Consumption and Portfolio Policies
With Incomplete Markets and
Short-Sale Constraints:
The Finite Dimensional Case

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Abstract

We employ a martingale approach to study a consumption-portfolio problem in a discrete time model with dynamically incomplete markets and short-sale constraints. In doing so we show how the approach is useful when markets are incomplete. The solution of the original dynamic problem is identical to the solution of a static problem of choosing among consumption bundles satisfying budget constraints formed using all Arrow-Debreu state prices consistent with no arbitrage. The budget feasible set can be defined by finitely many constraints even though infinitely many state price vectors are consistent with no arbitrage, and the approach is convenient for computation.
1 Introduction

The classical approach to the analysis of optimal intertemporal consumption and portfolio policies is stochastic dynamic programming, the use of which in this context was pioneered by Mossin (1968), Samuelson (1969), and Hakansson (1970) in discrete time and Merton (1969, 1971) in continuous time. Recently Pliska (1982, 1986), Cox and Huang (1987a, 1987b), and Pagès (1987) have used a martingale representation technology instead of dynamic programming to study optimal intertemporal consumption and portfolio policies, while Chamberlain (1988), Duffie and Huang (1985), and Huang (1987) have used it in a general equilibrium setting. However, to date little is known about how the martingale approach may be useful when markets are dynamically incomplete. With the exception of the paper by Pagès, those mentioned above assume that markets are complete, while Pagès makes an assumption about the asset price process and the nature of the incompleteness that excludes the cases of real interest.

In this paper we use the martingale approach to study optimal intertemporal consumption and portfolio policies in a general discrete time, discrete state space "finite dimensional" economy with dynamically incomplete markets and short-sale constraints. A dynamically incomplete (henceforth simply "incomplete") market is one in which not all contingent claims can be created by dynamic trading in the existing securities. We show how the original dynamic problem can be reformulated as a static problem which may then be attacked with the Lagrangian theory. The static formulation is useful for computation, has a natural economic interpretation, and has a geometry which makes it easy to understand the effect of incomplete markets. While our results of course apply only to the finite dimensional case, we impose no additional assumptions on either the asset price process or the nature of the incompleteness of markets.

In the martingale approach one solves the consumption and portfolio problem by separating it into two parts. First one identifies the set of attainable consumption bundles and solves a static optimization problem in order to select the consumption bundle most preferred by the consumer-investor. Then one determines the trading strategy needed to generate the most preferred consumption bundle. Pliska (1982) explicitly carries out these computations for a simple example in a discrete time finite probability space in which the agent consumes only at a single terminal date. Pliska (1986) extends the analysis to a more general continuous time stochastic environment (though again with consumption only at a single terminal date) in which asset prices are semi-martingales and consumption can be either positive or negative.

Cox and Huang (1987a) allow intermediate consumption and provide an easily verifiable set of sufficient conditions for existence weaker than those required in dynamic programming when
asset prices follow a diffusion process. They find the unique set of Arrow-Debreu state prices (or, after normalization by the bond price, the unique equivalent martingale measure) that is consistent with the absence of arbitrage, and show that the solution of a static problem of maximizing utility subject to a single budget constraint formed using this set of state prices and the portfolio policies needed to implement it are identical to the optimal consumption and portfolio policies given by dynamic programming. Cox and Huang (1987b) characterize the optimal policies and compute explicitly the optimal consumption and portfolio policies in certain situations in which it is difficult if not impossible to use dynamic programming. An appealing feature of their approach is that the Lagrangian theory may be used to study the static problem.

While Cox and Huang only consider the continuous time case, the use of the martingale approach in the discrete time finite dimensional case with complete markets follows immediately from their analysis. Just as in the continuous time case, one finds the Arrow-Debreu state prices implied by the asset prices, solves a static problem of maximizing utility subject to a budget constraint formed using the state prices, and then implements the solution of the static problem.

A limitation of the analyses of Pliska and Cox and Huang is that these authors assume that markets are complete. In the papers by Cox and Huang the completeness of markets gives a unique set of Arrow-Debreu state prices or a unique equivalent martingale measure that is used to form the budget constraint in the static problem that comprises the first part of the martingale approach. When markets are incomplete there are infinitely many sets of state prices or equivalent martingale measures that are consistent with the absence of arbitrage, and the static problem of maximizing utility subject to the requirement that consumption be feasible with respect to budget constraints formed using all state prices consistent with no arbitrage involves infinitely many budget constraints. Hence it is not immediately obvious that the Lagrangian theory can be used to attack the problem.

The problem we study is that of a consumer-investor who selects his most preferred consumption bundle from among those that can be generated by a dynamic trading strategy using his endowment and the available securities. We first characterize the set of Arrow-Debreu state prices that are consistent with the given securities prices and the absence of arbitrage, and then show that the set of consumption bundles that can be generated by a dynamic trading strategy is identical to the set of consumption bundles that are budget feasible with respect to all of the state prices consistent with no arbitrage. Therefore the solution of the agent's original problem is identical to the solution of a static problem of maximizing utility subject to the constraint that the consumption bundle lies in the budget feasible set.

This static problem involves infinitely many budget constraints. We show that the set of
feasible consumption bundles is generated by a finite number of budget constraints corresponding to the extreme points of the closure of the set of state prices consistent with no arbitrage. That is, the budget feasible set can be defined by a finite number of constraints even when markets are incomplete and there are infinitely many state price vectors consistent with no arbitrage. Hence the solution of the consumer-investor’s problem can be easily characterized using the Lagrangian theory and numerical solutions can be computed using standard nonlinear programming techniques. We find that the set of feasible consumption bundles has a nice geometry and a pleasing interpretation in terms of prices. We also extend our results to include short-sale constraints.

If the agent’s preferences have an expected utility representation the optimal consumption plan may be obtained using dynamic programming. In this case the agent’s original program and the static problem we obtain have identical solutions. An advantage of our approach vis-a-vis dynamic programming is that our approach requires significantly less computation.

In our finite dimensional setting, a static formulation of Breeden (1987) represents another alternative to the martingale approach. In our setting the relationship between consumption and the trading strategies (shares of securities held in various states) is given by a set of linear equations. Breeden uses these to eliminate consumption from the problem and obtains a static problem in which maximization is performed with respect to the shares of the securities held. Breeden’s approach has the same computational advantages vis-a-vis the dynamic programming that ours has. However, Breeden’s approach does not provide the interpretation of the feasible consumption set in terms of the prices of consumption in the various states that ours does. Our formulation is also more convenient for computation with the utility functions commonly assumed in finance. In addition, our characterization of the solution of the general problem makes it easy to obtain closed form solutions for the optimal consumption policy in the case of an agent with a time-additive, state-independent utility function of the hyperbolic absolute risk aversion (HARA) class (with no nonnegativity restrictions on consumption) when the investment opportunity set is constant and there are no short-sale constraints (see Hakansson (1970) for these solutions obtained via dynamic programming).

The balance of the paper is organized as follows. Section 2 presents an example with which we introduce the main ideas of the paper and illustrate the geometry of our approach. Section 3 describes the economy and its vector space representation and introduces most of the notation. We exploit the vector space representation to show that the set of consumption bundles that can be generated by a dynamic trading strategy using the agent’s endowment and the available securities is identical to the set of consumption bundles that are budget feasible with respect to all of the state prices that are consistent with the absence of arbitrage, and show that the budget feasible
set can be defined by a finite number of constraints. This program is carried out in Section 4. In Section 5 we present a simple characterization of the solution of the agent’s problem, indicate how to implement the optimal consumption policy using a dynamic trading strategy, and discuss the advantages of our approach for computations. Section 6 consists of an example which illustrates our approach. Section 7 contains a few concluding remarks.

2 An Example

We consider an economy with one consumption good, only two dates \( t = 0,1 \) and three possible outcomes \( \{ \omega_1, \omega_2, \omega_3 \} \) at time 1. In order to be able to graph the feasible consumption set we require that the good be consumed only at time 1, and use \( c = (c_1, c_2, c_3)' \) to denote consumption in the three states.

As a point of departure, we first suppose that there are three securities available for trading, so markets are complete. These securities have current prices \( S_0^1 = 1, S_0^2 = 2, \) and \( S_0^3 = 3, \) and state contingent payoffs at time 1 of \( S_1^1 = (1,1,1), S_1^2 = (3,2,1), \) and \( S_1^3 = (1,3,5). \) The given securities prices imply that the Arrow-Debreu state prices are \( (1/3,1/3,1/3). \) If we require that consumption be nonnegative and let the agent have an endowment of 1 at time 0, the feasible consumption set is

\[
\{ c \in \mathbb{R}^3 \mid c \geq 0, 1/3c_1 + 1/3c_2 + 1/3c_3 \leq 1 \}. 
\]

This set is shown in Figure 1.

Now suppose that the third security is not available for trading. When only the first two securities are available for trading, the feasible consumption set is defined by the inequalities

\[
c \leq \varphi_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \varphi_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \tag{1}
\]

\[
\varphi_1 + 2\varphi_2 \leq 1, \tag{2}
\]

where \( \varphi_1 \) and \( \varphi_2 \) denote the number of shares of the two securities. The first inequality says essentially that the agent’s consumption must be spanned by the payoffs of the two securities,\(^1\) and the second inequality is the budget constraint.

Equations (1) and (2) imply

\(^1\)The inequality appears because we have not yet excluded the possibility that the agent becomes satiated; we do this below.
\[c_2 \leq 1, \quad 1/2c_1 + 1/2c_3 \leq 1,\] (3) (4)

so that when only the two securities are available for trading the feasible consumption set is

\[\{c \in \mathbb{R}^3 \mid c \geq 0, c_2 \leq 1, 1/2c_1 + 1/2c_3 \leq 1\}.\]

The bundles a nonsatiated agent might choose lie in the intersection of the two planes \(c_2 = 1, 1/2c_1 + 1/2c_3 = 1\), or in the segment connecting \((0, 1, 2)\) and \((2, 1, 0)\). This segment is also shown in Figure 1. It is a subset of the feasible consumption set for the complete markets case.

The "1" on the right hand side of the two inequalities (3) and (4) is the initial wealth. The coefficient "1" on \(c_2\) in the inequality \(c_2 \leq 1\) and the coefficients "1/2" and "1/2" on \(c_1\) and \(c_3\) in the inequality \(1/2c_1 + 1/2c_3 \leq 1\) have a natural economic interpretation. These coefficients can be thought of as the prices of consumption in the two states. To see this, note that the least costly way to obtain an additional unit of consumption in the state \(\omega_1\) without decreasing consumption in any state is to take a long position of 1/2 unit of the second security and a short position of 1/2 unit of the first. The cost of this portfolio is 1/2. The least costly way to obtain an additional unit of consumption in the state \(\omega_2\) without decreasing consumption in any state is to take a long position of 3/2 units of the first security and a short position of 1/2 unit of the second security. The cost of this position is 1/2. Both of these positions yield a positive amount in the second state, which is why the coefficient on \(c_2\) is zero in the inequality \(1/2c_1 + 1/2c_3 \leq 1\). That is, the two portfolio strategies that generate consumption in the first and third states also generate consumption in the second state, so if one is undertaking either of these strategies there is a range of state \(\omega_2\) consumption for which the cost of consumption in that state is zero.

An additional unit of consumption in state \(\omega_2\) can be obtained either by purchasing 1 unit of the first security or 1/2 unit of the second security. Both of these strategies have a cost of one, and both also yield positive amounts in the first and third states.

It turns out that we can make precise the interpretation of the coefficients of the two inequalities as prices. When markets are incomplete there is not a unique state price vector. Rather, there are infinitely many state price vectors consistent with the absence of arbitrage. In our example, the state prices \(\pi = (\pi_1, \pi_2, \pi_3)\) consistent with the absence of arbitrage satisfy

\[\pi_1 + \pi_2 + \pi_3 = 1,\] (5)
\[ 3\pi_1 + 2\pi_2 + \pi_3 = 2, \quad (6) \]

\[ \pi_1 > 0, \quad (7) \]

\[ \pi_2 > 0, \quad (8) \]

\[ \pi_3 > 0, \quad (9) \]

or

\[
\begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix} = \alpha \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} + (1 - \alpha) \begin{pmatrix}
1/2 \\
0 \\
1/2
\end{pmatrix},
\]

where \(0 < \alpha < 1\).

In the complete markets case one defines the set of feasible consumption bundles by forming a budget constraint using the single state price vector. Reasoning by analogy with the complete markets case, one might conjecture that when markets are incomplete one would define the feasible set by forming budget constraints using all of the (infinitely many) state price vectors consistent with the absence of arbitrage. This is a correct definition of the feasible set, but most of the constraints are redundant. We can define the feasible set using only the extreme points of the closure of the set of state price vectors consistent with the absence of arbitrage. These extreme points are \((0,1,0)\) and \((1/2,0,1/2)\). The components of these vectors are the coefficients on \(c_1\), \(c_2\), and \(c_3\) in the inequalities (3) and (4).

This result provides a natural interpretation of the feasible consumption set in terms of prices and budget constraints. The agent's consumption in the second state, for example, is constrained by the fact he must sacrifice one unit of his initial wealth in order to increase his state \(\omega_2\) consumption by one unit. This interpretation makes it much easier to understand the feasible consumption set. In addition, it is easy to see exactly how closing markets affects the feasible set. In our example, disallowing trading in the third security increased the dimension of the linear span of the set of state prices consistent with no arbitrage from one to two and added another plane defining the feasible consumption set.

The extreme points are relatively easy to calculate even for large problems. Once we have them, the agent's problem is simply to maximize utility subject to budget constraints formed using the extreme points. The problem is almost identical to the complete markets case, the only difference being that there is more than one budget constraint when markets are incomplete.

Our approach also works when there are short-sale constraints. If we now suppose that the second security may not be sold short, the set of feasible consumption bundles is defined by the inequalities
\[ c \leq \varphi_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \varphi_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \]

\[ \varphi_1 + 2\varphi_2 \leq 1, \]
\[ \varphi_2 \geq 0. \]

We obtain

\[ c_2 \leq 1, \]
\[ 1/2c_1 + 1/2c_3 \leq 1, \]
\[ c_3 \leq 1. \]

The last of these inequalities comes from the short-sale constraint. The triple \((0, 0, 1)\), the components of which are the coefficients on \(c_1, c_2, \text{ and } c_3\) in the last inequality, is also an extreme point. The bundles a nonsatiated agent might choose lie in the segment connecting \((1, 1, 1)\) and \((2, 1, 0)\). We defer discussion of the details of short-sale constraints and simply observe that \((0, 0, 1), (0, 1, 0)\) and \((1/2, 0, 1/2)\) are the extreme points of the set of state prices consistent with no arbitrage when the second security may not be sold short. The triples \((0, 1, 0)\) and \((1/2, 0, 1/2)\) satisfy (5)–(9) above, while \((0, 0, 1)\) satisfies \(3\pi_1 + 2\pi_2 + \pi_3 \leq 2\) along with (5) and (7)–(9). Although the role of these relations will not be clear until Section 4, we note that the securities prices are supermartingales with respect to the measure \(\pi = (0, 0, 1)\) and martingales with respect to \((0, 1, 0)\) and \((1/2, 0, 1/2)\).

### 3 The Finite Market Economy

We consider the following model of an agent's intertemporal consumption and portfolio policy under uncertainty. There is a finite number \(N\) of states of the world, indexed by \(\omega \in \Omega\). There is a finite number of time periods, indexed by \(t = 0, 1, 2, \ldots, T\). The information in the economy is exogenously specified and is represented by a sequence of partitions of \(\Omega\), \(\{F_t; t = 0, \ldots, T\}\). The interpretation is that at time \(t\) the agent knows which cell of \(F_t\) contains the true state. Information increases through time; \(F_{t+1}\) is at least as fine as \(F_t\). Without loss of generality we assume \(F_0\) is trivial and \(F_T\) is the discrete partition (i.e., \(F_0 = \Omega\) and \(F_T = \{\omega | \omega \in \Omega\}\)). The \(\sigma\)-field of events generated by \(F_t\)
is denoted \( \mathcal{F}_t \), and the family of \( \sigma \)-fields \( \mathcal{F} = \{ \mathcal{F}_t; t = 0, 1, \ldots, T \} \) is called the filtration generated by the sequence of partitions \( F_t \). This information structure can be easily and intuitively represented by an event tree, and we make use of this representation when it is convenient to do so.

There is a single consumption good which the agent consumes at each date and cannot store. The agent's consumption can depend only upon the information he has at time \( t \). We formalize this by defining a consumption bundle to be a stochastic process \( c = \{ c_t; t = 0, 1, \ldots, T \} \) that is adapted to \( \mathcal{F} \); this means that \( c_t \) is measurable with respect to \( \mathcal{F}_t \). Similarly, the agent receives a wage income stream which is an \( \mathcal{F} \)-adapted stochastic process \( y = \{ y_t; t = 0, 1, \ldots, T \} \).

At each date there are \( K + 1 \) securities available for trading in frictionless markets, with \( K \geq 0 \). It is convenient, but certainly not necessary, to assume that all \( K + 1 \) securities are available for trading at each date. Only trivial modifications are needed if some securities are not available for trading at some dates, and we indicate them below. We use \( S^k_t \) to denote the price of security \( k \) at time \( t \), and use \( S = \{ S_t; t = 0, 1, \ldots, T \} \), where \( S_t = (S^0_t, S^1_t, \ldots, S^K_t)' \), to denote the securities price process. For convenience we assume that the securities pay no dividends; this affects nothing. We assume that the securities price process \( S \) is adapted to \( \mathcal{F} \). We make no further assumptions on the price system besides requiring that \(|S^k_t| < +\infty \) for \( t = 0, 1, \ldots, T \) and \( k = 0, 1, 2, \ldots, K \).

In particular, we require neither that one of the securities be a riskless bond nor that there are sufficiently many linearly independent securities that markets are (dynamically) complete.

The agent's problem is to manage a portfolio of these \( K + 1 \) securities in order to obtain his most preferred consumption plan. We use a vector space representation to formalize this as follows.

Let \( N_t = |F_t| \), the number of cells (or nodes or events) in partition \( F_t \) at date \( t \). Clearly \( N_0 = 1 \) and \( N_T = N \). If we let \( L = N_0 + N_1 + \ldots + N_T \), then \( L \) is the total number of cells from time 0 to time \( T \). It is clear that any \( \mathcal{F} \)-adapted stochastic process can be characterized by its values at these \( L \) cells. This allows us to represent any \( \mathcal{F} \)-adapted process by an \( L \)-dimensional Euclidean vector space \( \mathbb{R}^L \) with its coordinates properly defined. We allocate the first coordinate to date 0's cell, the next \( N_1 \) coordinates to date 1's cells, the next \( N_2 \) coordinates to date 2's cells, etc.

Now we can see that any consumption bundle can be viewed as a vector in \( \mathbb{R}^L \) with each coordinate representing the consumption in some cell of some partition (in some state at some date). Similarly, a wage income stream can also be viewed as a vector in \( \mathbb{R}^L \). From now on we use \( c, y \in \mathbb{R}^L \) to denote consumption bundles and income streams respectively.

A trading strategy is a predictable \( K + 1 \) dimensional process \( \varphi = \{ \varphi_t; t = 1, \ldots, T \} \), where \( \varphi_t = (\varphi^0_t, \varphi^1_t, \ldots, \varphi^K_t)' \). The components of the trading strategy \( \varphi^k_t \) can be interpreted as the number of shares of security \( k \) held by the investor between \( t - 1 \) and \( t \). The assumption that the trading strategy is predictable means that \( \varphi_t \) is measurable with respect to \( \mathcal{F}_{t-1} \). This is the natural
information constraint.

We can model the trading strategy as a vector in a Euclidean space in a fashion similar to the way we handled the consumption bundles and wage income streams. Since at each time \( t < T \) the \( K+1 \) components of the trading strategy \( \varphi^0, \varphi^1, \ldots, \varphi^K \) have to be determined at each of the \( N_i \) cells of partition \( F_1 \), the total number of values to be determined is \( M = (K + 1)(N_0 + N_1 + \cdots + N_{T-1}) \). Thus a trading strategy can be represented by a vector \( \varphi \in \mathbb{R}^M \). Each of the coordinates of \( \varphi \) represents a long or short position in one of the securities at some date \( t < T \) and subset of \( \Omega \). We allocate the first \( K + 1 \) coordinates to the investment strategies a date 0, the next \( K + 1 \) coordinates to the investment strategies at the first cell of \( F_1 \) at date 1, the next \( K + 1 \) to be the strategies at the second cell of \( F_1 \), etc.

The trading strategies and payoffs are related through a payoff matrix \( X \in \mathbb{R}^{L \times M} \). For \( t = 0, 1, \ldots, T - 1 \), we think of each security at each date and cell of \( F_t \) as a distinct investment opportunity. We have already seen that there are \( M = (K + 1)(N_0 + N_1 + \cdots + N_{T-1}) \) such opportunities. We represent each such investment opportunity as a vector in \( \mathbb{R}^L \), where we recall that \( L = N_0 + N_1 + \cdots + N_T \), so that a vector in \( \mathbb{R}^L \) can represent the payoff of an investment in each cell of \( F_t \) for each date \( t = 0, 1, \ldots, T \). As with \( c \) and \( y \), we allocate the first coordinate to date 0's cell, the next \( N_1 \) coordinates to date 1's cells, the next \( N_2 \) coordinates to date 2's cells, etc.

We use \( x^m \in \mathbb{R}^L \) to denote the vector corresponding to the \( m \)-th investment opportunity and use \( x^m(A_t) \) to denote the element of \( x^m \) corresponding to \( A_t \in F_t \). Suppose the \( m \)-th investment opportunity \( x^m \) is the opportunity to invest in the \( k \)-th security in the event \( A_t \). Following Breeden (1987), we represent this opportunity by setting \( x^m(A_t) = -S^k_t(A_t) \), setting \( x^m(A_{t+1}) = S^k_{t+1}(A_{t+1}) \) if \( A_{t+1} \subset A_t \), and setting \( x^m(A_{t+1}) = 0 \) if \( A_{t+1} \not\subset A_t \). All other elements of \( x^m \) (i.e., those corresponding to other dates) are set equal to zero. For a common stock, the vector representing an investment opportunity will have one negative element corresponding to the cost of buying the stock at a particular date and state and a number of positive elements corresponding to the payoffs from closing out the position next period. The matrix \( X \) is formed by adjoining the \( M \) column vectors \( x^m \), with the \( m \)-th column of \( X \) corresponding to the \( m \)-th investment opportunity.

Each component of the vector \( \varphi \) also corresponds to one investment opportunity. The consumption bundle produced by a trading strategy \( \varphi \) given a matrix \( X \) and income vector \( y \) is \( c = X \varphi + y \). If certain securities are not available for trading in certain events at certain dates, there is no vector \( x^m \) corresponding to the missing investment opportunities and we form the matrix \( X \) by adjoining \( M < (K + 1)(N_0 + N_1 + \cdots + N_{T-1}) \) columns \( x^m \). Similarly, the vector \( \varphi \) would contain no component corresponding to the missing investment opportunity.
We use $C(y)$ to denote the set of feasible consumption bundles that can be generated by a
dynamic trading strategy using income $y$. That is,
\begin{equation}
C(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \exists \varphi \in \mathbb{R}^M \text{ s.t. } c = X\varphi + y \}. \tag{10}
\end{equation}
The equation $c = X\varphi + y$ appears with an equality instead of an inequality because we will assume
below that the agent can never be satiated and therefore will never choose a consumption bundle
such that $c \leq X\varphi + y$ and $c \neq X\varphi + y$.\footnote{An earlier version of this paper also considers the case where the agent may reach satiation and the set of feasible consumption bundles is defined $C(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \exists \varphi \in \mathbb{R}^M \text{ s.t. } c \leq X\varphi + y \}$ and the budget feasible set (see equation (13) below) is defined $B(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \exists \pi \in \Pi, \pi(c - y) \leq 0 \forall \tau \in \Pi \}$. We obtain identical results using slightly more elaborate arguments. This earlier version also contains a more extensive discussion of computational issues, and is available from the authors upon request.} Since $\varphi$ represents a dynamic trading strategy, it is easy
to see that $C(y)$ is identical to the set of feasible consumption bundles in a dynamic programming
formulation of the consumption and portfolio problem. Were all of the elements of $y$ save the first
zero the condition $c = X\varphi + y$ would be the usual self-financing constraint. The agent chooses
among various consumption bundles $c$.

Specifically, we consider the consumer-investor's problem
\begin{equation}
\max_{c \in C(y)} u(c, p), \tag{11}
\end{equation}
where $p$ is an $L$–vector of the probabilities of the various states and $u(c, p)$ represents a non-
decreasing utility function. We assume that the agent's preferences are time consistent in the sense
that a consumption plan that is initially optimal remains optimal at all future dates and states.
Weller (1978) shows that a sufficient condition for time consistency is that the agent's preferences
have an expected utility representation with respect to the uncertainty remaining at every date and
state, but this condition is stronger than is necessary [Donaldson, Rosman, and Selden (1980),
Johnsen and Donaldson (1985)]. Time additive functions $u(c, p) = u(c_1) + \sum_{i=2}^{L} p_i u(c_i)$ are of course
included in the class of utility functions we consider, but certainly do not exhaust the class.

An example will make the formulation clear. Consider a three date economy with two
securities and the following partitions of $\Omega$: $F_0 = \Omega$, $F_1 = \{ \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\} \}$, and $F_2 =
\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. One of the securities (security 0) is riskless and always has a price of
1. The other (security 1) is risky and has the following prices in the various cells of the partitions
of $\Omega$: $S^1_0(\Omega) = 2$; $S^1_1(\{\omega_1, \omega_2\}) = 4$, $S^1_1(\{\omega_3, \omega_4\}) = 1$; $S^1_2(\omega_1) = 8$, $S^1_2(\omega_2) = 2$, $S^1_2(\omega_3) = 2$, and $S^1_2(\omega_4) = 1/2$. The numbers of cells at each date are $N_0 = 1$, $N_1 = 2$, and $N_2 = 4$, and we have
$L = N_0 + N_1 + N_2 = 7$. Hence any consumption bundle can be represented by a vector in $\mathbb{R}^7$, i.e.,
$c = (c(\Omega), c(\{\omega_1, \omega_2\}), c(\{\omega_3, \omega_4\}), c(\{\omega_1\}), c(\{\omega_2\}), c(\{\omega_3\}), c(\{\omega_4\}))^T$, where the $c(\{\omega_i\})$ denote
consumption at the various dates and subsets of Ω. This securities price process can be represented by the event tree in Figure 2.

There are two investment opportunities at date 0, the riskless bond and the risky security. Similarly, at date 1, there are two investment opportunities at each of the cells {ω₁, ω₂} and {ω₃, ω₄}. Thus a trading strategy is φ ∈ ℝ⁶, and the payoff matrix X has six columns. For this example, we have

\[
X = \begin{pmatrix}
-1 & -2 & 0 & 0 & 0 & 0 \\
1 & 4 & -1 & -4 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 8 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 1/2
\end{pmatrix}
\]

The consumption bundle produced by a trading strategy φ given an income vector y is c = Xφ + y, and C(y) = {c ∈ ℝ⁷ | c ≥ 0, ∃φ ∈ ℝ⁶ s.t. c = Xφ + y).

Incomplete markets can be represented quite easily in this set-up. For example, if the risky security were not available for trading at time 1, the matrix X would be missing its fourth and sixth columns.

4 Characterization of the Set of Feasible Consumption Bundles

We now make precise and prove the claims made in Section 2. In particular, in this section we show that the set of feasible consumption bundles C(y) consisting of the c ≥ 0 that can be generated by a dynamic trading strategy using income y is identical to a set B(y) consisting of the c ≥ 0 that are budget feasible with respect to all of the state prices that are consistent with the absence of arbitrage, and then show that this set B(y) can be characterized using finitely many constraints. Hence the solution of (11) is identical to the solution of the static problem

\[
\max_{c ∈ B(y)} u(c, p),
\]

and the trading strategies that implement the solutions are identical.

Breeden (1987) used a representation of the set of feasible consumption bundles essentially identical to (10) in advancing a numerical technique to solve the consumption-portfolio problem as a static problem. Breeden used the equality c = Xφ + y to eliminate c and solved a static problem where the maximization was over φ. By characterizing the set of feasible consumption bundles in terms of the budget feasible set B(y) we provide an alternative characterization of the set of
feasible consumption bundles that both is more useful for computation and has a natural economic interpretation.

We first consider incomplete markets when short-sales of the available securities are permitted, and then turn to short-sale constraints.

4.1 Incomplete Markets

If our consumption-portfolio problem is to be well-posed we certainly do not want the securities price process to admit any arbitrage opportunities. That is, we want to exclude the possibility that something might be created from nothing, or that a trading strategy which produces positive payoffs and never requires any investment might exist. We begin by making precise this notion of an arbitrage opportunity.

Definition 1 An arbitrage opportunity is a vector \( \varphi \in \mathbb{R}^M \) such that \( X \varphi \geq 0 \) and \( X \varphi \neq 0 \).

This definition of an arbitrage opportunity is the same as the "simple free lunch" of Harrison and Kreps (1979). We assume that no such opportunities exist. That is, we assume that \( X \varphi \geq 0 \) implies \( X \varphi = 0 \).

We also define a set of state prices \( \Pi \), or, after normalization by the price of any security, a set of equivalent martingale measures:

\[
\Pi = \{ \pi \in \mathbb{R}^L \mid \pi X = 0, \pi > 0, \pi_1 = 1 \}.
\]

Here \( \pi_1 \) denotes the first component of the vector \( \pi \). We explain these after we present Proposition 1.

Proposition 1 If there are no arbitrage opportunities then \( \Pi \) is non-empty. Moreover the linear span of \( \Pi \) is identical to \( X^\perp \), the largest linear subspace orthogonal to the linear subspace spanned by the column vectors of \( X \). Therefore it has dimension \( L - \text{rank}(X) \), i.e. \( \dim(\text{span}(\Pi)) = L - \text{rank}(X) \).

Proof. A special case of Motzkin's Transposition Theorem [Schrijver (1986), Corollary 7.1k] says that there is a vector \( z \) with \( z > 0 \) and \( zA = 0 \) if and only if \( Aw \geq 0 \) implies \( Aw = 0 \). Letting \( A = X \) and \( w = \varphi \), there exists \( \hat{\pi} > 0 \) such that \( \hat{\pi}X = 0 \). If we normalize \( \hat{\pi} \) so that its first component \( \pi_1 = 1 \) the normalized vector \( \pi \) will be an element of \( \Pi \).

Since it is obvious that \( \Pi \subset X^\perp \), all we need do to prove the second claim is show that any \( y \) satisfying \( yX = 0 \) can be represented as a linear combination of elements of \( \Pi \). Let \( w = \hat{\pi} - \epsilon y \). For sufficiently small \( \epsilon \), \( w > 0 \) and \( w/w_1 \in \Pi \), so it is easy to see that \( y \) can be represented as a linear combination of the \( \hat{\pi} \) and \( w/w_1 \), i.e. \( y \in \text{span}(\Pi) \).

\[ \Box \]
The relationship between this proposition and the securities markets is as follows. If markets are dynamically complete then at each node of the event tree describing the economy there must be available for trading as many linearly independent securities as branches leaving the node, and the dimension of the linear span of $\Pi$ is one — i.e., there is a unique state price vector. Each column of $X$ corresponds to one investment opportunity or security at one node, so if there are fewer linearly independent securities than branches at some nodes $X$ will have fewer linearly independent columns than in the complete markets case. If we count each "missing" linearly independent security at each node as an unavailable market, the dimension of the linear span of $\Pi$, or $\dim X^\perp$, is equal to the number of unavailable markets plus one.

The $\pi \in \Pi$ can be interpreted as Arrow-Debreu state prices, and the requirement that $\pi_1 = 1$ says simply that the price of consumption at time 0 is one. These state prices comprise a measure on $\Omega$ such that when we integrate a security price $S^k_t$ over the intersection of the current cell and any partition $F_s$ ($s > t$) with respect to this measure the integral is equal to the current security price multiplied by the price of the current state. Letting $A_t$ denote an element of $F_t$, we have

$$\pi(A_t)S^k_t(A_t) = \sum_{A_{t+j} \in F_{t+j} \cap A_t} \pi(A_{t+j})S_{t+j}(A_{t+j}) \quad \forall t \in \{0, \ldots, T - 1\}, \forall j \in \{1, \ldots, T - t\}.$$  

If the elements of $X$ have been normalized by the price of any security, or if the rate of return on one of the securities is always zero, then the $\pi \in \Pi$ may also be interpreted as probabilities. In this case, for each $A_t \in F_t$ we have a security with unit current price and unit price in every state next period, so the $\pi$ indeed are probabilities. The set $\Pi$ will consist of the equivalent probability measures under which the securities prices are martingales. Letting $Q_\pi$ be the new probability measure given by $\pi$, $Q_\pi$ is equivalent to the original measure because $\pi > 0$ and the securities prices are martingales because

$$E^{Q_\pi}(S^{k*}(t+1) \mid F_t) = \sum_{A_{t+j} \in F_{t+j} \cap A_t} \frac{\pi(A_{t+j})}{\pi(A_t)} S^*_{t+1}(A_{t+1}) = S^*_{t+1},$$

where $S^{k*}_t$ denotes the time $t$ normalized price of the $k$-th security. Normalization is really not necessary in our model because even if we do not normalize the securities prices we can still treat $\pi$ as a finite measure and define conditional measure in the way one defines conditional probability.

The sum of these conditional measures may not be one, but this is the only difference.

Next, we define the budget feasible set

$$B(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \pi(c - y) = 0 \ \forall \pi \in \Pi \}. \quad (13)$$

The following theorem shows that the budget feasible set is identical to the set of feasible consumption bundles.
Theorem 1 \( B(y) = C(y) \).

**Proof.** Let \( D = \{ c \in \mathbb{R}^L \mid \exists \varphi \in \mathbb{R}^M \text{ s.t. } c - y = X\varphi \} \) and let \( E = \{ c \in \mathbb{R}^L \mid \pi(c - y) = 0 \forall \pi \in \Pi \} \).

We first show \( D = E \).

(i) \( D \subseteq E \). Consider \( c \in D \). For \( \pi \in \Pi \), we have \( \pi X\varphi = 0 \) giving us \( \pi(c - y) = 0 \). Therefore \( c \in E \).

(ii) \( E \subseteq D \). From Proposition 1, \( E = \{ c \in \mathbb{R}^L \mid \pi(c - y) = 0 \forall \pi \in X^\perp \} \). Thus there exists a vector \( \varphi \in \mathbb{R}^M \) such that \( c - y = X\varphi \).

Together, (i) and (ii) yield \( D = E \). Finally, we have \( B(y) = E \cap \{ c \in \mathbb{R}^L \mid c \geq 0 \} = D \cap \{ c \in \mathbb{R}^L \mid c \geq 0 \} = C(y) \). \( \blacksquare \)

Theorem 1 provides a complete characterization of the feasible consumption set when markets are incomplete. It allows one to see exactly how the incomplete markets case differs from the complete markets case where the feasible consumption set can be defined by a single budget constraint. This characterization of the budget feasible set is not yet useful for characterizing optimal policies or for computation because \( B(y) \) is defined by infinitely many constraints. A more useful characterization of the set of feasible consumption bundles follows.

Define \( \overline{\Pi} = \{ \pi \in \mathbb{R}^L \mid \pi X = 0, \pi \geq 0, \pi_1 = 1 \} = \text{cl}(\Pi) \). The set \( \overline{\Pi} \) is a closed, bounded (convex) polyhedron contained in \( X^\perp \), and therefore is generated by (or is the convex hull of) finitely many extreme points. If we define \( \Pi^e = \{ \pi \mid \pi \text{ is an extreme point of } \overline{\Pi} \} \), it is clear from the discussion in Section 7.2 of Schrijver (1986) that the following lemma is true.

**Lemma 1** \( \overline{\Pi} = \text{conv.hull}(\Pi^e) \).

Define

\[
B^e(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \pi(c - y) = 0 \forall \pi \in \Pi^e \}.
\]

Then

**Corollary 1** \( B^e(y) = B(y) \).

**Proof.** From lemma 1, it is easy to see that \( B^e(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \pi(c - y) = 0 \forall \pi \in \overline{\Pi} \} = \{ c \in \mathbb{R}^L \mid c \geq 0, \pi(c - y) = 0 \forall \pi \in \text{cl}(\Pi) \} = B(y) \). \( \blacksquare \)

This is one of the key results of the paper. The fact that the set of the feasible consumption bundles can be characterized by finitely many constraints involving the extreme points of \( \Pi \) allows us to attack the problem

\[
\max_{c \in B^e(y)} u(c, p)
\]
using the Lagrangian theory. Furthermore, the budget feasible set has a nice geometric interpretation as a polyhedron defined by budget hyperplanes formed using the extreme points of the set of state prices consistent with the absence of arbitrage. Section 6 consists of an example which illustrates this approach.

For computational purposes, if the number of extreme points, \( k \), of \( \text{cl}(\Pi) \) is greater than \( \dim(X^\perp) \), then \( k-\dim(X^\perp) \) of the constraints \( \pi(c - y) = 0 \) will be redundant, and \( k-\dim(X^\perp) \) of them can be omitted. In fact, we can perform maximization using \( \dim(X^\perp) \) constraints of the form \( g_i(c - y) = 0 \), for \( i = 1, \ldots, \dim(X^\perp) \), where the \( g_i \) are any \( \dim(X^\perp) \) linearly independent vectors that lie in \( \Pi \) or its affine hull. A set of \( \dim(X^\perp) \) extreme points of \( \Pi \) form a particularly convenient set, but any set may be used. This may make certain problems easier computationally. Standard algorithms exist for generating a basis for the orthogonal space of a matrix, and given a basis for \( X^\perp \) it is easy to find \( \dim(X^\perp) \) points that lie in the affine hull of \( \Pi \). For some problems it may be easier to use these algorithms than to generate the extreme points of \( \Pi \).\(^3\)

4.2 Short-sale Constraints

It is straightforward to extend our approach to include short sale constraints on some of the securities. The arguments are either identical to or slight generalizations of those in the previous section. In our formulation short sale constraints (or restrictions on long positions) appear in the form of inequality constraints on the components of \( \varphi \), i.e. \( \varphi_i \geq 0 \) or \( \varphi_i \leq 0 \). A short-sale constraint on one security will involve constraints on more than one of the elements of \( \varphi \). In particular, if a security is available for trading at \( k \) different dates and states and it may never be sold short, then \( k \) of the \( \varphi_i \) are constrained to be greater than or equal to zero.

Rearrange the rows of \( X \), and the components of \( \varphi, c \) and \( y \) so that

\[
\varphi = \begin{pmatrix}
\varphi_r \\
\varphi_s \\
\varphi_t
\end{pmatrix},
\]

\(^3\)If one uses the Householder method to construct the \( QR \) decomposition of \( X, X = QR \), then the last \( L-M \) columns of \( Q \), say \( (q^{M+1}, \ldots, q^L) \), will span the left null space of \( X \) (see, e.g., Golub and Van Loan, Section 6.2 (1983)). Given a basis for the left null space of \( X \) we can easily construct \( \dim(\Pi) \) linearly independent vectors that lie in \( \{ \pi \in \mathbb{R}^L \mid \pi X = 0, \pi_1 = 1 \} \). Consider the \( q^i, i \in \{M + 1, \ldots, L\} \), such that \( q^i_1 \neq 0 \). (Note that at least one of the \( q^i \) must have a nonzero first component, for if \( q^i_1 = 0 \) for all \( i \in \mathbb{N} \), then \( \{ \pi \in \mathbb{R}^L \mid \pi X = 0, \pi_1 = 1 \} = \emptyset \), which contradicts Proposition 1.) Normalize these so that the first components are equal to one, i.e., define \( \tilde{q} = q^i / q^i_1 \). Pick any one of these \( \tilde{q}^i \), say \( \tilde{q}^k \), and construct \( \tilde{q} = \tilde{q}^k - \tilde{q} \) for those vectors with \( q^i_1 \neq 0, i \in \{M + 1, \ldots, L\}, i \neq k \). Then let \( \tilde{q}^i = q^i \) if \( q^i_1 = 0, i \in \{M + 1, \ldots, L\} \). Consider the space given by

\[
\tilde{q}^k + \sum_{i \in \{M + 1, \ldots, L\}; i \neq k} \alpha_i \tilde{q}^i,
\]

where \( \sum_i \alpha_i = 0 \). This space is the affine hull of \( \Pi \). That is, it is exactly \( \{ \pi \in \mathbb{R}^L \mid \pi X = 0, \pi_1 = 1 \} \).
\[ X = (X_r \ X_s \ X_t), \]

with \( \varphi_r \geq 0, \varphi_s \leq 0, \) and \( \varphi_t \) unconstrained. Define

\[ C_1(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \exists \varphi \text{ s.t. } c = X\varphi + y, \varphi_r \geq 0, \varphi_s \leq 0 \}. \]

Also define

\[ \Pi_1 = \{ \pi \in \mathbb{R}^L \mid \pi > 0, \pi X_r \leq 0, \pi X_s \geq 0, \pi X_t = 0, \pi_1 = 1 \} \]

and

\[ B_1(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \pi(c - y) = 0 \forall \pi \in \Pi, \pi(c - y) \leq 0 \forall \pi \in \Pi_1 \setminus \Pi \}. \]

Similar to the interpretation of \( \Pi \), we can establish the following relationship between the lack of arbitrage opportunities and \( \Pi_1 \).

**Proposition 2** If there are no arbitrage opportunities for dynamic trading strategies satisfying \( \varphi_r \geq 0 \) and \( \varphi_s \leq 0 \) then \( \Pi_1 \) is non-empty.

**Proof.** Motzkin’s Transposition Theorem [Schrijver (1986), Corollary 7.1k] implies that there is a vector \( \hat{\pi} \) with \( \hat{\pi} > 0, \hat{\pi} X_r \leq 0, \hat{\pi} X_s \geq 0 \) and \( \hat{\pi} X_t = 0 \) if and only if \( X\varphi \geq 0 \) implies \( X\varphi = 0 \), where \( \varphi_r \geq 0 \) and \( \varphi_s \leq 0 \). If we normalize \( \hat{\pi} \) so that its first component \( \pi_1 = 1 \) the normalized vector \( \pi \) will be an element of \( \Pi \).

Similar to the incomplete markets case, we want to show that \( C_1(y) = B_1(y) \).

**Theorem 2** \( B_1(y) = C_1(y) \).

**Proof.** Let \( D = \{ c \in \mathbb{R}^L \mid \exists \varphi \text{ s.t. } \varphi_r \geq 0, \varphi_s \leq 0, c = X\varphi + y \} \) and let \( E = \{ c \in \mathbb{R}^L \mid \pi(c - y) = 0 \forall \pi \in \Pi, \pi(c - y) \leq 0 \forall \pi \in \Pi_1 \setminus \Pi \} \). Similar to the proof of Theorem 1, we only need to show \( D = E \). \( D \subseteq E \) is obvious, so we only have to show \( E \subseteq D \).

Since \( \Pi \subseteq \Pi_1 \), Theorem 1 implies that there exists a vector \( \varphi \) such that \( c - y = X\varphi \). We first claim that \( \varphi_r \geq 0 \). If this is not true, we may assume for simplicity that the first component of \( \varphi_r, \varphi_1 \), is strictly negative. Find a vector \( z \) such that \( zX_1 \neq 0 \) and \( zX^1 = 0 \), where \( X_1 \) is the first column of \( X \) and \( X^1 \) is the submatrix of \( X \) formed by removing \( X_1 \) from \( X \). (Such a vector \( z \) always exists because the lack of arbitrage opportunities implies that \( \Pi \) is non-empty.)

Now, choose a vector \( \pi \in \Pi \) and a sufficiently small real number \( \epsilon \) such that \( \pi_\epsilon = \pi + \epsilon z > 0 \) and \( \pi_\epsilon X_1 < 0 \) and \( \pi_\epsilon X^1 = 0 \). Let \( \hat{\pi}_\epsilon \) be a normalization of \( \pi_\epsilon \) such that \( \hat{\pi}_\epsilon = \pi_\epsilon / \pi_\epsilon 1 \in \Pi_1 \). We obtain that \( \hat{\pi}_\epsilon (c - y) = \hat{\pi}_\epsilon X_1 \varphi_1 > 0 \), a clear contradiction. Therefore \( \varphi_r \geq 0 \), and similarly \( \varphi_s \leq 0 \).
What is interesting here is that with short-sale restrictions, the set \( \Pi_1 \) consists not only of the equivalent measures under which securities prices are martingales, but also those under which the prices of the securities for which short-sale is restricted are supermartingales. This is similar to the supermartingale result of Dybvig and Ross (1986). An intuition for the result is as follows. If a security price follows a supermartingale with non-zero expected change, then that security is "overpriced," and offers an apparent arbitrage opportunity. However, if short-sales are not possible the opportunity cannot be exploited and nothing precludes the security price from following a supermartingale.

Next define

\[
\bar{\Pi}_1 = \{ \pi \in \mathbb{R}^L \mid \pi \geq 0, \pi X_r \leq 0, \pi X_s \geq 0, \pi X_I = 0, \pi_1 = 1 \}.
\]

The set \( \bar{\Pi}_1 \) is a closed, bounded (convex) polyhedron and is generated by finitely many extreme points. Let \( \Pi_1^* \) denote the set of extreme points of \( \bar{\Pi}_1 \). Define

\[
B_1^*(y) = \{ c \in \mathbb{R}^L \mid c \geq 0, \pi(c - y) = 0 \forall \pi \in \Pi_1^*, \pi(c - y) \leq 0 \forall \pi \in \Pi_1^* \}.
\]

Then, similar to the incomplete markets case, we have

**Corollary 2** \( B_1(y) = B_1^*(y) \).

**Proof.** Identical to the proof of Corollary 1. \( \square \)

This corollary allows us to attack the problem

\[
\max_{c \in B_1^*(y)} u(c, p)
\]

using the Lagrangian theory. Theorems 1 and 2 and Corollaries 1 and 2 combine incomplete markets and short-sale restrictions in a very general way and show how a dynamic consumption and portfolio problem can be solved as a static maximization problem with finitely many equality and inequality constraints.

### 4.3 Generating Extreme Points

As the last step in characterizing the feasible consumption set we consider how to generate the extreme points of \( \bar{\Pi} \) and \( \bar{\Pi}_1 \) in order to form the budget constraints.\(^4\) In general it can be difficult to generate the extreme points of an arbitrary polyhedron. However, the event tree information

\(^4\)As indicated in Section 4.1, when there are no short-sale constraints we can form the budget constraints by using \( \dim(X^{-}) \) vectors that lie in the affine hull of \( \Pi \).
structure gives the payoff matrix $X$ a special structure that considerably simplifies the task of generating the extreme points. We need the following proposition which follows from Theorem 8.4 in Schrijver (1986) and the discussion immediately following that theorem.

**Proposition 3** Let $\pi$ be a vector in $\mathbb{R}^L$, let $A$ and $B$ be matrices, let $b$ be a vector, and let $P$ be the bounded (convex) polyhedron defined by the system $\pi A \geq 0$, $\pi B = b$. A point $\pi^e$ is an extreme point of $P$ if and only if $\pi^e \in P$ and $\pi^e$ satisfies $\pi B = b$ along with $L - \text{rank}(B)$ linearly independent equations from the subsystem $\pi A = 0$.

Let

$$J = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

where $J$ is $L \times (L - 1)$ and $I$ is an $(L - 1) \times (L - 1)$ identity matrix, and let $X = ( z \quad X )$, where $z^T = ( 1 \quad 0 \ldots \quad 0 )$. We see that $\Pi$ is defined by

$$\pi J \geq 0,$$

$$\pi X = z^T.$$

We set up a system of the form

$$\pi Z = (z^T, 0),$$

where $Z$ is a matrix formed from the columns of $X$ and $L - \text{rank}(X)$ columns from $J$. This is essentially constraining $L - \text{rank}(X)$ of the elements of $\pi$ to equal zero and solving for the remaining elements such that $\pi X = 0$. Applying the above proposition with $A = J$, $B = X$, and $b = z^T$, we see that any solution $\hat{\pi}$ of the above system is a candidate extreme point. If $\hat{\pi} \in \Pi$ then it is an extreme point; otherwise it is not and we discard it. We continue this procedure until we have formed all possible matrices $Z$.

We could apply this proposition directly to the entire system in order to generate the extreme points of $\Pi$. However, this would not exploit the special structure of the event tree and the payoff matrix $X$. Instead, we use this proposition to generate the extreme points of the sets of conditional state prices at the price subsystems at each node of the event tree, and then combine these conditional state prices in the obvious fashion. The following proposition shows how this can be done.
Proposition 4. The set of extreme points for the whole price system $\Pi^c$ can be generated through multiplications of the extreme points of the price subsystems at each node of the event tree. That is, the extreme points of the price subsystems can be interpreted as conditional probabilities (or measures), and they can be used to define the probabilities (or measures) for each node using the rule $P(A \cap B) = P(B|A)P(A)$.

Proof. We only sketch the basic idea. Suppose we have an extreme point $\pi$ for the whole price system. Take any sub-tree, and project $\pi$ to the price subsystem associated with this sub-tree. One obtains either a zero probability at the node that generates this tree or a positive probability. In the latter case, we claim that the projection defines an extreme point for the sub-tree. If this is not true, the projection can be written as a convex combination of two different points of the subsystem. This in turn defines two different points in $\Pi$ which have the property that $\pi$ can be represented as a convex combination of these two points. This contradicts the assumption that $\pi$ is an extreme point.

Conversely, any point $\pi$ generated from the extreme points of the subsystems through multiplications must be an extreme point of the whole system, for otherwise it could be represented as a convex combination of two different points in $\Pi$. The projection of $\pi$ on any sub-tree (which is either zero or an extreme point for the subsystem by assumption) could be written as a convex combination of the projection of the other two points. The projection of the other two points must differ on at least one sub-tree. This violates the assumption that $\pi$ is generated from extreme points of the subsystems through multiplications.

With this procedure, the problem of finding the extreme points of a polyhedron is reduced to a series of comparatively simple problems that involve finding the extreme points at each node of the event tree. The solution for $\pi$ can then be obtained by the appropriate multiplications of the solutions of these smaller problems. This procedure is illustrated in the example in Section 6.

A similar procedure can be employed for the extreme points of $\Pi_1$. The polyhedron $\Pi_1$ is defined by the systems

\[
\begin{align*}
\pi J & \geq 0, \\
\pi X_r & \geq 0, \\
\pi X_s & \leq 0, \\
\pi X_t & = z^T,
\end{align*}
\]

where $J$, $X_t$, and $z$ correspond to the $J$, $X$, and $z$ defined earlier. By Proposition 3 we can find the extreme points of this polyhedron by forming a system

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\( \pi Z = 0, \)

where \( Z \) is an \( L \times L \) matrix formed from the columns of \( X_t \) and \( L - \text{rank}(X_t) \) linearly independent columns taken from \( J, X_r \) and \( X_s \). The solution \( \hat{x} \) of this system is a candidate extreme point. If \( \hat{x} \in \bar{\Pi}_1, \hat{x} \) is an extreme point, otherwise we discard it. Again, we continue this procedure until we have formed all possible matrices \( Z \).

Just as before, the actual computation of the extreme points is simplified by the fact that the matrix \( X \) has a special structure and the extreme points of the whole system can be obtained through multiplications of the extreme points of the price subsystems at each node.

5 Solution of the Optimization Problem

Given our characterization of the feasible set, the agent's problem (11) becomes

\[
\max \ u(c, p) \tag{14}
\]

subject to the constraints

\[
\begin{align*}
  c & \geq 0, \tag{15} \\
  \pi(c - y) & = 0 \quad \forall \pi \in \Pi^e, \tag{16} \\
  \pi(c - y) & \leq 0 \quad \forall \pi \in \Pi^f. \tag{17}
\end{align*}
\]

The \( \pi \in \Pi^f \) and the \( \pi \in \Pi^e \) are the extreme points of the sets

\[
\bar{\Pi}_1 = \{ \pi \in \mathbb{R}^L | \pi \geq 0, \pi X_r \leq 0, \pi X_s \geq 0, \pi X_t = 0, \pi_1 = 1 \}
\]

and

\[
\bar{\Pi} = \{ \pi \in \mathbb{R}^L | \pi \geq 0, \pi X = 0, \pi_1 = 1 \}.
\]

The problem (14)–(16) is a nonlinear program subject to linear inequality and equality constraints and can be solved using standard algorithms (see, for example, Gill, Murray, and Wright (1981) and Fletcher (1981)).
A useful characterization of the solution of our static problem (14)-(16) can be obtained from a simple manipulation of the first order conditions. The solution of (14)-(16) will satisfy a system of equations

\[
\begin{align*}
    u(x, (c, p)) - \eta_i &= \sum_{r=1}^{R} \lambda^r \pi_i^r \quad i = 1, \ldots, L, \\
    \pi^r(c - y) &= 0 \quad r = 1, \ldots, \bar{r}, \\
    \pi^r(c - y) &\leq 0 \quad r = \bar{r} + 1, \ldots, R, \\
    c_i &\geq 0 \quad i = 1, \ldots, L, 
\end{align*}
\]

(18) \hspace{1cm} (19) \hspace{1cm} (20) \hspace{1cm} (21)

where we have indexed the extreme points associated with the constraints (17) and (16) using the integers 1, \ldots, R, the \(\eta_i\) are the Lagrange multipliers of the constraints \(c_i \geq 0\), and the \(\lambda^r\) are the Lagrange multipliers of the constraints \(\pi^r(c - y) \leq 0\). We can rewrite (18) as

\[
\frac{\partial}{\partial c_i} u(c, p) - \eta_i = \lambda \sum_{r=1}^{R} \alpha_r \pi_i^r \quad i = 1, \ldots, L, 
\]

(22)

where \(\sum_{r=1}^{R} \alpha_r = 1\), \(\alpha_r \geq 0\). Examining (22), we see that this is one of the first order conditions of a problem with one budget constraint formed using a convex combination of the extreme points. The particular convex combination depends upon the utility function. That is, at the solution it is as though there were complete markets, where the Arrow-Debreu state price vector depends on the utility function.

This characterization of the solution is very simple. The unique state price vector is just a separating hyperplane. (The budget feasible set \(B(y)\) and the upper contour sets of the utility function are convex sets.) However, the characterization, while not very deep, is useful.\(^5\)

Associated with this single Arrow-Debreu state price are, at every node, conditional state prices for consumption at the nodes next period. For utility functions of the HARA class, the conditional state prices are the same at every node when the uncertainty in the prices of the securities is multiplicative, there is a bond, and the investment opportunity set is constant (and there are no binding short-sale constraints or non-negativity constraints on consumption). Further, the single Arrow-Debreu price for the whole system can be recovered by multiplying together the conditional state prices as in Proposition 4. Since the conditional state prices are the same at every node, we can recover the price for the whole system by solving only a one-period problem. Once we have

\(^5\)This characterization in terms of a single state price also motivated some of the research in the continuous time case [He and Pearson (1989)].
obtained the single Arrow-Debreu price for the whole system it is easy to calculate the optimal consumption and portfolio policies.

A direct proof of the claim that the conditional state prices are the same at all nodes is both cumbersome and tedious, for it involves writing out all of the first order conditions for the static problem with budget constraints formed using the elements of \( \Pi^c \) and recognizing certain symmetries. However, it is easy to see why the claim should be true. It is well known that in the setting described above the proportion of the agent’s risky asset portfolio invested in each of the risky securities is independent of wealth (e.g., Hakansson (1970); see also Merton (1971)). Hence the optimal proportions will not depend on the current node. But the portfolio choices can be the same at all nodes only if the conditional state prices are the same at all nodes.

5.1 Implementing an optimal solution.

Once an optimal consumption bundle is found, the remaining issue is how to implement it through a dynamic trading strategy, i.e., how to solve explicitly for \( \varphi \). From the equality \( c - y = X\varphi \) it is clear that one can solve for \( \varphi \) in terms of \( c - y \) and the elements of \( X \). This turns out to be fairly simple to do. Working backwards, given the optimal consumption pattern at date \( T \) one determines the trading strategy for each node at time \( T - 1 \) and the wealth level required to carry out this strategy. This requires only that one solve a system of linear equations. After finding the wealth required at each node at time \( T - 1 \) and adding the time \( T - 1 \) consumption at that node one can determine the trading strategies for each of the nodes at \( T - 2 \) and the wealth levels required at these nodes. Repeating the procedure \( T \) times gives the trading strategies for all dates and nodes. This procedure is illustrated in the example in Section 6.

5.2 Computational considerations

In our formulation the agent’s original problem (11) becomes the static problem (14)–(16), which can be solved using standard nonlinear programming algorithms. An alternative formulation of the problem is due to Breeden (1987). Breeden also considers the intertemporal consumption and portfolio problem (11), uses the relationship \( c = X\varphi + y \) to eliminate \( c \), and solves

\[
\max_{\varphi} u(X\varphi + y, p)
\]

subject to

\[
\varphi \geq 0,
\]

\[
(23)
\]

\[
(24)
\]
\[ X \varphi + y \geq 0. \] (25)

This is also a nonlinear program subject to linear inequality and equality constraints and can be solved using standard algorithms. Dynamic programming is yet another alternative for solving the original problem if the agent's preferences have an expected utility representation.

Breeden (1987) discusses why his approach is better for computation than dynamic programming when the agent has a time additive state independent utility function, and his discussion also applies to the static problem (14)–(16) we obtain. The computational inefficiency of dynamic programming stems from the fact that computing optimal policies using dynamic programming requires that at each node one compute the optimal policies for all possible levels of wealth, including those that will never be realized at the node. The relative superiority of the static formulations is even greater when the agent's utility function is not time additive (or more generally of the linear Koopmans aggregator type discussed by Bergman (1985)). For utility functions in this class optimal policies depend upon wealth but not upon past consumption histories. In general optimal policies depend upon past consumption histories as well as wealth, and computing optimal policies using dynamic programming requires that at each node one compute the optimal policies for all possible levels of wealth and past consumption histories. This may involve computing optimal policies for all possible realizations of a high dimensional random variable, and Breeden's discussion of the computational inefficiency of dynamic programming applies with even more force.

If the agent's preferences do not have an expected utility representation dynamic programming cannot be used to compute optimal policies and the static formulations of us and Breeden are the only way to compute optimal intertemporal consumption and portfolio policies.

In general, there is little that can be said about the relative merits of the two static formulations, Breeden's (23)–(25) and ours (14)–(16). Computational efficiency is likely to be problem, data, and algorithm-dependent. However, our formulation is superior for additively separable utility functions, a class which includes the utility functions most commonly used in finance.

If the agent's utility is additively separable, the objective function (14) of our formulation will be of the form

\[ u(c, p) = u(c_1) + \sum_{i=2}^{L} p_i u(c_i), \] (26)

which is additively separable in the variables \( c_i \). The objective function in Breeden's formulation is

\[ u(X \varphi + y, p) = u(x_1 \varphi + y_1) + \sum_{i=2}^{L} p_i u(x_i \varphi + y_i), \] (27)

23
where \( x_i \) denotes the \( i \)-th row of the matrix \( X \) and \( y_i \) denotes the \( i \)-th component of the vector \( y \). Even though the agent's utility is additively separable in consumption, the objective function is not additively separable in the variables \( \varphi_i \) over which maximization is to be performed. This lack of additive separability, compared to the additive separability in the \( c_i \) in our formulation, makes our approach preferred for computation.

The problem (14)–(16) with an additively separable utility function is a \textit{separable programming problem} (see, e.g., Walsh (1975), p. 5). General nonlinear programming algorithms tend to be more efficient when applied to such problems.\(^6\) More importantly, special methods are available for the solution of such problems (see, e.g., Hadley (1964) and Beale (1970); a more recent algorithm suited for such problems is described in Fourer (1986)). In essence, these methods reduce the nonlinear programming problem to an approximating linear programming problem via piecewise linear approximations of the nonlinear functions.\(^7\) The approximating linear program is much easier to solve than the original nonlinear program. While it only yields approximate solutions, more accurate approximations can be obtained by repeated solution user finer approximations in the regions surrounding the solutions of the previous approximations. The relative efficiency of this approach is of no importance in small examples, but is likely to be of importance in problems of large enough size to be of practical interest.

6 An Example

The economy has three dates \( t = 0,1,2 \) and the following partitions of \( \Omega \): \( F_0 = \Omega \), \( F_1 = \{ \{\omega_1,\omega_2,\omega_3\}, \{\omega_4,\omega_5,\omega_6\}, \{\omega_7,\omega_8,\omega_9\} \} \), and \( F_2 = \{ \omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8,\omega_9 \} \). There are two securities, a riskless bond with price \( S_0^0(\omega) = 1 \) for \( t = 0,1,2 \), and a risky security with prices \( S_0^1(\{\omega_1,\omega_2,\omega_3\}) = 2 \), \( S_0^1(\{\omega_4,\omega_5,\omega_6\}) = 1 \), \( S_0^1(\{\omega_7,\omega_8,\omega_9\}) = 1/2 \), \( S_1^1(\omega_1) = 4 \), \( S_1^1(\omega_2) = 2 \), \( S_1^1(\omega_3) = 1 \), \( S_1^1(\omega_4) = 2 \), \( S_1^1(\omega_5) = 1 \), \( S_1^1(\omega_6) = 1/2 \), \( S_1^1(\omega_7) = 1 \), \( S_1^1(\omega_8) = 1/2 \), and \( S_1^1(\omega_9) = 1/4 \). This securities price process can be represented by the event tree shown in Figure 3.

Using our convention for allocating the components of \( \pi \), \( c \), and \( y \) to the various events, we have

\[
\pi = (\pi(\Omega), \pi(\{\omega_1,\omega_2,\omega_3\}), \pi(\{\omega_4,\omega_5,\omega_6\}), \pi(\{\omega_7,\omega_8,\omega_9\}),
\]

\[
\pi(\omega_1), \pi(\omega_2), \pi(\omega_3), \pi(\omega_4), \pi(\omega_5), \pi(\omega_6), \pi(\omega_7), \pi(\omega_8), \pi(\omega_9)),
\]

\(^6\)A limited set of computational experiments using an algorithm for general nonlinear programming problems, subroutine E04VCF from the NAG FORTRAN Library – Mark 11 [Numerical Algorithms Group Limited (1982)], confirmed that this is true for power and exponential utility functions in a simple example.

\(^7\)An example which illustrates the basic idea may be found in Shapiro (1979), Section 1.5.
\[ c = (c(\Omega), c(\{\omega_1, \omega_2, \omega_3\}), c(\{\omega_4, \omega_5, \omega_6\}), c(\{\omega_7, \omega_8, \omega_9\}), c(\omega_1), c(\omega_2), c(\omega_3), c(\omega_4), c(\omega_5), c(\omega_6), c(\omega_7), c(\omega_8), c(\omega_9)' ), \]

and similarly for \( y \). The matrix \( \mathbf{X} \) is

\[
\mathbf{X} = \begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 2 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & -1 & 0 \\
.5 & 1 & 0 & 0 & 0 & 0 & -.5 \\
0 & 0 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & .5 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & .5 \\
0 & 0 & 0 & 0 & 0 & 0 & .25 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}. \\
\]

We first must find the extreme points of \( \bar{\Pi} \).

The matrix \( \mathbf{Z} \) consists of \( \mathbf{X} \) along with four columns taken from \( \mathbf{J} \). The columns from \( \mathbf{J} \) have the effect of constraining certain elements of \( \pi \) to equal zero. Any \( \pi \) such that \( \pi \mathbf{X} = 0 \) and \( \pi_1 = 1 \) can be written

\[
\pi = \left\{ 1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13} \right\}. \tag{28}
\]

The components \( \pi_2, \pi_3, \pi_4 \) satisfy the equations

\[
\begin{pmatrix}
1 & \pi_2 & \pi_3 & \pi_4 \\
1 & \pi_2 & \pi_3 & \pi_4 \\
0.5 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & -1 \\
2 & 1 \\
1 & 1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}, \tag{29}
\]

along with a constraint requiring that one of the \( \pi_i \) equal zero arising from one of the columns of \( \mathbf{J} \). The components \( \pi_5/\pi_2, \pi_6/\pi_2, \pi_7/\pi_2 \) satisfy the equations

\[
\begin{pmatrix}
1 & \pi_5/\pi_2 & \pi_6/\pi_2 & \pi_7/\pi_2 \\
1 & \pi_5/\pi_2 & \pi_6/\pi_2 & \pi_7/\pi_2 \\
0.5 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
-2 & -1 \\
4 & 1 \\
2 & 1 \\
1 & 1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}. \tag{30}
\]

25
along with a constraint requiring that one of the terms $\frac{\pi_5}{\pi_6}$ equal zero. The values of $\pi_5$, $\pi_6$, and $\pi_7$ are obtained from the obvious multiplications. Similar equations hold for the other components of $\pi$.

Solving these equations amounts to finding the extreme points for the price subsystems at each node of the event tree in Figure 3. It is easy to see that $(0, 1, 1)$ and $(1/3, 0, 2/3)$ are the only two extreme points that satisfy (29). Similarly, $(0, 1, 0)$ and $(1/3, 0, 2/3)$ are the only two extreme points that satisfy (30), and we find the same two extreme points for the other two subsystems.

Now we find the elements of $\Pi^e$, the extreme points of the closure of the set of equivalent measures for the whole price system, by multiplying together the extreme points of the subsystems as we would usually do for conditional probabilities; how to do this is clear from equation (28). We obtain

$$
\Pi^e = \left\{ \begin{pmatrix}
1 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1/3 \\
1/3 \\
2/3 \\
0 \\
1/3 \\
1/3 \\
2/3 \\
4/9
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
2/3 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
2/3 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
4/9 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
2/3 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
4/9 \\
0
\end{pmatrix} \right\}.
$$

These six extreme points are not linearly independent and any five of them may be used to define the constraints in the maximization problem.

The nonzero components of these extreme points may be interpreted as the prices of consumption in the various states. For example, the least costly way to obtain an additional unit of consumption in the state $\{\omega_1, \omega_2, \omega_3\}$ without decreasing consumption in any state is to take a long position in $2/3$ share of the risky security and a short position in $1/3$ unit of the riskless bond. This position yields 1 unit of consumption in the state $\{\omega_1, \omega_2, \omega_3\}$ and $1/2$ unit of consumption in the state $\{\omega_1, \omega_2, \omega_3\}$. Entering this position costs $1/3$ unit of current consumption, and $\pi^e_5 = \pi^r(\{\omega_1, \omega_2, \omega_3\}) = 1/3$ for the extreme points with $\pi^e_5 \neq 0$ (here $\pi^r$ denotes the $r$-th element of $\Pi^e$). This set of transactions is also the cheapest way to obtain consumption in the state $\{\omega_7, \omega_8, \omega_9\}$, and $\pi^e_4 = \pi^r(\{\omega_7, \omega_8, \omega_9\}) = 2/3$ for the extreme points with $\pi^e_4 \neq 0$.

With the extreme points we are now in a position to solve the static utility maximization problem and determine the optimal consumption bundle. We suppose that all outcomes are equally likely
and that the consumer has log utility and an endowment of 10 now and zero in both future periods. We solve

\[
\max_c \left[ \log c_1 + 1/3 \sum_{i=2}^{4} \log c_i + 1/9 \sum_{i=5}^{13} \log c_i \right]
\]

subject to

\[\pi^r c = 10 \quad \text{for } r = 1, \ldots, 5.\]

Here we have used the first five elements of \(\Pi^r\) to define \(B(y)\). The optimal consumption plan is

\[c = (3^{1/3}, 5, 3^{1/3}, 2^{-1/2}, 7^{-1/2}, 5, 3^{3/4}, 5, 3^{1/3}, 2^{-1/2}, 3^{3/4}, 2^{-1/2}, 1^{-7/8}).\]

Given the optimal consumption plan we work backward to determine the trading strategies. For example, at the node \(\{\omega_1, \omega_2, \omega_3\}\), we determine the trading strategy by solving the (overdetermined) system

\[
\begin{align*}
\varphi^0(\{\omega_1, \omega_2, \omega_3\}) + 4\varphi^1(\{\omega_1, \omega_2, \omega_3\}) &= 7^{1/2}, \\
\varphi^0(\{\omega_1, \omega_2, \omega_3\}) + 2\varphi^1(\{\omega_1, \omega_2, \omega_3\}) &= 5, \\
\varphi^0(\{\omega_1, \omega_2, \omega_3\}) + \varphi^1(\{\omega_1, \omega_2, \omega_3\}) &= 3^{3/4},
\end{align*}
\]

where \(\varphi^1(\{\omega_1, \omega_2, \omega_3\})\) denotes the number of units of the risky security to hold at node \(\{\omega_1, \omega_2, \omega_3\}\) and \(\varphi^0(\{\omega_1, \omega_2, \omega_3\})\) denotes the number of units of the bond. Solving these we obtain

\[
\begin{align*}
\varphi^0(\{\omega_1, \omega_2, \omega_3\}) &= 2^{1/2}, \\
\varphi^1(\{\omega_1, \omega_2, \omega_3\}) &= 1^{1/4}.
\end{align*}
\]

The total cost at \(\{\omega_1, \omega_2, \omega_3\}\) is then \(\varphi^0(\{\omega_1, \omega_2, \omega_3\}) + 2\varphi^1(\{\omega_1, \omega_2, \omega_3\}) = 5\). Performing the calculations at the other nodes in the obvious way yields

\[
\begin{align*}
\varphi^0 &= (3^{1/3}, 2^{1/2}, 1^{-2/3}, 1^{1/4}), \\
\varphi^1 &= (1^{2/3}, 1^{-1/4}, 1^{2/3}, 2^{-1/2}),
\end{align*}
\]

where \(\varphi^0 \equiv (\varphi^0(\Omega), \varphi^0(\{\omega_1, \omega_2, \omega_3\}), \varphi^0(\{\omega_4, \omega_5, \omega_6\}), \varphi^0(\{\omega_7, \omega_8, \omega_9\}))\) and \(\varphi^1\) is defined analogously.
7 Conclusion

In this paper we show how the martingale approach can be used to determine optimal intertemporal consumption and portfolio policies in a general finite dimensional economy with incomplete markets and short-sale constraints. When markets are incomplete the feasible consumption bundles satisfy budget constraints formed using all Arrow-Debreu state prices consistent with the absence of arbitrage. We show that the set of feasible consumption bundles is identical to the budget feasible set defined by the budget constraints formed using the extreme points of the closure of the set of state prices consistent with no arbitrage. This result makes the martingale approach useful even when markets are incomplete and there are infinitely many Arrow-Debreu state prices or equivalent measures consistent with the absence of arbitrage.

When there are short-sale constraints securities prices must be super-martingales under the set of Arrow-Debreu state prices or equivalent measures consistent with the absence of arbitrage, and the feasible consumption bundles must satisfy budget constraints formed using all of these state prices. In this case we also show that this set of feasible consumption bundles may be defined using only budget constraints formed from the extreme points of the set of state prices consistent with no arbitrage.

Our approach to the consumption-portfolio problem is very convenient for computation. It also provides an alternative approach to obtain the optimal consumption policies for time-additive, state-independent, utility functions of the HARA class (with no nonnegativity restrictions on consumption) when there is a constant investment opportunity set and there are no short-sale constraints.

All of our results of course apply only to the discrete time finite dimensional case. How the martingale approach can be useful in the infinite dimensional continuous time case with incomplete markets and short-sale constraints is the subject of a companion paper, He and Pearson (1989).
References


Figure 1
Graphical representation of the feasible consumption sets
The complete markets feasible consumption set is the polyhedron consisting of the convex hull of $(0,0,0)$, $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$; the bundles a nonsatiated agent might choose lie on the face consisting of the convex hull of $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$. If the third security is not available for trading the bundles a nonsatiated agent might choose lie on the line segment connecting $(0,1,2)$ and $(2,1,0)$. If the third security is not available for trading and the second security may not be sold short the bundles a nonsatiated agent might choose lie in the segment connecting $(1,1,1)$ and $(2,1,0)$. 
Figure 2
Event tree representation of the securities price process
The two components of the ordered pairs indicate the prices of the riskless and risky securities, respectively.
Figure 3
Event tree representation of the securities price process
The two components of the ordered pairs indicate the prices of the riskless and risky securities, respectively.