The Relationship Between Non-Arbitrage and Recursive Competitive Equilibrium Pricing

by

Richard Breen
Gregory Connor

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The Relationship Between Non-Arbitrage and Recursive Competitive Equilibrium Pricing

Richard Breen
The Economic and Social Research Institute
4 Burlington Road
Dublin 4  Ireland

and

Gregory Connor
Haas School of Business
University of California, Berkeley
Berkeley, CA  94620

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Abstract

Kreps (1981) shows that, in a finite horizon economy, any set of asset prices obeying non-arbitrage restrictions are also competitive equilibrium prices for some chosen configuration of investor preferences and endowments. Therefore, in the absence of preference and endowment restrictions, non-arbitrage and competitive equilibrium analysis are equivalent. This paper demonstrates that Kreps' equivalence result does not extend unchanged to an infinite horizon recursive economy. We apply the results to the Cox-Ross-Rubinstein options pricing model.

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1. Introduction

Most asset pricing theories rely either on non-arbitrage or competitive equilibrium analysis. In non-arbitrage analysis the economist specifies the set of available assets and range of trading opportunities and derives the set of possible price relationships which do not permit investors to form zero-cost portfolios with strictly positive payoffs (arbitrage portfolios). Competitive equilibrium models additionally specify all investors' preferences and endowments. Competitive equilibrium prices are such that the demand for each asset equals its supply given that each investor demands a budget-optimal portfolio of the assets.

It is trivial that competitive equilibrium prices obey non-arbitrage conditions, since arbitrage portfolios are inconsistent with budget-optimal portfolios. Kreps (1981) proves that in an important sense the converse also holds. Consider a set of prices obeying non-arbitrage conditions. Kreps shows that there exists some set of preferences and endowments such that these prices will satisfy competitive equilibrium conditions. This result means that if the economist does not restrict preferences or endowments in his model, arbitrage analysis and competitive equilibrium analysis are equivalent.

Kreps sets his pricing theory in a finite-horizon model. The economy lasts from time 0 to T and then ends. A state of nature in this framework is a complete realization of all relevant variables from time 0 to T. Lucas (1978) and Prescott and Mehra (1980) suggest an alternative approach to
asset pricing theory in which the economy lasts forever. A state of nature in their model is a particular realization of the economy at any time t. The same fundamental states of nature occur repeatedly through time, and follow a Markov process. All relevant observable variables (dividends, prices, chosen portfolios) at time t are a function of the time t state of nature only, not of earlier or later states of nature. Prescott and Mehra call this a recursive model, and the related supply-demand equilibrium condition they call recursive competitive equilibrium. Recursive models have notable advantages over finite horizon models of asset pricing, particularly their greater compatibility with standard estimation techniques.

This paper shows that Kreps' equivalence theorem does not extend directly to recursive models. The problem is that a recursive model places restrictions on the ratios of marginal utilities of consumption between pairs of states which occur at different dates. Because of this, in a recursive model, the set of possible competitive equilibrium prices is smaller than the set of non-arbitrage prices. For example, in recursive competitive equilibrium, an investor's marginal utility of consumption is necessarily equal at two different dates if the same state of nature occurs at the two dates. Therefore equilibrium pricing connecting the same state at different dates must be risk-neutral pricing. This need not be true of non-arbitrage prices.

Section two reviews Kreps' equivalence result in a finite horizon model. Section three provides a formal statement of
the non-equivalence in recursive models. Section four applies the result to the Cox, Ross, Rubinstein (1979) binomial option pricing model. Section five demonstrates how, by the use of an expanded, unobserved state space, the equivalence can be reestablished in recursive models. All proofs are relegated to an appendix.

2. Kreps' Equivalence Theorem in a Finite-Horizon Model

In this section we give a simple version of Kreps' equivalence theorem in a discrete-time, finite horizon economy. Let the economy consists of T+1 trading and consumption dates 0, 1, ..., T. The information structure of the economy is given by a strictly increasing filtration of possible events $\theta_t$, $t=0, ..., T$ and associated probability measure. The economy includes $n$ assets with dividend payments given by an $n$-vector stochastic process $d(\theta_t)$. Asset prices are an $n$-vector stochastic process $p(\theta_t)$. We assume that prices and dividends are adapted to $\theta$ and have finite second moments: $E[d(\theta_{t+1})^2|\theta_t]<\infty$; $E[p(\theta_{t+1})^2|\theta_t]<\infty$ for all $\theta_t$. The economy ends at T when a liquidating dividend is paid on each asset. Hence $p(\theta_T) = 0$ for all $\theta_T$.

An arbitrage portfolio in state $\theta_t$ is an $n$-vector, $a_t$, such that:

$$a_t'p(\theta_t) \leq 0 \quad (1)$$

$$\Pr[a_t'(p(\theta_{t+1}) + d(\theta_{t+1})) < 0 | \theta_t] = 0 \quad (2)$$

$$\Pr[a_t'(p(\theta_{t+1}) + d(\theta_{t+1})) > 0 | \theta_t] > 0. \quad (3)$$

The following result originally comes from Ross (1978)
and was extended by Kreps (1981) and Hansen and Richards (1987). Let \( \rho \) denote any scalar such that \( 0 < \rho < 1 \).

**Theorem 1**: There do not exist any arbitrage portfolios for any \( \theta_t, t=0, \ldots, T-1 \), if and only if there exists a stochastic process, \( \lambda(\theta_t, \theta_{t+1}) \) \( t=0, \ldots, T-1 \), such that
\[
\Pr(\lambda(\theta_t, \theta_{t+1}) > 0 | \theta_t) = 1 \text{ for all } \theta_t, \text{ and}
\]
\[
p(\theta_t) = \rho E[\lambda(\theta_t, \theta_{t+1})(p(\theta_{t+1}) + d(\theta_{t+1}))) | \theta_t] \text{ for all } \theta_t. \tag{4}
\]

Next we verify the equivalence of non-arbitrage and competitive equilibrium analysis. Without loss of generality we assume that all investors are identical. Assume that the representative investor has time discount factor \( \rho \) and time-constant, state-independent, risk-averse utility function \( u(C_t) \) where \( C_t \) denotes time \( t \) consumption. The utility function has positive first derivative and negative second derivative everywhere on its domain (which can be either \( (0, \infty) \) or \( (-\infty, \infty) \)). We impose the following weak restrictions on preferences \( \lim_{C \to 0} u'(C) = 0 \), and either \( \lim_{C \to 0} u'(C) = \infty \) or \( \lim_{C \to \infty} u'(C) = \infty \). The representative investor chooses his/her portfolio to maximise the discounted sum of expected lifetime utility.

Let \( X_t \) denote the cash flow to the representative investor given that he holds the per-capita supply of all assets. The economy can include additional assets other than the \( n \) assets under consideration and the representative investor can also have non-financial sources of cash flow such as labour income, so that \( X_t \) need not equal the per capita dividend cash flow of
the n assets. We assume that $X_t$ is adapted to $\theta_t$. Competitive equilibrium prices are defined as any n-vector stochastic process $p(\theta)$, adapted to $\theta_t$ with finite second moment, which solves the following set of $T$ backwards-inductive first order conditions:

$$p(\theta_t) - \rho E\left[\frac{u'(X(\theta_{t+1}))}{u'(X(\theta_{t+1}))} (p(\theta_{t+1}) + d(\theta_{t+1})| \theta_t)\right] \text{ for all } \theta_t. \quad (5)$$

Theorem 2: Price process $p(\theta)$ does not permit arbitrage if and only if there exists a consumption cash flow process $X(\theta)$ such that, given $X(\theta)$, $p(\theta)$ is a competitive equilibrium price process.

For given non-arbitrage price process $p(\theta)$ we define a supporting equilibrium as a representative investor, defined by $(\rho, u(\cdot), X(\cdot))$, such that the investor is in competitive equilibrium under the prices. Theorem 2 says that we can always construct a supporting equilibrium in a finite horizon model by appropriately choosing the stochastic process $X(\theta)$. Kreps also varies $u(\cdot)$, but this is not necessary in our version. (Kreps specifies preferences for net trades whereas we specify preferences for consumption, and a consumption endowment. He varies preferences for net trades which is equivalent to varying both $u(\cdot)$ and $X(\cdot)$ in our formulation.) Kreps' result is more general than ours since he allows for multiple goods and continuous-time trading whereas we have a single good and discrete time periods.
3. Non-Equivalence in Recursive Models

In this section we demonstrate that the equivalence of non-arbitrage and competitive equilibrium analysis does not hold in recursive asset pricing models. Uncertainty in a recursive model is described by a k-vector of real-valued state variables $\theta = (\theta_1, \ldots, \theta_k)$. The state variables follow a Markov process: the probability density of $\theta_{t+1}$ depends only on $\theta_t$ and not on $t$ or $\theta_{t-s}$, $s > 0$. The economy includes $n$ assets with dividends $d(\theta)$ and prices $p(\theta)$. We assume that both prices and dividends are functions of $\theta_t$ only, and have finite one-step-ahead second moments: $E[d(\theta_{t+1})^2 | \theta_t] < \infty$, $E[p(\theta_{t+1})^2 | \theta_t] < \infty$ for all $\theta_t$.

Since $k$ can be any finite number, the Markov assumption is not too restrictive; see Mehra and Prescott (1985), Connor and Korajczyk (1989), and Boessarts and Green (1989) for diverse examples. The non-arbitrage condition for prices is analogous to that in the finite-horizon case. Let $\rho$ denote any scalar such that $0 < \rho < 1$.

**Corollary to Theorem 1:** There do not exist any arbitrage portfolios if and only if there exists a scalar function, $\lambda(\theta_t, \theta_{t+1})$, such that

$$Pr(\lambda(\theta_t, \theta_{t+1}) > 0 | \theta_t) = 1 \text{ for all } \theta_t$$

(6)

and

$$p(\theta_t) - \rho E[\lambda(\theta_t, \theta_{t+1}) | p(\theta_{t+1}) + d(\theta_{t+1}) | \theta_t] \text{ for all } \theta_t$$

(7)

Define a non-arbitrage pricing function as any $\lambda(\theta_t, \theta_{t+1})$ obeying (6) and (7). As in section 2, we assume that the
representative investor has time-constant, state-independent, risk-averse utility function \( u(C_t) \) and time discount factor \( \rho \). At each date \( t \), the representative investor chooses his/her portfolio to maximise the discounted sum of expected lifetime utility; the portfolio choice at time \( t \) can only depend upon \( \Theta_t \). Recursive competitive equilibrium prices are defined as an \( n \)-vector function of \( \Theta \) obeying the following:

\[
p(\Theta_t) = \rho E\left[ \frac{u'(X(\Theta_{t+1}))}{u'(X(\Theta_t))} (p(\Theta_{t+1} + d(\Theta_{t+1}) | \Theta_t) \right) \text{ for all } \Theta_t. \tag{8}
\]

The conditions for recursive competitive equilibrium are stricter than those for recursive non-arbitrage, as shown in the next theorem.

**Theorem 3**: A recursive price process, \( p(\Theta) \), has a supporting recursive competitive equilibrium if and only if there exists a non-arbitrage pricing function \( \lambda(\Theta_t, \Theta_{t+1}) \) such that:

\[
Pr[(\lambda(\Theta_t, \Theta_{t+1}) - \frac{\gamma(\Theta_{t+1})}{\gamma(\Theta_t)}) | \Theta_t] = 1 \text{ for all } \Theta_t \tag{9}
\]

for some function \( \gamma(\Theta) \).

Theorem 3 shows that in a recursive competitive equilibrium model the pricing function \( \lambda(\Theta_t, \Theta_{t+1}) \) must be "reciprocal-symmetric" (the two argument function equals the ratio of a one-argument function to itself evaluated at a two points). This implies, for example, that the competitive
equilibrium price of a dollar in state $\theta^*$ at some future time
given the investor is currently in the same state $\theta^*$ will be
the risk-neutral price ($\lambda = 1$). This "reciprocal-symmetry"
condition need not hold for non-arbitrage prices.

4. Application to Cox-Ross-Rubinstein

The non-equivalence of recursive non-arbitrage and
competitive equilibrium pricing has consequences for
non-arbitrage models such as futures and options pricing
models. For example, the binomial option pricing model of
Cox, Ross, and Rubinstein (1979) (hereafter CRR) is a
non-arbitrage pricing model which can be embedded in a
recursive setting. We show that it cannot be a recursive
competitive equilibrium model except for particular values of
the parameters.

The CRR model has two modelled assets: a one-period
riskless bond and a risky stock. The bond has a time and
state invariant gross return $1+\tau$. The stock has a return of
$1+\phi$ with probability $\pi$ and $1/(1+\phi)$ with probability $1-\pi$.

This recursive model has a countable infinity of
possible stock prices given by $S(1+\phi)^n$ where $n$ is any integer
(positive or negative) and $S$ is any possible stock price. A
state of nature at time $t$ is indexed by the scalar value $S_t$.
The time-invariant transition probabilities are

$$\Pr(S_{t+1} = S_t(1+\phi)) = \pi$$  \hspace{1cm} (10)

$$\Pr(S_{t+1} = S_t(1+\phi)^{-1}) = 1-\pi,$$  \hspace{1cm} (11)
given $S_t$ all other outcomes for $S_{t+1}$ have zero probability.

Applying equation (4) from Theorem 1 to the stock and
bond, non-arbitrage pricing requires that there exist positive scalars, $\lambda^u$ and $\lambda^d$ such that the following equations hold:

\begin{align*}
1 &= \rho [\pi \lambda^u (1+r) + (1-\pi) \lambda^d (1+r)] \\
1 &= \rho [\pi \lambda^u (1+\phi) + (1-\pi) \lambda^d (1+\phi)^{-1}] \\
\end{align*}

(12)  
(13)

From Theorem 3, a supporting recursive competitive equilibrium requires that condition (9) is met. Condition (9) taken together with (12) and (13) implies

$$\lambda^u = 1/\lambda^d$$  
(14)

In order for conditions (12), (13) and (14) to hold, we must place restrictions on the return process of the stock and the riskless asset. Rearranging (12), (13), and (14) and using $0 < \rho < 1$ gives:

$$0 < \left( \frac{1}{\pi (1-\pi)} \right) \left( \frac{1+\phi}{1+r} - 1 \right) \left( 1 - \frac{(1+\phi)^{-1}}{(1+r)} \right) \left( 1 + \phi - (1+\phi)^{-1} \right)^{-2} < 1. \quad (15)$$

This restriction is not generally compatible with the CRR model. Suppose that we observe a risk-free asset with annual return of 6% and a risky asset with expected annual return of 16% and standard deviation of 20%. Assume that one period of the model represents one month. Calibrating the CRR model gives $r = .00487$, $\phi = .01681$ and $\pi = .86918$. These values are inconsistent with recursive competitive equilibrium because they violate (15). This particular risk free asset, risky asset pair cannot be represented by the CRR model in a recursive competitive equilibrium environment.

5. An Alternative Version of the Equivalence Theorem

Kreps' equivalence theorem can be salvaged in a more
limited form by letting the representative investor observe a more detailed economic environment than the econometrician. For example, the representative investor's income can depend upon random shocks which are not explained by the econometrician's observed state variables \( \theta \). We show below that there exists a competitive equilibrium over some larger set of state variables which is consistent with the observed non-arbitrage prices on the restricted set observed by the econometrician.

We define a refinement of \( \theta \) as an expanded vector of state variables \( \theta^* = (\theta^*_1, \ldots, \theta^*_k, \theta^*_k+1, \theta^*_\infty) \) where the first \( k \) terms are the original state variables \( \theta \). For particular time \( t \) values we say that \( \theta^*_t \in \Theta_t \) if the first \( k \) terms of \( \theta^*_t \) equal \( \theta_t \). We continue to assume that prices and dividends are adapted to \( \theta \), but allow representative investor income to be adapted to \( \theta^* \). We write \( p(\theta^*) \) and \( d(\theta^*) \) to mean \( p(\theta^*_t) = p(\theta_t) \) and \( d(\theta^*_t) = d(\theta_t) \) for all \( \theta^*_t \in \Theta_t \).

**Theorem 4:** A price process \( p(\theta) \) does not permit arbitrage if and only if there exists a refinement of \( \theta, \theta^* \), and a consumption cash flow process, \( X(\theta^*) \), such that given \( X(\theta^*) \), \( p(\theta^*) \) is a recursive competitive equilibrium price process.

Theorem 4 only partly revives the Kreps equivalence theorem, since it requires an expansion of the state space which may not be well-motivated in particular applications. In some cases, though, it can be appropriate. One possible motivation is the presence of real investment opportunities in
the economy (as opposed to our assumption of fixed supplies of asset endowments). Adding real investment opportunities requires an expansion of the representative investor's state space since this period's income depends upon last period's investment choice. This influence of past investment on current income is not captured by the current state variables determining asset payoffs. Consider the CRR model. We might replace and/or supplement the existing assets with linear investment technologies, as is done in Cox, Ingersoll and Ross (1985). The risk-free asset could be replaced with a linear technology with a time and state invariant return $1+r$, and the risky asset with a linear technology with state-dependent but time-invariant return $1+r_s$. It might be possible to make the CRR model fully consistent with recursive competitive equilibrium by adding real investment along these lines.
BIBLIOGRAPHY


APPENDIX

Proof of Theorem 1: The result is a direct application of Kreps (1981). Given current event \( \theta_t \), define the normed linear space over time \( t+1 \) payoffs, \( z \), by \( \|z\| = E[z^2|\theta_t] \). The time \( t+1 \) payoffs of event \( \theta_t \) portfolios, \( a'(p(\theta_{t+1}) + d(\theta_{t+1})) \), constitute a subspace of this normed linear space. Define \( K \) as the strictly positive cone in this linear space and then apply Kreps, 1981, lemma 5.

Proof of Theorem 2: Suppose \( p(\theta) \) is a competitive equilibrium price process for some \( X(\theta) \). For an arbitrary portfolio \( a \) in state \( \theta_t \), consider \( a'p(\theta_t) \) using (5):

\[
a'p(\theta_t) = pE[(u'(X(\theta_{t+1}))/u'(X(\theta_t)))a'(p(\theta_{t+1}) + d(\theta_{t+1}))|\theta_t]\quad (A1)
\]

Note that \( u'(X(\theta_{t+1}))/u'(X(\theta_t)) > 0 \) everywhere by the assumption that \( u' > 0 \). Given (2) and (3) the RHS of (A1) is strictly positive and so (1) does not hold.

Now suppose \( p(\theta) \) does not permit arbitrage. By Theorem 1 there exists \( \lambda(\theta_t, \theta_{t+1}) \) strictly positive almost surely and satisfying (4). By assumption \( u' \) is continuous and monotonic everywhere on its domain and ranges over \( (0, \infty) \). Let \( f(\cdot) \) denote the inverse of \( u' \). From the nature of a filtration, given event \( \theta_t \), then every \( \theta_{t-s} \) is determined, since each \( \theta_{t+1} \) is associated with a unique \( \theta_t \). Define a cash flow process, \( X(\theta_t) \), as follows:

\[
X(\theta_0) = f(1)
\]

\[
X(\theta_t) = f\left(\prod_{s=0}^{t-1} \lambda(\theta_s, \theta_{s+1})\right) \quad (A2)
\]
Substituting from (A2) into (4) yields (5) which defines competitive equilibrium prices.

**Proof of Corollary:** Same as the proof of Theorem 1.

**Proof of Theorem 3:** Suppose that there exists a non-arbitrage pricing function \( \lambda(\theta_t, \theta_{t+1}) \) which satisfies (9). Let \( f(.) \) denote the inverse of \( u' \). Define the cash flow process \( X(\theta_t) = f(\gamma(\theta_t)) \). Using this \( X(\theta) \), the recursive competitive equilibrium condition (8) holds since \( \lambda(\theta_t, \theta_{t+1}) = \gamma(\theta_{t+1})/\gamma(\theta_t) \) obeys the non-arbitrage condition (7).

Suppose that there exists a supporting recursive competitive equilibrium for some consumption cash flow process \( X(\theta) \). Define \( \lambda(\theta_t, \theta_{t+1}) \) as follows:

\[
\lambda(\theta_t, \theta_{t+1}) = \frac{u'(X(\theta_{t-1}))}{u'(X(\theta_t))}
\]

and note that \( \lambda(\theta_t, \theta_{t+1}) \) satisfies the non-arbitrage condition (7) since \( X(\theta) \) obeys the recursive competitive equilibrium condition (8). Obviously we set \( \gamma(\theta) = u'(X(\theta)) \).

**Proof of Theorem 4:** Suppose that for some refinement \( \theta^* \) there exists \( X(\theta^*) \) which supports competitive equilibrium. By the definition of competitive equilibrium:

\[
p(\theta^*_t) - \rho E \left[ \frac{u'(X(\theta^*_{t+1}))}{u'(X(\theta_t^*))} \left( p(\theta^*_{t+1}) + d(\theta^*_{t+1}) \mid \theta^*_t \right) \right] \quad (A3)
\]
Conditioning on the coarser information, \( \theta_t \), gives:

\[
P(\theta_t) - \rho E \left[ \frac{u'(X(\theta^*_{t+1}))}{u'(X(\theta^*_t))} \left( p(\theta^*_{t+1}) + d(\theta^*_t) \right) \mid \theta_t \right] \quad (A4)
\]

Consider an arbitrary \( n \)-vector, \( a \), applied to both sides of (A4):

\[
a'P(\theta_t) - \rho E \left[ \frac{u'(X(\theta^*_{t+1}))}{u'(X(\theta^*_t))} a'(p(\theta^*_{t+1}) + d(\theta^*_t) \mid \theta_t) \right] \quad (A5)
\]

If (2) and (3) hold for some \( a \), then it follows from (A5) that \( a'P(\theta_t) > 0 \), so that (1) does not hold. Hence there do not exist any arbitrage portfolios.

Suppose there do not exist any arbitrage portfolios. Then by Corollary 1 there exists a function \( \lambda(\theta_t, \theta^*_{t+1}) \) obeying (4). Using \( \lambda(\theta_t, \theta^*_{t+1}) \), define the stochastic process \( z \) as follows:

\[
z_t = 1 \text{ for } t = 0 \\
z_t = \lambda(\theta_{t-1}, \theta_t)z_{t-1} \text{ for } t > 0.
\]

Although the stochastic process \( z \) may not be adapted to \( \theta \), it is adapted to the refinement \( \theta^* \) defined by \( \theta^*_t = (\theta_t, \theta^*_{t-1}, z_{t-1}) \). Let \( f(.) \) denote the inverse of \( u' \). Define a consumption cash flow process, \( X(\theta^*) \) as follows:

\[
X(\theta^*_{0}) = f(1) \text{ for } t = 0, \\
X(\theta^*_{t}) = f(z(\theta^*_{t})) \text{ for } t > 0.
\]

It is easy to show, by the same steps as in Theorem 3, that \( X(\theta^*) \) supports recursive competitive equilibrium under \( p(\theta^*) \).