MARKET LIQUIDITY, HEDGING AND CRASHES

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Introduction

Immediately following the stock market crash of October 19, 1987, both practitioners and academics sought an explanation based on external events. While a number of trends were clearly "bad news" for the market, these trends had been revealing themselves over the previous months. It was difficult to isolate new events occurring between October 16th and October 19th which were of sufficient importance to explain the magnitude of the price fall.

The Brady Commission's examination of the October break therefore centered on internal market causes rather than external events. In particular, the Commission focused attention on a number of large institutions following "price insensitive" strategies such as portfolio insurance.¹

In dramatic language, the Brady Report painted a picture of enormous waves of institutional selling driving down prices excessively. The report claimed such sellers suffered from an "illusion of liquidity". And it buttressed this conclusion by pointing out that such large sellers alone sold about $6 billion of stock and stock index futures.

Although there has been a unanimous positive response to the Brady Commission's marshalling of facts, there has not been a unanimous acceptance of their interpretations of these facts. Formal portfolio insurance strategies were

¹ Report of the Presidential Task Force on Market Mechanisms [1988] (the "Brady Report"). Portfolio insurance strategies are dynamic hedging strategies which provide protection by replicating a put option: see Rubinstein and Leland [1982]. These strategies have the property that they tend to sell after the market has declined, and buy after market rises.
followed by less than 3% of stock market funds. While $6 billion seems a
large amount, it represented only 15% of total stock and stock index futures
volume on October 19th. And in absolute terms, the $6 billion amounts to less
than 0.2% of the roughly $3.5 trillion of equity value at the beginning of the
day. Is it reasonable to think that selling 0.2% can drive down prices by over
20%—that selling $6 billion can cause losses of $700 billion? Of course the
answer depends upon the elasticity of demand for stocks. But traditional
models imply an elasticity much greater than the market exhibited on October
19th.

A recent study by Brennan and Schwartz [1987] suggests that a 5% use of
portfolio insurance by investors would have a minimal impact on market prices
and volatility. Their model, and other commonly studied portfolio/consumption
models, suggest elasticities of demand for stock far greater than 1—more than
100 times the elasticity implied by the Brady Commission’s conclusions.
Information changes rather than selling by portfolio insurers are needed to
explain October 19th in these standard models.

Other evidence does not seem to confirm a strong connection between the crash
and portfolio insurance. If short-run selling were the cause of the decline,
we might expect a quick reversal. But this did not occur. And Richard Roll’s
[1988] cross-market studies showed little correlation between October 1987
performance and various aspects of markets, including whether portfolio

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2 Best estimates suggested $70-$100 billion in funds were following
formal portfolio insurance programs. On a pre-crash total stocks value of
about $3.5 trillion, this represents 2 to 3 percent. Of course, informal
hedging strategies such as stop-loss selling may have amounted to considerably
more than this: see the survey of Shiller [1987].
insurance was used.\textsuperscript{3}

In sum, the crash of 1987 presents the following dilemma to current financial models: the amount of selling seems insufficient to explain the large price drop observed on October 19th. Thus, information changes seem necessary to explain the drop. But no such information changes can be documented.

Parallels with the crash of 1929 may be useful in understanding the crash of 1987. Like 1987, no significant economic news was associated with the period immediately surrounding the earlier crash. Several large declines preceded the crash of 1929—as they did in 1987. Volatility increased markedly in the weeks preceding both drops. In both cases, hedging strategies were discussed as a possible contributing factor: stop-loss orders in 1929, and portfolio insurance in 1987. In 1929, stop-loss strategies were used for portfolio protection, but were also triggered by margin calls—which led to greater controls over margined stock buying following the crash.

Because of these similarities, we would hope that an explanation of the 1987 crash would also be relevant to the 1929 crash. The explanation cannot be entirely in terms of futures markets and portfolio insurance, since neither existed in 1929.

In this paper we develop an explanation of market "crashes" which reconciles the strands of evidence above. This explanation is not based on important changes in information, and therefore is consistent with the failure to observe

\textsuperscript{3} The fact that foreign markets fell despite the absence of portfolio insurance is not proof that portfolio insurance had nothing to do with the crash. Clearly, the crash in the US market was believed to reflect negative information, and other markets fell on this belief.
any significant events which directly "caused" the 1987 (or 1929) decline. In fact we define a "crash" to be a discontinuity in the relationship between the underlying environment and stock prices: an infinitesimal shift in information (or other small shock) can lead to a major change in stock market level.\(^4\)

Our explanation of crashes is based on unobserved plans of investors to hedge against losses. In 1929, stop-loss strategies were used. In 1987, portfolio insurance and stop-loss strategies were followed. As such, we develop a "price pressure" argument akin to Grossman [1988a]. But this argument must meet two criticisms:

1) How can relatively small amounts of hedging drive down prices significantly; and

2) Why didn't stock prices rebound the moment such selling pressure stopped?

Our model answers the first question by examining the determinants of market liquidity. An important aspect of financial markets is that only a small proportion of investors actively gather information on future economic prospects and/or asset supply.\(^5\) Other investors look to current prices to impute information about future prices. This dual role of prices--affecting demand both through the budget constraint and through expectations--leads to very different price elasticities than traditional models, in which prices play

\(^4\) Such discontinuities are commonly observed in physical systems and have been the recent subject of study by mathematicians examining "catastrophe theory". In a preliminary paper, Gennette and Leland [1987] considered a simple model of stock market discontinuities.

\(^5\) This point is also emphasized in Leland and Rubinstein [1988].
only the first role. Only recently have financial economists begun to explore
the implications of markets where prices play both roles.\(^6\)

In such environments, there is an important difference between observed and
unobserved supply changes. If there are relatively few informed investors,
markets may be much less liquid--and therefore more fragile--than traditional
models predict when unobserved supply shocks occur.\(^7\) We show that relatively
small unobserved supply shocks can have pronounced effects--more than 100 times
greater than the effects of observed supply shocks--on current market prices.

Our model answers the second question by showing how a discontinuity in market
prices can occur if most investors are unaware of the magnitude of other
investors' hedging plans. A small change in information can lead to a dramatic
fall in prices--with no immediate rebound occurring\(^8\). This feature of crashes
distinguishes our results from Grossman [1988a,b].

Finally, our model suggests that some changes in market organization can
radically reduce the likelihood of crashes. The most important such change is
the informing of markets about the size and trading requirements of hedging
programs. Pre-announcement of trading requirements can lessen the impact of
such trades by a factor greater than 100. To the extent that the specialist's
book helps reveal the nature of order flow, this information should be made
available to all traders. And as suggested by Grossman, the use of put options

Diamond and Verrecchia [1981], Kyle [1985], and Admati [1985]. These papers
focus on the role of prices in aggregating information.

\(^7\) This point is discussed informally in Leland and Rubinstein [1988].

\(^8\) An informal description of this possibility is given in Gennette and
Leland [1987].
to implement hedging may also serve to smooth markets.

The model builds from the work of Grossman and Stiglitz [1980], Hellwig [1980], Diamond and Verrecchia [1981], Admati [1985], Gennette [1985] and Kyle [1985]. We postulate a subset of informed investors who receive signals about future market values. Random supply keeps these investors from perfectly inferring the aggregate information. Some investors, however, whom we characterize as market-makers, receive information about the size of the random supply. This information enables market-makers to distinguish, at least partially, selling which is information-based from selling which is motivated by other reasons. The ability to examine the market-making role, and its effect on market liquidity, is new to our model.

We also allow for the presence of hedging programs such as stop-loss orders or portfolio insurance. These hedging strategies are usually nonlinear functions of the equilibrium price; hence the resulting Rational Expectations Equilibrium is in general a nonlinear function of the signals. We can examine the effects of these strategies on market equilibrium and stability for alternative specifications about the observability of hedging. The possibility of price discontinuities, with the implication of crashes, is also new to our model.
I. Informed and Uninformed Investors

We assume that investors may be informed or uninformed. The informed investors can be subdivided into two types, who differ in terms of the signals they are able to observe.

Thus in all there are three classes of investors:

- **Uninformed** investors (denoted U), who observe only the equilibrium price $p_0$;

- **Price-informed** investors (I), who observe a personal, unbiased signal $p_i$ on future price $p$ and also observe $p_0$; and

- **Supply-informed** investors (SI), who observe a common supply signal $S$ and the equilibrium price $p_0$.\(^9\)

The price-informed investors can be thought of as having (personal) information about economic fundamentals which are noisy predictors of future price.

Supply-informed investors can be thought of as market-makers who have information about the sources of order flows--the size of new issues, portfolio restructurings, and other elements of liquidity trading.

\(^9\) Investors who are both price-informed and supply-informed can easily be incorporated in this framework. Adding an I/SI investor has the same impact as adding an I and an SI investor separately.
Our model allows arbitrary proportions of investors in each class. The relative proportion of investors who are informed vs. uninformed is a key determinant of market liquidity. Informed investors--and particularly supply-informed investors--will absorb a substantial proportion of liquidity trading demands. Even when they are relatively few, informed investors constitute an important fraction of the supply of liquidity.

Thus an important empirical question is the relative number of investors of each type. While data on this question is difficult to gather, we do have some evidence that informed traders--and particularly supply-informed traders--are relatively small as a fraction of total market capital.

Among supply-informed investors are specialists and other market makers (including "upstairs" desks) who adjust their positions in response to changing demand for liquidity. Because of their role as market makers, these investors have special information on the nature of demand. We presume that, through the order book or simply their knowledge of institutional trading, these investors can learn (perhaps imperfectly) about the volume of non-information (or "liquidity") trading versus information trading.

The funds committed to supply-information gathering (and providing liquidity) depend upon the return to this activity. In some cases, competitive forces will determine the amount provided. In other cases, institutional factors such as the specialist system may limit the number of potential entrants. We shall see below that such limitations can importantly affect the stability of markets.

There is no way to provide an exact quantification of such funds in the market.
But they clearly are small relative to the $3.5 trillion of equity investment. For example, total capital of New York Stock Exchange specialists, including lines of credit, is approximately $3 billion.\textsuperscript{10} Total capital committed by "upstairs" trading houses and other forms of market-making may be four or five times this number.\textsuperscript{11} But at $15-$20 billion, supply-informed funds would represent about 0.5% of total market capital.

Capital devoted to "price-informed" market timing is even more difficult to estimate. It would include funds which explicitly gather information about future economic prospects ("fundamentals") and engage in market timing strategies reflecting this information. While many funds actively alter their exposures to individual stocks, most do not actively alter their total stock exposure based on information about future economic trends—perhaps because long-term success stories have been so rare.\textsuperscript{12} But a few do. The single most prominent market timing strategy is "Tactical Asset Allocation", utilized by perhaps $20 billion of assets. A total of $70 billion, or about 2%, might be a guess for price-informed funds which actively gather information about future prices and trade on it\textsuperscript{13}.

This leaves most investors in the class we term "uninformed". "Passive" might be a somewhat less pejorative description of these investors, who participate


\textsuperscript{11} Conversations with officers at major investment banking firms.

\textsuperscript{12} See R. Henriksson [1984].

\textsuperscript{13} It should be noted here that we assume that the informed investors have information of value. It is clearly difficult to assess the fraction of the market trading on "quality" fundamental information, but it is most probably smaller than the fraction engaged in timing in general.
in the market "for the long haul" and do not move in and out based on
information about fundamentals or current liquidity trading. The relative lack
of popularity of information-based market timing strategies suggests most
investors belong to this class.
II. Market Equilibrium

A single risky security is traded. Its future price $p$ is a normally distributed random variable with unconditional variance $\Sigma$ and unconditional expectation $\bar{p}$. All investors share this prior distribution of future price. Current price is determined by supply and demand. Riskless bonds are also traded and the riskless rate is normalized to zero$^{14}$.

a. Supply

The net supply of stocks is a fixed amount $\bar{m}$, modified by three additional factors:

1) A random and exogenously determined net supply created by "liquidity traders". This shock is comprised of two pieces:

   > An unobserved liquidity shock

   $L$, distributed $N(0, \Sigma_L)$;

   > A partially-observed liquidity shock

   $S$, distributed $N(0, \Sigma_S)$.

   which is observed by all supply-informed investors.$^{15}$

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$^{14}$ All the results extend in a straightforward way to the case of non-zero interest rates.

$^{15}$ An equivalent formulation would allow the supply-informed investors to receive a noisy signal about total liquidity trading, and not distinguish $S$ from $L$. A simple transformation of variables allows the alternative interpretation.
S and L are assumed to be independently distributed.

2) A deterministic supply by hedgers, rebalancers, and others who utilize dynamic strategies akin to portfolio insurance. The supply of stock from these strategies is a given function of the current market price $p_0$. We denote this supply by PI, where PI may depend upon $p_0$. This hedging demand will be observed by different market participants in the alternative environments which are considered below.

Thus the total supply is:

$$\bar{m} + L + S + PI$$

or

$$m + PI$$

where $m = \bar{m} + L + S$.

b. Demand

As discussed above, there are three classes $j$ of investors characterized by the information signals (if any) they receive. All investors maximize expected utility of terminal wealth over a single period. Preferences are assumed to exhibit constant absolute risk aversion. The utility function of each investor in class $j$ is given by:

$$U_j(W) = -\exp \{-W/a_j\}.$$
Expectations depend upon the signals which investors observe. Each price-informed investor \( i \) observes

\[ p'_i = p + \epsilon_i, \]

where \( p \) is the true future price and \( \epsilon_i \) denotes a noise term uncorrelated with other random variables and uncorrelated across price-informed investors. Both \( p \) and \( \epsilon_i \) are assumed to be normally distributed as \( N(\bar{\rho}, \Sigma) \) and \( N(0, \Sigma_p) \), respectively. \(^{16}\)

Supply-informed investors receive a common signal

\[ S, \]

where \( S \) is the liquidity supply (distributed \( N(0, \Sigma) \)) observed only by supply-informed (SI) investors.

All investors can observe the current market price, and use this (and their other information) to determine their conditional distributions of future price.

Investors in each class \( j \) have the same conditional price variance \( Z_j \). The expectation of the future price conditional on the information available to an agent \( i \) belonging to class \( j \) is denoted \( \bar{p}_j^i \). It has been shown elsewhere that the portfolio optimization problem is equivalent to the maximization of the certainty equivalent and leads to a demand for shares by investor \( i \) in class \( j \) equal to:

\[ n_j^i = a_j Z_j^{-1} (\bar{p}_j^i - p_0). \]

\(^{16}\) For simplicity and notational convenience, we assume that the \( \epsilon_i \) are i.i.d across agents. Differences in information precision would not affect our results.
There are $w_j$ investors of type $j$. Demand per investor in class $j$, 
$n_j = \Sigma n_j^i / w_j$, is equal to 
\[ n_j = a_j Z_j^{-1} (\bar{p}_j - p_0) \]
where $\bar{p}_j$ is the mean expected future price for investors in class $j$. 
All supply informed investors observe the same signals and hence have the same 
conditional expected price. The same holds for uninformed investors. Price 
informed investors differ in terms of the error terms $\epsilon_i$ but as their number 
increases, the mean expected future price converges to the actual future price 
$p$ by the Law of Large numbers\footnote{Hellwig [1980] shows the error terms $\epsilon_i$ do cancel in the limit of a 
sequence of finite economies where the relative proportion of investors in each 
class remains fixed and the total number of investors as well as the supply 
parameters grow without bound at the same rate. Individual agents are thus price 
takers and, importantly, the individual error terms $\epsilon_i$ do not affect prices.}.

The relative market power of investor class $j$, $k_j$, is defined as the ratio of 
the weighted risk tolerance of the class to the sum of the weighted tolerances: 
\[ k_j = a_j w_j / \Sigma a_j w_j. \]
Define the normalized total demand $D$ as the sum of the individual classes' 
demands divided by the weighted sum of the three classes' risk tolerance, 
\[ D = \frac{\Sigma j w_j n_j}{\Sigma j w_j a_j} = \Sigma j k_j Z_j^{-1} (\bar{p}_j - p_0) \]  
(II.2)

Similarly, the supply parameters: $PL$, $M$, $S$, and $L$ are normalized by dividing 
the original parameters by $\Sigma j w_j a_j$; we do however keep the same notation. Our 
analysis focusses on relative proportions and is thus unaffected by this 
normalization.
c. Equilibrium

Equilibrium of supply (II.1) and demand (II.2) yields the equation for equilibrium price:

\[ Z^{-1} p_o + PI(p_o) = \sum_j k_j Z_j^{-1} \bar{p}_j - m \]

where \( Z^{-1} \) is \( \sum k_j Z_j^{-1} \). We may now characterize the equilibrium price function relating current price to future price (as revealed by the average of individual signals \( \epsilon_i \)), total non-hedging supply \( m \), and observed supply \( S \). The theorem states that there is a unique rational expectations equilibrium price function\(^{18}\):

Theorem

Provided that the parameters of the model verify assumption \( A1 \) below there exists a unique Rational Expectations Equilibrium (REE) of the form:

\[ p_o = F \{ p - \bar{p} - H L - I S \}, \quad (II.3) \]

where \( F \) is a real valued monotonically increasing function; \( H \) and \( I \) are real constants; and \( F, H \) and \( I \) depend only on the agent's preferences and on the mean and variances of the random variables.

\(^{18}\) A Rational Expectations Equilibrium price function is a market clearing price function implied by investor behavior which is the same as the price function on which investor decisions are based (Lucas [1978]).
**Assumption A1:** \( Z^{-1}p_0 + PI \) is a strictly monotonic function of \( p_0 \).

The proof and all derivations are provided in the Appendix as well as expressions for \( F, H, \) and \( I \).

In the absence of hedgers or if the hedging supply is a linear function of equilibrium prices, \( F \) is a linear function (equation VI.3 in the Appendix).

The REE function \( F \) is non-linear if the hedging supply is a nonlinear function of equilibrium prices \( p_0 \).

We discuss the economic meaning of assumption A1 in Section VI.
III. The Nature of Equilibrium Pricing: An Example

Consistent with our earlier discussion, we consider an example in which there are relatively few price-informed and supply-informed investors. We assume 0.5% of investors ("market makers") are supply-informed, and 2% of investors are price-informed. There is no hedging supply PI: This will be introduced in Section V.

Several other parameters must be specified before the model is complete. A key parameter is the quality of the information signal received by the price-informed investors. The better the signal, as expressed by the signal-to-noise ratio, the lower the conditional variance $Z_1$ for the price-informed investors.

We assume that the quality of the signal received by each price-informed investor is not very high. Specifically, we assume that the price-informed investors' signal-to-noise ratio is 0.2. Thus if $\Sigma$ is the ex-ante expected variance of future price, and $\Sigma_e$ is the variance of each price-informed investors' future price signal, then $\Sigma_e$ is 5 times $\Sigma$. This assumption implies that (in equilibrium) the price-informed investors' conditional standard deviation for future price is 19%, rather than the 20% of uninformed investors, who observe price only. This slight improvement seems consistent with the perceived difficulty in predicting future market prices.\(^{19}\)

\(^{19}\) While each individual signal about future price is quite noisy, the average signal perfectly reflects future price, as in Hellwig [1980]. But individual investors cannot "back out" the true future price from current price because supply is noisy.
Also important is the fraction of total liquidity supply shock which, on average, can be observed by supply-informed investors. Since $S$ is observed and $L$ is not, this fraction can be parameterized by the ratio $\Sigma_S/\Sigma_L$. If the ratio is high, then supply-informed investors on average will observe most of the total liquidity shock. We assume the ratio is one: On average, supply-informed investors receive a signal which reveals information about half the total liquidity supply shock. Conditional on the supply signal, a supply-informed investor estimates a 17.5% standard deviation for future price.

Our example is consistent with a rational expectations price function as in Theorem 1. We seek $\Sigma$, the ex ante market variance, and $m$, the fixed supply, to equate expected return on future stock price to 6% and the standard deviation of future stock price to 20%. Finally, we find a variance of supply $\Sigma_m$ consistent with the variance of $p_0$, the current price, equalling the variance of the future price $p$, conditional on $p_0$. This provides the example with intertemporal consistency. Parameters for the example are summarized in the Appendix.

The rational expectations equilibrium price function (III.3) for our example is

$$p_0 = 0.5 \left[ p - \bar{p} - 19.95 L - 8.14 S \right] + 1.$$  \hspace{1cm} (III.1)

Given this price function and the volatilities of future price and

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$^{20}$ Recall the interest rate is normalized to zero. Thus, the assumed return of 6% represents a 6% premium over the riskless interest rate.
liquidity surprises, the standard deviation of $p_0$ is 20%, as is the standard deviation of $p$ conditional on $p_0$. 
IV. Stock Market Liquidity

Because of the Brady Commission's focus on limited market liquidity, we wish to examine the impact of changes in supply on market price. We postulate a small percentage change in supply, and determine the resulting percentage change in the equilibrium price from the pricing relation (II.3). This will then determine the price elasticity of stocks. The greater the price elasticity, the more liquid we term the market. In this section, we continue to assume there is zero net hedging: \( P_I = 0 \).

We study three possibilities: supply increases, and

1) The increase in supply is known to all investors;

2) the supply-informed investors (only) receive an accurate signal about the increase in supply;

3) no signal is received by the supply-informed investors (or anyone else).

The first possibility is modeled by letting the expected supply \( \bar{m} \) change. A change in expected supply will be common knowledge and will not affect investors' expected future price. From equation (II.3) and (VI.3) in the Appendix,

\[
\text{Elasticity} = - \frac{\delta \bar{m}}{\delta P_0} = \frac{P_0}{\bar{m}} Z^{-1}
\]
Given the example (III.1), we find an elasticity of 17: a 1% observed supply increase will lead to a 0.06% fall in price. Such a high elasticity is very much in line with the predictions of traditional models, which do not postulate that investors learn from market prices. Investor classes participate proportionately to their number \( k_i \) in absorbing the increase in supply.

The second possibility is modeled by a small increase the random supply \( S \) which is observed only by the supply-informed investors. In this case,

\[
\text{Elasticity} = - \frac{\delta S}{\delta P_0} = \frac{P_0}{\bar{m}} \frac{1}{F I}
\]

In the example (III.1), we find elasticity is 0.16: a 1% partially-observed increase in supply will lower price by 6%. Remarkably, this is only one percent of the elasticity above. This is because investors, with the exception of the supply-informed, revise downward their expectations (which are conditional on \( p_0 \)) as price falls. Thus they are less willing to absorb the increased supply. Indeed, in our example, the supply-informed investors absorb about 54% of any increase in liquidity supply—even though they constitute only 0.5% of investors!\(^21\)

Price-informed traders, who are 2% of investors, absorb another 18%. They are more willing to buy as prices fall because (on average) they receive signals about future price which moderate the fall of expected future price.

\(^{21}\) The exponential utility model does not limit purchases by investors to their initial wealth. If we imposed a "no leverage" condition, elasticity in this case would be even lower. This is because in our equilibrium, supply-informed investors will buy tremendous amounts of stock (on a per-capita basis) when prices fall. This would only be possible if they can undertake levered stock positions.
Uninformed investors have no such signals, and can only infer from current price. Because they impute a lower future price as current price falls, they absorb but 28% of the increased supply, despite the fact that they constitute 97.5% of investors.

The final possibility is where \( L \) increases. In this case elasticity falls even further, since the supply-informed traders will not observe the increase in supply and will not increase their demand. From (II.3), we can determine that

\[
\text{Elasticity} = - \frac{\delta L}{\delta p_0} = \frac{P_0}{m} \frac{1}{F H}
\]

In the example (III.1), elasticity will be 0.07, or about 1/250th of the elasticity predicted by traditional portfolio/consumption models. A 1% unobserved increase in supply will lower prices by 14%! In this case price-informed investors will absorb about 40% of the supply increase, and uninformed investors absorb the remaining 60%. But price must fall precipitously to induce them to absorb the extra supply.

The model therefore resolves the paradox of low vs. high demand elasticity. If supply changes are unobserved, all investors will revise downward their expected future price and will absorb the increased supplies only after price has fallen substantially. Price-informed investors will have somewhat greater elasticity of demand than uninformed investors, since they receive independent information about future prices. But their contribution will be minimal if they are few, or if their price information is very noisy.
When supply changes are partially observed, only the supply-informed investors will have a high elasticity of demand and the consequent ability to absorb liquidity trades. Uninformed investors will provide relatively little support in absorbing large liquidity supply, since they believe a falling price is likely to reflect bad news.

In sum, traditional models will grossly overestimate the liquidity of financial markets, unless all investors observe the increase in supply.
V. Hedging Strategies and Market Stability

Hedgers sell as stock prices fall. They do so to protect themselves against further potential losses. Whether hedge programs are carried out by portfolio insurance programs, or by less formal means such as stop-loss orders, the result is the same: stocks are sold as prices fall. This selling must be absorbed by other investors.

It is generally believed that hedge programs can make markets more volatile. But can they lead to a crash or "meltdown", where selling begets selling and prices plunge without stop? Phrased more formally, can the function relating price to underlying information become discontinuous?

We examine these questions by adding a hedging supply (PI) to a market which previously did not have such a supply. At the initial equilibrium price \(p_0 = 1\), we normalize hedging supply to be zero. But for prices below \(p_0 = 1\) the hedging supply will be positive; for prices above, negative.

For markets with relatively few price-informed and supply-informed investors, we show that:

1) If all investors know hedging strategies (PI), markets will be more volatile than without PI. But markets will be continuous (e.g. no crashes) under most environments.

2) If only supply-informed investors know PI hedging strategies,
markets will be more volatile than in case 1); if PI is followed by a substantial fraction of investors, there is the possibility of market crashes (discontinuities).

3) If no investors know PI hedging strategies, market crashes can occur even when a modest fraction of investors follow hedging strategies.

To examine the effects of hedging in detail, we must specify the nature of hedging strategies. We assume that a fraction \( \omega \) of assets are protected by a put-option replicating strategy.\(^{22}\) The required amount of supply created by this hedging strategy will depend on the current stock price \( p_0 \). The incremental hedging supply when future price is \( p_0 \), relative to the supply at the initial equilibrium price (one) is given by

\[
PI = \omega [N(d_1(1)) - N(d_1(p_0))]
\]

where \( \omega \) is the fraction of assets subject to the hedging strategy, \( N(\cdot) \) is the cumulative normal distribution function, and \( d_1 \) is given from the Black-Scholes formula

\[
d_1(p_0) = \frac{\ln \left( \frac{p_0}{K} \right) + \frac{1}{2} \sigma^2}{\sigma}
\]

\(^{22}\) Put-replicating strategies are just one possible type of hedging. Others might include stop-loss, "constant proportion of surplus" policies, or do-it-yourself strategies. We examine put-option replication because it was the most prevalent of formal protection strategies on October 19th.
where \( K \) is the striking price of the option, and \( \sigma \) the standard deviation of \( \Delta \) conditional on \( p_0 \).

Note that the derivative of the hedging supply with respect to \( p_0 \), \( \Pi' \), is negative, and becomes initially more negative as \( p_0 \) falls, before eventually approaching zero as \( p_0 \) falls to zero.

The three environments above can be analyzed as follows:

(i) When all investors observe the function \( \Pi \), we replace \( \bar{m} \) by \( \bar{m} + \Pi \).

(ii) When supply-informed investors observe \( \Pi \), then \( \Pi \) comes as a partially observed shock: we replace \( S \) by \( S + \Pi \).

(iii) When no investors observe \( \Pi \), then \( \Pi \) comes as an unobserved liquidity shock and we replace \( L \) by \( L + \Pi \).

Cases (ii) and (iii) differ from (i) in the nature of the resulting equilibrium. In case (i), all investors know the function \( \Pi \), and thus can predict the amount of hedging supply as prices change. The equilibrium is a (non linear) Rational Expectations Equilibrium (REE). In case (ii), supply-informed investors observe the amount of \( \Pi \), but (erroneously) believe it is a random shock \( S \). The equilibrium is not a full REE in the sense that investors think that the equilibrium price function \( F \) is the linear function which would obtain if there were no hedging supply. In case (iii), investors observe neither the amount nor the function \( \Pi \). Thus, they are surprised by insurance and (erroneously) believe it represents a change in either expectations or

\[ 23 \text{ See Black and Scholes [1973]. Our formula assumes the interest rate has been normalized to zero, and assumes a one-year time horizon.} \]
supply. This equilibrium also is not a full REE.

With the three alternative specifications above, we can derive the excess demand functions (demand minus supply) as \( p_0 \) varies. The equations for excess demand are given by

\[
\begin{align*}
XD_A &= \frac{1}{H} \left( p - \bar{p} - H L - I S + \frac{Z^{-1}(\bar{p} - p_0) - (\bar{m} + PI)}{Z^{-1} - \Sigma^{-1}} \right) \quad (V.1i) \\
XD_p &= \frac{1}{H} \left( p - \bar{p} - H L - I S + \frac{Z^{-1}(\bar{p} - p_0) - \bar{m}}{Z^{-1} - \Sigma^{-1}} \right) - \frac{I}{H} \text{ PI} \quad (V.1ii) \\
XD_u &= \frac{1}{H} \left( p - \bar{p} - H L - I S + \frac{Z^{-1}(\bar{p} - p_0) - \bar{m}}{Z^{-1} - \Sigma^{-1}} \right) - \text{ PI} \quad (V.1iii)
\end{align*}
\]

These three functions are graphed in Figure 1, for the parameters in our earlier example, with 5% of investors following a hedge strategy (\( \omega = .05 \)). Note that the fully anticipated excess demand function is the flattest; neither it nor the partially anticipated excess demand function are "backward bending". But the unanticipated excess demand function is backward bending.

The three curves in Figure 1 intersect at \( p_0 = 1 \). Thus, in the absence of future price or liquidity shocks, the price \( p_0 = 1 \) is an equilibrium in all three cases. Now say that information signals about future price become slightly more pessimistic: \( p - \bar{p} = -.01 \), a 1% downward shock in future price. This will cause demand to fall slightly, thereby shifting all three curves to the left by the same small amount.\(^{24}\) Figure 2 depicts this shift.

\(^{24}\) Curiously, current price \( p_0 \) falls by less than 1% if all investors can observe PI. This is because of rational expectations. The fall in price could be due to additional liquidity trading as well as to lowered expectations. It
To restore equilibrium, price will fall in all three cases, until excess demand again is zero. But the excess demand curve for the unobservable hedging case has the steepest slope at $p_0 = 1$. The resulting price drop (2.7%) to restore equilibrium will be greater than the price drop (0.7%) to restore equilibrium in the partially observable case—which in turn will be greater than the price drop (0.5%) to restore equilibrium in the fully observable case.

In short, the market price is more volatile in response to future price shocks when hedging supply is unobservable. It will also be greater in all cases if $\omega$, the proportion of hedgers, becomes larger.

a. Prelude to a Crash

We continue to examine market behavior as information about future price becomes (continuously) more pessimistic. Figure 3 indicates the situation where $p - \bar{p}$ is now -.016. Relative to our initial equilibrium (at $p_0 = 1$), the average of future price signals is now 1.5% more pessimistic than the case in Figure 2. Of course, price must drop further to restore equilibrium. This in turn creates further portfolio insurance selling.

How far does price drop? This depends on how completely portfolio insurance selling is observed. If every investor observes PI, the price falls from 1 to .992, or 0.8%. If it is observed only by the supply informed, price falls by 1.2%. But if no one can observe PI selling, price will fall 7.25% in response

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is rational to assume that not all the price is caused by lower future price expectations—even though (with our superior knowledge) this is the case.
to the signal \(-0.016\), or almost ten times as far as when PI is fully observed.

Note that in the case of unobserved PI, the market also becomes more sensitive to future price signals as price falls. This can be seen in Figures 2 and 3 by the fact that excess demand is becoming a steeper function of \(p_0\). Thus, volatility of the market is increasing as \(p_0\) falls.

The move from the situation in Figure 1 to the situation in Figure 3 seems to correspond to the steady erosion of confidence that occurred during the month leading up to October 19, 1987. As the Brady Commission Report documented, a number of negative economic trends came to light over this period. Interest rates were rising. The dollar was weakening. Tensions in the Middle East were increasing. And so on. In our model, this is reflected by a sequence of negative signals about future price.

As the market fell, portfolio insurance programs became more active. At higher market levels, not much hedging was necessary, given the relatively low levels of protection chosen by many pension funds. But as the market fell closer to the desired protection level, greater hedging was needed. The market became more volatile. Yet portfolio insurance--although beginning to attract some public attention--was largely unknown to the majority of investors, and not fully understood even by market professionals. It was Friday, October 16, 1989.
b. The Crash

Figure 3 shows the market at a critical point when hedging strategies are not observed. Prices had fallen strongly over the previous several trading sessions, with great volatility. Over the weekend, a bit more negative news came into the market—nothing earthshaking, but enough to shift the backward bending excess demand curve a fractional amount further to the left.

Figure 4 illustrates the situation as it may have been on Monday morning, October 19, 1989. \((p - \bar{p})\) has fallen to -.018, slightly below its previous level. The marginally negative news over the weekend, coupled with further portfolio insurance selling (including some resulting from Friday’s decline) led to rapidly falling prices.

Observing these falling prices, uninformed investors (rationally) concluded that highly negative information must have been received by the price-informed investors. (Indeed, the following day’s newspapers vainly sought the information event which "must" have triggered the crash.) As reported by Shiller [1987], the majority of investors stood on the sidelines or bought only limited amounts—consistent with a conviction that something unknown but terrible must have happened. Meanwhile, hedgers were selling ever larger amounts.

As Figure 4 shows, excess supply actually increased as prices began to fall, leading them to fall even further. The feared meltdown was actually happening.
Only when hedgers had largely completed their hedging programs did the market restabilize itself—but at a much lower level. In Figure 4, our example shows a post-crash equilibrium price of \( p_0 = 0.64 \): a 30% drop from its previous closing price in Figure 2. While the market on October 19th did not fall quite this far, it also is the case that many hedgers scaled back the size of their hedging programs in the face of extraordinarily high transactions costs.\(^{25}\)

A similar story could be told about the 1929 crash. The only difference is that portfolio insurance programs would be replaced by "stop-loss programs". While less exact in delivering desired results, stop-loss orders have the effect of increasing liquidity supply as prices fall. It is no accident that investigators focused on the role of stop loss orders and margined stock buying—since the latter created forced stop-loss orders as the market descended.

It should be emphasized that a crash in our model is not due to a discontinuous change in the underlying information. Rather, the market reaches a critical point, and a "catastrophe" occurs—both in practice and in theory.

While hedging strategies are an important part of our explanation of the crash, equally important is the market structure which precludes observing these hedging strategies. Figures 1 - 4 also plot the excess demand functions associated with partial or complete observability. These "regular" (i.e., not backward-bending) excess demand functions eliminate the possibility of crashes

\(^{25}\) For a description of how hedging programs were modified in the presence of high trading costs, see Leland [1988b].
in our example. Indeed, if PI programs had been fully observable, prices would have fallen only about 1%. And the fall would have been about 1.5% if supply-informed investors had observed the extent of PI sales.

That is not to say that crashes are impossible with partial observability. If we had assumed a 15% use of portfolio insurance (ω = 0.15), the excess demand curve with partial observation would be backward bending: See Figure 5. A 15% use would represent over $500 billion, or more than 5 times the total amount estimated for formal programs. Shiller's survey suggested formal portfolio insurance programs were "the tip of the iceberg" relative to total hedging, so it is possible that the crash could have occurred in our example even with supply-informed traders aware of hedging supply.26

c. After the Fall

A new low-price equilibrium is established after the crash. If information about future prices now reverses itself, returning it to pre-crash levels of optimism, will the market shoot back to its former level?

The answer is no. In Figure 4, a small rightward shift of the excess demand function will lead to a small increase in equilibrium price \( p_0 \) from the 0.64 level. Even if the upper branch of the excess demand curve intersects the zero excess demand line, implying the possibility of multiple equilibria, the lower equilibrium price is locally stable and can be expected to prevail.

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26 Alternatively, if supply-informed traders were aware of the PI function (rather than just the amount of PI at the then-current price), they may have been able to predict the possibility of a crash. With this possibility for prediction, they would then abstain from trading (or even "front-run"), thus making the market less stable even than the unobservable PI case.
Since the slope of the excess demand curve is less steep at $p_0 = .64$ than just before the crash (when $p_0 = .9275$), volatility will return to lower levels.

But eventually, if information becomes still more favorable (to $p - \bar{p} = .026$, well above pre-crash levels), the excess demand curve will shift sufficiently to the right such that its lower branch is just tangent to the vertical zero excess demand line (see Figure 6). This will be accompanied by higher volatility. Any further increase in future price expectations could lead to an upward jump in prices: a "meltup" rather than a "meltdown." In our example, the discontinuous jump would commence at $p_0 = 0.74$ (15% above the market low) and jump to $p_0 = 1.043$.

Is such an upward jump possible only in the mind of the theorist? Perhaps. But over the period 1928-1988, 22 of the 38 one-day stock market moves that exceeded 7% were upward jumps. And the financial press occasionally remarks on such a possibility.27

Figure 7 graphs the equilibrium price function relating $p_0$ to the future price surprise $p - \bar{p}$, for the three different observability cases. For the case with unobserved PI, we see the point of discontinuity on the upper branch of the function is at $p_0 = .927$; and the discontinuity on the lower branch at $p_0 = .740$. For the other two cases, there are no discontinuities given our example's parameters.

Future price surprises are not the only possible sources of discontinuous price behavior. A large random liquidity supply shock could also lead to

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discontinuous behavior. But whatever the cause, the critical price—where the discontinuity occurs—will remain the same. This leads us to examine the general nature of critical points: when do they occur, and what determines their level?
VI. Discontinuities: Some General Results

We wish to characterize price levels for which the price function becomes discontinuous. These critical points depend upon the size and nature of hedging programs, and whether they can be partially or fully observed.

First consider the case in which hedging strategies are unobservable: investors are unaware of hedging strategies and thus do not distinguish them from unobservable liquidity trades. The equilibrium price $p_0$ is the price level for which excess demand is equal to zero (equation V.1iii). Discontinuities will occur if the root (or roots) of V.1iii are discontinuous functions of the variables $p$, $L$ and $S$. Since excess demand is continuous and differentiable in $p_0$, discontinuities will take place at points where the function reaches an extremum. Differentiating V.1iii with respect to $p_0$ yields

$$\frac{\delta XD_u}{\delta p_0} = - \left( \frac{1}{F} + P'I' \right).$$

where $F = 1 - Z/\Sigma$ is the coefficient which obtains in the case of no hedging because agents’ view of the equilibrium equation ignores hedging, and where

$$P'I' = \omega \frac{N'(d_1)}{p_0 \sigma}.$$

The derivative of the demand for hedging ($P'I'$) tends to zero as prices become large and as prices become small; hence the derivative is negative at both ends.
The function will be discontinuous if and only if the equation:

\[ 1 + F H \pi' = 0 \]  

(admits a solution. Then, given that \( \pi' \) has a unique inflexion point in our case, the equation has two solutions, the critical points \( c_1 \) and \( c_2 \) (\( c_2 > c_1 \)). Excess demand is an increasing function of equilibrium price \( p_0 \) in the interval \( [c_1, c_2] \) and decreasing elsewhere. Since \( \pi' \) is negative and decreasing-increasing, the first critical point, \( c_1 \), decreases as \( F H \) increases and the second, \( c_2 \), increases as \( F H \) increases. Equation V.2 has 2 roots if and only if:

\[ \omega F H > K e^{-\frac{3}{2} \sigma^2} \left( \frac{1}{2\pi \sigma^2} \right)^{\frac{1}{2}} = \phi_{\min}, \]

implying

\[ \omega_{\min} = (FH)^{-1} \phi_{\min} \]

The root is unique (and there is no discontinuity) when equality obtains. \( \omega_{\min} \) represents the smallest proportion of hedgers which could create a market crash.

\( c_1 \) and \( c_2 \) are given by:

\[ c_1 = K e^{-2\sigma^2} e^{-\left[2\sigma^2 \ln \left( \frac{\omega}{\phi_{\min}} \right) \right]^{\frac{1}{2}}}; \quad c_2 = K e^{-2\sigma^2} e^{-\left[2\sigma^2 \ln \left( \frac{\omega}{\phi_{\min}} \right) \right]^{\frac{1}{2}}} \]

The difference \( c_2 - c_1 \) is the range of prices \( p_0 \) for which no equilibrium exists, the amount of the price drop when \( c_1 \) is reached from above is larger than that difference. In our base case, the discontinuity occurs for a value of \( \phi_{\min} \) of 0.425, which is reached for a \( \omega_{\min} = 4.26\% \). This percentage of
hedgers will can create a market crash in our example. Conversely, the market melt-up takes place when the equilibrium price reaches \( c_2 \) from below and the price jumps to a level higher than \( c_1 \).

Now consider the case where there is partial observation of hedging strategies: supply-informed agents investors can observe the sum of \( S \) and \( PI \). The same reasoning leads to an equation analogous to \( V.2 \):

\[
1 + F I P I' = 0
\]

and to:

\[
\omega_{\text{min}} = (FI)^{-1} \phi_{\text{min}}.
\]

The critical points are obtained by substituting \( H \) with \( I \) in \( (V.3) \). Since \( H > I \) in all cases, the minimum fraction \( \omega_{\text{min}} \) (10.4\% in our example) is higher than in the previous case, \( c_1 \) is larger, \( c_2 \) smaller and ceteris paribus the price drop is smaller.

Finally, in the anticipated case identical results obtain provided that \( FH \) is replaced with \( Z \). Again, \( Z \) is smaller than both \( H \) and \( I \), and \( \omega_{\text{min}} \) is larger, in fact very large in our base case: over 10 times expected supply.\(^{23}\)

In summary, crashes are most likely to occur in the unobserved case, since the inequality is satisfied for the lowest values of \( \omega_{\text{min}} \). Since the critical point difference \( c_2 - c_1 \) is greatest in this environment, the "crashes"

\[^{23}\text{This means that the selling (as prices fall to zero) of an amount equal to the expected supply by investors following a Black-Scholes put option-replicating strategy would be met by the buying of investors as prices fell continuously. Selling would have to be over 10 times more intensive before the market would crash (i.e. fall discontinuously).}\]
associated with this environment will also be the largest in size, ceteris paribus.
VII. Making Markets More Stable

Our analysis suggests that unobserved hedging strategies can destabilize a market, leading to greater volatility and ultimately to a crash. Are there private or governmental policies which will lessen the chances of such an event in the future?

Outlawing hedging strategies is one such possibility, but it is neither practical nor desirable. It isn’t practical because it isn’t enforceable. An investor following a stop-loss or portfolio insurance hedging strategy can always claim he’s doing so for other reasons—an anticipated expense, a forecast of weak markets, etc. Short of prohibiting selling for any reason, it is impractical to prohibit selling for hedging purposes. Nor would it be desirable. Investors are willing to participate in a market because they can sell whenever they wish to—including for risk avoidance purposes.

We should note that the market is partially self-correcting. Stop-loss and dynamic hedging strategies are fully effective only when prices move continuously. The possibility of a crash will limit the use of dynamic protection strategies. Of course if relatively few investors follow such strategies, crashes are unlikely to occur.

But portfolio protection is a legitimate aim of private investors. Is there a way in which investors can achieve protection without contributing to—or suffering from—discontinuous markets? Our analysis provides some clues.

The most important result is that widespread knowledge of dynamic hedging
usage can minimize its impact on markets. The preceding section showed that the unobserved hedging which created a 30% crash in market prices would have less than a 1% impact on prices if it were observed by all investors. Does this seem preposterous? Some post-crash evidence suggests it is not. On October 19, 1988, exactly one year after the crash, the Japanese government sold over $24 billion of a single stock, Nippon Telephone & Telegraph. This was four times the amount portfolio insurers had sold of all stocks the year before. Yet NT&T stock did not decline by a significant amount (either at sale or at the time of initial announcement), because investors had prior knowledge that the sale did not reflect an informational change.

Interestingly, portfolio insurers were anxious to disseminate information about their trading requirements prior to the crash. But events happened more quickly than regulatory approval.²⁹

An alternative is for hedgers to use static instruments which provide the same results as dynamic hedging strategies. For example, put options provide protection without requiring further trading. They would seem the ideal instrument to avoid the problems of trading in uninformed (and therefore illiquid) markets. A criticism of this argument is that it simply pushes the problem back one level: the sellers of the put option will need to protect themselves through a dynamic hedging strategy. Even if this is true, however, at least there will be publicly available information about the number of outstanding put options. Astute observers can "reverse engineer" the dynamic

²⁹ A major portfolio insurance firm (LDR) and the NYFE had requested the right to publicize large futures sales in advance. The theory behind the request was that preannouncement would allow time for the market to organize a competitive response. An objection by a major futures exchange claiming that preannouncement was equivalent to prearranged trading led to delays in approval by the CFTC.
strategies which the open interest in such options imply. If this information is widely disseminated, we will have nearly-universal observation of PI strategies.

Short of all investors being aware of hedging plans, our analysis also shows that the stability of markets is strongly affected by the number of supply-informed traders who can observe these plans. These "market makers" play a role far beyond their numbers in increasing market liquidity. The crash which occurred in our example with no investors observing hedging could have been prevented if there had been as few as 0.03% supply-informed investors (given $\omega = .05$) observing hedging supply PI.

To the extent that stock-exchange specialists have privileged access to information on the nature of order flows, they play a key role in providing stability. Rules which limit free-entry to this activity will leave markets considerably more vulnerable than otherwise. Electronic "open books" should be a seriously considered reform, and other forms of market organization (such as single-price auctions) should be examined.

Would price limits help? The answer is "no"—unless such limits (and the trading halts caused by their being reached) permitted better dissemination of information on hedgers' selling. Absent this, price limits would only delay the ultimate crash by a bit, without modifying its magnitude. Certainly the market did not seem to benefit from the "trading halt" created by the weekend of October 17th and 18th.
VIII. Conclusion

We have shown that information differences amongst market participants can cause financial markets to be relatively illiquid. A small unobserved supply shock can create a large fall in prices. This is because the fall in prices affects investors' expectations as well as their budgets. Traditional models which do not recognize that many investors are uninformed will grossly overestimate the liquidity of stock markets.

A consequence of diminished liquidity is that even relatively small unobserved supply shocks from hedging programs can have a destabilizing effect. We developed an example in which a market crash occurred when only 5% of investors were following an insurance-replicating hedging program. The model provides a very general analysis of when markets can have crashes.

Our model also suggests some policies to minimize the chance of future crashes. These include the wide dissemination of knowledge about hedgers' actions, and the use of put options or related securities which provide hedging without requiring dynamic trading. This recommendation supports a similar contention by Grossman [1988]. Allowing wider access to the information in specialists' books might also help to stabilize the market. In contrast, price limits are unlikely to have useful effects unless combined with greater dissemination of trading information at the time limits are reached.
VI Appendix

Notations and example parameters in parenthesis

Prices

$p_0$: current equilibrium price
$p$: realized end of period price
$p$: unconditional expected end of period price (1.06)
$p$: investor i's expected end of period price
$\Sigma$: unconditional variance of end of period price (0.08)
$Z_j$: class j investor conditional variance of p
$Z$: market power-weighted average conditional variance of p

Information

$m$: supply of shares divided by the sum of risk tolerance coefficients, variance $\Sigma_m$ (0.00034)
$p_i^T$: price signal observed by investor i in class I
$\epsilon_i$: price signal noise, uncorrelated across investors, uncorrelated with other random variables, ex-ante variance $\Sigma$, (0.4)
$S$: supply signal observed by investors S1, variance $\Sigma_S$ (0.00017)
$L$: m-S liquidity supply, variance $\Sigma_L$ (0.00017), L and S are independent

Investors

$S1$: supply informed investor class, observe $p_0$ and i
$I$: price informed investor class, observe $p_0$ and $p$
$U$: uninformed investor class, observe $p_0$
$a_j$: investor class j risk tolerance
$w_j$: number of investors in class j
$k_j$: relative market power of class j: ratio of the products of $w_j$ and $a_j$ to the sum across classes: $k_j = a_j w_j / \sum a_j w_j$ ($k_i$: 0.02, $k_{SI}$: 0.005 $k_j$: 0.975)
$PI$: hedging share supply
$\omega$: fraction of share total hedged (5%)

Rational Expectation Equilibrium (REE) prices

$p_0 = F \{ p-p - H L - I S \}$

Auxiliary variables

$K_L = \Sigma / k_{SI}$
$\rho_p = \Sigma / \Sigma_p$: price signal to noise ratio (0.2)
$\rho_s = \Sigma / \Sigma_L$: supply signal to noise ratio (1)
The variance-covariance matrix of \[
\begin{bmatrix}
    p \\
p' \\
S \\
F^{-1}(p_0)
\end{bmatrix}
\] is given by:

\[
\begin{bmatrix}
\Sigma & \Sigma & 0 & \Sigma \\
\Sigma & \Sigma + \Sigma_\varepsilon & 0 & \Sigma \\
0 & \Sigma & -\Sigma_\varepsilon & \Sigma + H^2 \Sigma_L + I^2 \Sigma_\varepsilon \\
\Sigma & \Sigma & -\Sigma_\varepsilon & \Sigma + H^2 \Sigma_L + I^2 \Sigma_\varepsilon
\end{bmatrix}
\]

Denote \( V \) the 3*3 variance-covariance submatrix of the signals and \( \Delta \) its determinant, we have:

\[
\Delta = \Sigma_{p'} (\Sigma_0 \Sigma_0' - \Sigma_0^2) - \Sigma \Sigma_0^2.
\]

\[
V^{-1} \Delta = \begin{bmatrix}
\Sigma_0 \Sigma_0' & \Sigma_0 \Sigma_0' & -\Sigma_0 \Sigma_0' \\
\Sigma_0 \Sigma_0' & \Sigma_0 \Sigma_0' & \Sigma_0 \Sigma_0' \\
-\Sigma_0 \Sigma_0' & \Sigma_0 \Sigma_0' & \Sigma_0 \Sigma_0'
\end{bmatrix}
\]

where \( \Sigma_x \) and \( \Sigma_{xy} \) denote the variance of variable \( x \) and its covariance with \( y \) respectively with \( x=p, S \) and \( F^{-1}(p_0) \)(denoted 0).

The distribution of end of period prices conditional on all three signals is normal with expectation and variance \(^31\):

\[
\begin{align*}
\bar{p}_{II} &= \bar{p} + \text{Cov}(p;(p',S,F^{-1}(p_0)))^t \cdot V^{-1} \cdot \begin{bmatrix}
p' - \bar{p} \\
S \\
F^{-1}(p_0)
\end{bmatrix} \\
\bar{p}_{II} &= \bar{p} + (A_{II}, B_{II}, C_{II})^t \\
Z_{II} &= \Sigma - \text{Cov}(p;(p',S,F^{-1}(p_0)))^t \cdot V^{-1} \cdot \text{Cov}(p;(p',S,F^{-1}(p_0))) \\
Z_{II} &= \Sigma - (A_{II} \Sigma + C_{II} \Sigma)
\end{align*}
\]

\(^{31}\) See Mood and Graybill(1963), for example.
where $'$ denotes the transpose of a vector or matrix, and the second equation defines the coefficients $A_{II}$, $B_{II}$, and $C_{II}$. We dropped the subscript for the price signal $p'$ which differs across investors.

Straightforward and lengthy (!) manipulation of the equations leads to:

$$Z_{II} = \left[ \frac{1}{\Sigma} + \frac{1}{\Sigma_{e}} + \frac{1}{H^2 \Sigma_{L}} \right]^{-1}$$

$$Z_{II}^{-1} A_{II} = \frac{1}{\Sigma_{e}}$$

$$Z_{II}^{-1} B_{II} = \frac{1}{H^2 \Sigma_{L}}$$

$$Z_{II}^{-1} C_{II} = \frac{1}{H^2 \Sigma_{L}}$$

These parameters would obtain for an investor who could observe all the signals. To derive the corresponding parameters for the supply informed investors, SI, it suffices to take the limit of $\Sigma_{e}$ at infinity. For investors I and U, the parameters are obtained by replacing $H^2 \Sigma_{L}$, the variance of the noise term in $F^{-1}(p_{o})$, with $H^2 \Sigma_{L} + I^2 \Sigma_{S}$ in the expression for the corresponding parameters for II and SI respectively.

It yields:

$$Z_{SI} = \left[ \frac{1}{\Sigma} + \frac{1}{H^2 \Sigma_{L}} \right]^{-1}$$

$$Z_{SI}^{-1} A_{SI} = 0$$

$$Z_{SI}^{-1} B_{SI} = \frac{1}{H^2 \Sigma_{L}}$$

$$Z_{SI}^{-1} C_{SI} = \frac{1}{H^2 \Sigma_{L}}$$

$$Z_{I} = \left[ \frac{1}{\Sigma} + \frac{1}{\Sigma_{e}} + \frac{1}{H^2 \Sigma_{L} + I^2 \Sigma_{S}} \right]^{-1}$$

$$Z_{I}^{-1} A_{I} = \frac{1}{\Sigma_{e}}$$

$$Z_{I}^{-1} B_{I} = 0$$

$$Z_{I}^{-1} C_{I} = \frac{1}{H^2 \Sigma_{L} + I^2 \Sigma_{S}}$$

$$Z_{U} = \left[ \frac{1}{\Sigma} + \frac{1}{H^2 \Sigma_{L} + I^2 \Sigma_{S}} \right]^{-1}$$

$$Z_{U}^{-1} A_{U} = 0$$

$$Z_{U}^{-1} B_{U} = 0$$

$$Z_{U}^{-1} C_{U} = \frac{1}{H^2 \Sigma_{L} + I^2 \Sigma_{S}}$$
The corresponding market power weighted averages are given by:

\[ Z^{-1} = \sum_j k_j Z_j^{-1} = \frac{1}{\Sigma} + \frac{1 + H \Sigma_c + I \Sigma_s}{H^2 \Sigma_L + I^2 \Sigma_s} \]

\[ A = \sum_j k_j Z_j^{-1} A_j = \frac{k_j}{\Sigma_e} \]

\[ B = \sum_j k_j Z_j^{-1} B_j = k_{SI} \frac{I}{H^2 \Sigma_L} \]

\[ C = \sum_j k_j Z_j^{-1} C_j = k_{SI} \frac{I}{H^2 \Sigma_L} + \frac{k_j + k_{SI}}{H^2 \Sigma_L + I^2 \Sigma_s} \]

Proof of the theorem

The total demand for shares of the three classes of investors is equal to the total supply plus hedging supply:

\[ \sum_j k_j Z_j^{-1} (\bar{p}_j - p_0) = m + PI \]  \hspace{1cm} (VI.1)

Reorganizing terms yields, at the limit of economies with an infinite number of agents:

\[ \frac{Z^{-1} p_0 + PI - C F^{-1}(p_0) - Z^{-1} \bar{p} + \bar{m}}{A} = \frac{1}{A} \bar{p} - \frac{1}{A} L - \frac{1-B}{A} S \]  \hspace{1cm} (VI.2)

This equation is consistent with equation II.3 if and only if the following set of equations holds:

\[ H = \frac{1}{A}, \quad I = \frac{1-B}{A}, \quad F^{-1}(p_0) = \frac{Z^{-1} p_0 + PI - Z^{-1} \bar{p} + \bar{m}}{A + C} \]

This system of equations has a unique solution provided that assumption A1 holds.

The solutions are given by:

\[ H = \frac{\Sigma_e}{k_{SI}}, \quad I = H - \frac{H}{HK + 1}, \quad F^{-1}(p_0) = \frac{Z^{-1} p_0 + PI - Z^{-1} \bar{p} + \bar{m}}{Z^{-1} - \Sigma^{-1}} \]

with

\[ K_L = \frac{\Sigma_e}{k_{SI}} \]
Excess demand

Substitution of the solutions in equation VI.2 yields the excess demand (demand minus supply):

\[ XD_A = \frac{1}{\lambda} \left\{ p - \bar{p} - H L - I S + \frac{Z^{-1}(\bar{p} - p_0) - (\bar{m} + PI)}{Z^{-1} - \Sigma^{-1}} \right\} \]

The linear case

When the demand stemming from dynamic strategies is linear in \( p_0 \) (i.e. \( PI' \) is constant), \( F(z) \) is linear. In the case of no hedging supply (\( PI=0 \)):

\[ F(z) = \frac{Z^{-1} - \Sigma^{-1}}{Z^{-1}} z + \frac{Z^{-1} \bar{p} - \bar{m}}{Z^{-1}} = F(z) + f. \]

In this case, \( f \) is equal to the ex ante expectation of \( p_0 \).
Aggregate Excess Demand

Figure 1
Figure 2 1% downward shock
Figure 3 1.6% downward shock
Figure 4: 1.8% downward shock
Figure 5 15% portfolio insurance
Aggregate Excess Demand

2.6% upward shock

Figure 6 2.6% upward shock
Figure 7 Equilibrium price
References


