neq s^* \):

\[ E(\tilde{y} | \tilde{f} = f_{s^*}) = \pi_{s^*} / \text{Prob}(\tilde{f} = f_{s^*}); \] (A44)

\[ E(\tilde{y} | \tilde{f} = f) = 0 \text{ for all } f \neq f_{s^*}. \] (A45)

Let \( \lambda^* \) be the index corresponding to the unique rotated factor \( \tilde{\psi}_{\lambda^*} \) that takes the value 1 in state \( s^* \). Denote the \( \lambda^{th} \) row of A as \( (A_{\lambda^*})' \), a row-vector whose transpose is \( A_{\lambda^*} \). Then —

\[ E(\tilde{y} | \tilde{f} = f) = b'f \] (A46)

where

\[ b = [\pi_{s^*} / \text{Prob}(\tilde{f} = f_{s^*})] \times A_{\lambda^*}. \] (A47)

To see this, note that since \( \tilde{f} = A^{-1} \tilde{\psi} \), the only states where \( \tilde{\psi}_{\lambda^*} \) takes the value 1 are those with \( f_s = f_{s^*} \). So \( (A_{\lambda^*})'f = 1 \) only when \( f = f_{s^*} \), and zero otherwise. Therefore all pure securities are conditionally linear in \( \tilde{f} \). Any other payoff pattern is a linear combination of pure security payoffs, so it will also be conditionally linear in \( \tilde{f} \).
[Another, perhaps more enlightening way to prove this is to show that each pure security's \( \tilde{y} \) is linear in \( \tilde{\psi} \), and hence linear in \( \tilde{f} = A^{-1} \tilde{\psi} \).]

"ONLY IF":

There may be some groups of the S states where the random vector \( \tilde{f} \) has the same realization \( f_s \) for every state in the group. Let \( \mu (\leq S) \) be the number of distinct possible realizations of \( \tilde{f} \). Let \( Q \) be any set of \( \mu \) states each having a different realization of \( \tilde{f} \). An important part of the proof will be to show that if all patterns \( \tilde{y} \) are linear in \( \tilde{f} \) then \( \mu = \kappa \). \( \mu \) can not be less than \( \kappa \) because, thinking of \( \tilde{f} \) as a \( \kappa \)-by-\( S \) matrix —

\[ \kappa = \text{dimension of } \tilde{f}'s \text{ rowspace} = \text{dimension of } \tilde{f}'s \text{ columnspace} \leq \mu. \] (A48)

Construct a \( \mu \)-by-\( \kappa \) matrix \( A \) by letting each row be the transpose of

\[ \text{Prob}(\tilde{f} = f_s)/\pi_s \times b, \]
where \( b \) is such that for the payoff \( \tilde{y} \) on pure security number \( s \),

\[ E(\tilde{y} | \tilde{f} = f) = b' f, \] (A49)

and \( s \) ranges over all states in set \( Q \). Let \( \tilde{\psi} = A \tilde{f} \). Imagine shrinking the matrices \( \tilde{f} \) and \( \tilde{\psi} \) from having \( S \) columns to having \( \mu \) columns, by crossing out all columns except for those corresponding to states in \( Q \). (Note that since only duplicate columns are being crossed out, both for \( \tilde{f} \) and for \( \tilde{\psi} \), the columnspaces of each are unaffected.) Then we will show that the shrunken \( \tilde{\psi} \) (which is now \( \mu \)-by-\( \mu \)) is an identity matrix.
For any two states \( s_1 \) and \( s_2 \) in \( Q \), the \((s_1, s_2)\) element of the shrunken matrix \( \tilde{\psi} \) is

\[
[\text{Prob}(\tilde{f} = f_{s_1})/\pi_{s_1}] \times \text{E}(\tilde{y} | \tilde{f} = f_{s_2})
\]

(A50)

where \( \tilde{y} \) is the payoff on pure security \( s_1 \). If \( s_1 = s_2 \) then

\[
\text{E}(\tilde{y} | \tilde{f} = f_{s_2}) = \pi_{s_2}/\text{Prob}(f = f_{s_1})
\]

(A51)

so the diagonal elements all = 1. If \( s_1 \neq s_2 \) then \( f_{s_1} \neq f_{s_2} \) by the construction of \( Q \), so

\[
\text{E}(\tilde{y} | \tilde{f} = f_{s_2}) = 0.
\]

(A52)

Therefore the shrunken \( \tilde{\psi} \) is the \( \mu \)-by-\( \mu \) identity matrix. Thus:

\[
\mu = \text{dimension of columnspace of } \tilde{\psi}
\]

(A53)

\[
= \text{dimension of rowspace of } \tilde{\psi}
\]

(A54)

\[
\leq \text{dimension of rowspace of } \tilde{f} \quad \text{[since } \tilde{\psi} = \tilde{A}^{-1} \text{]}
\]

(A55)

\[
= \kappa
\]

(A56)

So we must have \( \mu = \kappa \). Thus the shrunken \( \tilde{f} \) has dimensions \( \kappa \) by \( \kappa \) and is the inverse of \( A \), and each column of \( \tilde{\psi} \) has a 1 and the rest zeroes.

Q.E.D.

Proof of Theorem 6

Immediate from Theorems 2 and 5.
ENDNOTES

1. This model determines relative prices but not the overall price level, and thus not the interest rate, which can be set arbitrarily.

2. Connor sets the interest rate to zero (see note 1 above).

3. This kind of counterexample would be impossible if the support of $\tilde{f}$ were a convex set.

4. One way to satisfy this restrictive condition would be to augment the set of factors to include new factors that are "general or multiple options" on the existing factors, as defined (and denoted "M") in Ross [1976]. Such a general or multiple option would be any mapping from the vector of existing-factor outcomes to a scalar.
REFERENCES


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