Consumption and Portfolio Decisions with Labor Income and Borrowing Constraints

by

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Consumption and Portfolio Decisions
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Abstract

We employ a duality approach to study intertemporal consumption and portfolio decisions when an individual has limited opportunities to borrow against future labor income and cannot totally insure the risk of income fluctuations. An individual’s optimal consumption-portfolio problem is cast in continuous-time under both certainty and uncertainty frameworks. The duality approach allows us to characterize in a simple way the individual’s optimal consumption and portfolio policies when there are labor income and borrowing constraints. Sufficient conditions for the existence of a solution to the individual’s consumption-portfolio problem are established.

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1 Introduction

The classical approach to the analysis of intertemporal consumption and portfolio decisions without labor income or borrowing constraints is to solve the individual’s indirect utility function using dynamic programming, see Merton (1971, 1973). However, this approach is difficult to implement when labor income and borrowing constraints are present, since the exact region in which these constraints are binding is unknown a priori and needs to be solved as a part of the solution. As a result, the classical analysis of optimal consumption and portfolio decisions, and the related intertemporal capital asset pricing models in the finance literature, have ignored the effect of borrowing constraints, see Merton (1971, 1973), Breeden (1978), and Cox, Ingersoll, and Ross (1985).¹

In this paper, we employ a duality approach to study intertemporal consumption and portfolio decisions when an individual has limited opportunities to borrow against future labor income and cannot totally insure the risk of income fluctuations. The basic idea of the approach is to transform the individual’s consumption-portfolio problem with borrowing constraints into an unconstrained dual minimization problem where the solution to the dual problem completely determines the solution to the original problem. Unlike the borrowing-constrained consumption-portfolio problem, the dual problem can be solved easily by dynamic programming. As a result, the duality approach allows us to characterize in a simple way the individual’s optimal consumption and portfolio decisions. It also enables us to address the issue of existence of a solution to the individual’s consumption-portfolio problem.

The roles of borrowing constraints and uninsurable risks have long been of concern to economists. Mehra and Prescott (1985) have suggested borrowing constraints as one of the market frictions that might be blamed for the equity premium puzzle. Bewley (1982), Mankiw (1986), and Scheinkman (1988) have argued that ignoring the possibility of uninsurable risks may lead to an over-estimation of the degree of relative risk aversion on the part of the representative agent when markets are incomplete. The effect of borrowing constraints and uninsurable risks on a single individual’s optimal consumption and investment behavior has been investigated by Schectman (1976), Bewley (1977), and Hellwig (1982) in discrete-time settings with a single riskless asset and a stream of uncertain labor income. The risks induced by the labor income cannot be insured. Due to the technical complexity of the problem, none of these authors was able to obtain explicit characterization of an individual’s optimal consumption and investment behavior, except for some limited asymptotic results. Bewley (1980, 1983) and Hernandez (1989) have also explored the existence of a

¹The solvency constraints examined in Håkansson (1970) are weaker than the borrowing constraints considered in this paper.
competitive equilibrium with borrowing constraints, but obtain little characterization of agents' optimal consumption and investment behavior.² More recently, Mariger (1987) has studied a simple discrete-time, certainty model and shown that borrowing constraints shorten an individual's effective planning horizon.³ Charpin (1989) has solved a continuous-time, certainty model under either a quantity constraint or an interest wedge between the borrowing and the lending rates; the functional form she assumes for income streams depends on two parameters which are adjusted to approximate observed wages profiles. Svensson (1988) has analyzed a continuous-time model with labor income under uncertainty, but individuals are allowed to borrow against their future income.

In this paper we study an individual's consumption-portfolio problem in the presence of labor income and borrowing constraints in a continuous-time framework under certainty and uncertainty. In addition to an initial wealth, the individual receives a stream of nonnegative labor income throughout his lifetime. In the certainty case, the individual receives a stream of deterministic labor income and can invest his wealth in a riskless asset, while in the uncertainty case, the individual receives a stream of stochastic labor income and can invest in both a riskless and a risky asset. The borrowing constraints limit the individual's ability to borrow against his future income in the sense that the individual cannot "sell" his labor income in the security market. Specifically, the individual optimally reallocates his wealth between consumption and investment, where his wealth at each point in time is constrained to be non-negative.

Unlike the models of Schectman (1976), Bewley (1977), and Hellwig (1982), our model allows the individual's labor income to be spanned by the existing assets. Although the introduction of labor income does not in itself constitute a new source of uncertainty in the economy, the risks associated with the labor income still cannot be totally insured because of the borrowing constraints. Therefore, our model analyzes the effect of borrowing constraints as well as the effect of uninsurable risks induced by the borrowing constraints. The assumption on spanning simplifies the analysis of borrowing constraints, and enables us to obtain closed form solutions for many of the cases that we consider.

If a complete set of Arrow-Debreu contingent claims markets existed for labor income, the individual could simply "sell" his labor income at the beginning of the planning horizon. His optimal policies then would be no different from those of an otherwise identical individual endowed with the same initial wealth supplemented with the cash value of his future labor income. The absence of a complete insurance market for labor income changes the individual's effective planning

²Scheinkman and Weiss (1986) characterize aggregate consumption and production decisions for a two agent economy with borrowing constraints in a specialized setting in which production decisions are stochastic and uncertainty is generated by a Poisson counting process.
³Mariger also tests the implications of borrowing constraints using U.S. families data and concludes that borrowing constrained families account for 16.7 percent of consumption in the population sampled.
horizon and gives rise to a precautionary demand for wealth.

In the certainty case, the individual divides his lifetime planning horizon into several sub-periods. At the beginning of the first (or odd) period, the individual is able to “sell” in the security market his labor income expected for that period. The individual can then solve the consumption-portfolio problem as if there were no borrowing constraints. Intuitively speaking, the individual expects his future labor income to fall, and therefore saves part of his wealth to efficiently allocate his consumption across time. In the second (or even) period, the individual expects his future labor income to rise and chooses to consume all of his labor income as it is received. In contrast to a world with complete contingent claims markets, the optimal consumption and investment decisions depend upon the entire profile of future labor income.

In the uncertainty case, the individual also divides his lifetime planning horizon into several sub-periods, but the duration of the time in which the individual is capable of “selling” his future labor income is a random variable and depends upon both the sample path of the asset price and the entire profile of the labor income. Unlike in the certainty case, the individual solves the consumption-portfolio problem as if there were no borrowing constraints in any of the sub-periods. Hence, the individual will never consume all of his labor income, as in the even periods of the certainty model.

Optimizing the dual problem has the economic interpretation of choosing the system of Arrow-Debreu state contingent prices that minimizes the individual’s maximum attainable utility level. The optimal control variable in the dual minimization problem forms an implicit system of Arrow-Debreu state contingent prices by which the individual does not want to choose a consumption-portfolio policy that would violate the borrowing constraints. If he could violate such constraints, the individual could increase his utility; therefore, preventing him from doing so amounts to decreasing the maximum utility level he can attain. The price system under which the individual would not purchase the restricted consumption claims is that system which minimizes the maximum attainable utility. These Arrow-Debreu state prices are similar to the minimax local martingale measure developed by He and Pearson (1989), who study optimal consumption and portfolio decisions in an incomplete securities market.

The paper is organized as follows. In Section 2, we consider an economy under certainty with a single riskless asset and study the individual’s optimal consumption and savings decisions. Section 3 studies an economy under uncertainty with two assets, a riskless asset and a risky asset. We provide an existence theory as well as a complete characterization of the individual’s optimal consumption and investment decisions. We summarize our results in Section 4.
2 The Certainty Case

In this section, we formulate an individual's consumption-saving problem when there are labor income and borrowing constraints in continuous-time under certainty. We establish a duality theory between the individual's consumption-saving problem and a dual minimization problem. We prove that the solution to the dual problem completely determines the solution to the original consumption-saving problem. We then provide a general characterization of the individual's optimal consumption and saving policies. Finally, we obtain closed form solutions for a class of utility functions with constant relative risk aversion (CRRA) when the labor income profile has certain simple structure.

2.1 The Economic Setting

We assume that the individual lives from time 0 to time $T$ ($T < \infty$) in a certain world in which there is a single good available for consumption at any time between 0 and $T$ ($T$ may be finite or infinite). The individual's preferences are represented by a time-additive utility function, $e^{-\delta t}u(c(t))$, where $\delta \geq 0$ is the discount factor expressing the individual's impatience, $c(t)$ is the consumption rate at time $t$, and $u$ is a continuous, increasing, and strictly concave function.

The individual receives an initial wealth $W_0 > 0$ at time 0 as well as a stream of non-negative labor income $y(t)$ at time $t$ from 0 to $T$. There is available for investment a single riskless asset with a return $r > 0$. Since the individual cannot borrow against his future labor income, he can only invest a non-negative amount in the riskless asset at each point in time. This implies that the individual's wealth at each point in time is non-negative. The individual allocates his wealth between consumption and saving in order to maximize his lifetime utility, subject to the constraint that his wealth is non-negative. Thus, the individual solves the following utility maximization problem:

$$\max_{c \geq 0} \int_0^T e^{-\delta t}u(c(t))dt$$

such that

$$\frac{dW(t)}{dt} = rW(t) + y(t) - c(t), \quad W(t) \geq 0.$$  

By direct integration, the dynamics of the wealth function can be written as

$$\int_0^tc(s)e^{-rs}ds + W(t)e^{-rt} = W_0 + \int_0^ty(s)e^{-rs}ds.$$

\(^4\text{In Section 3, we allow the riskless rate } r \text{ to be a function of } t, \text{ and allow the utility function to be of the form } u(z, t).\)
Hence, a feasible consumption policy that satisfies the borrowing constraints must be such that
\[
\int_0^t c(s)e^{-rs}ds \leq W_0 + \int_0^t y(s)e^{-rs}ds, \quad \forall t \in [0,T].
\]
The individual's problem \( (\ast) \) is equivalent to
\[
\max_{c \geq 0} \int_0^T e^{-\delta t}u(c(t))dt \tag{P}
\]
\[
\text{s.t. } \sup_{t \in [0,T]} \left[ \int_0^t (c(s) - y(s))e^{-rs}ds \right] \leq W_0.
\]
For the rest of this section, we need to make an assumption on the individual's labor income.

**Assumption 1**
\[
\int_0^T y(s)e^{-rs}ds < \infty,
\]
i.e., the present value of the future labor income is finite.

### 2.2 A Dual Minimization Problem: An Intuitive Derivation

We can consider the above consumption-saving problem \( (P) \) as a maximization problem with infinitely many inequality constraints. Therefore, we can solve \( (P) \) using the Lagrangian method. Let \( \lambda(t) \geq 0 \) be the Lagrangian multiplier for the non-negative wealth constraint at time \( t \), then
\[
\mathcal{L} = \int_0^T e^{-\delta t}u(c(t))dt + \lambda(T) \left( W_0 + \int_0^T (y(s) - c(s))e^{-rs}ds \right)
\]
\[
+ \int_0^T \lambda(t) \left( W_0 + \int_0^t (y(s) - c(s))e^{-rs}ds \right) dt
\]
\[
= \int_0^T e^{-\delta t}u(c(t))dt + \left( \lambda(T) + \int_0^T \lambda(s)ds \right) W_0 + \int_0^T \left( \lambda(T) + \int_t^T \lambda(s)ds \right) (y(t) - c(t)) e^{-rt}dt.
\]
Letting \( X(t) = \lambda(T) + \int_t^T \lambda(s)ds \), we obtain
\[
\mathcal{L} = \int_0^T e^{-\delta t}u(c(t))dt + X(0)W_0 + \int_0^T X(t)(y(t) - c(t)) e^{-rt}dt
\]
Note that \( X(t) \) is non-negative and decreasing.

According to the Saddle Point Theorem, cf., Holmes (1987) or Rockafellar (1975), the optimal consumption policy \( c \) can be found first by maximizing \( \mathcal{L} \) with respect to \( c \) (with no constraints) and then minimizing \( \mathcal{L} \) with respect to \( X \) subject to the requirement that \( X \) be non-negative and decreasing. To carry out these steps, we define the convex conjugate of \( u \) as follows:
\[
\bar{u}(y) = \max_{x \geq 0} \{ u(x) - xy \},
\]
and define the inverse of $u'_+$

$$f(x) = \inf\{y \geq 0 : u'_+(y) \leq x\},$$

where $u'_+$ denotes the right-hand derivative. It is easy to verify that $\bar{u}$ is strictly convex, strictly decreasing, and $\bar{u}'(y) = -f(y)$, cf., Rockafellar. Substituting the above notation yields

$$\max_{c \geq 0} \mathcal{L} = \int_0^T e^{-\delta t} \bar{u}(e^{(\delta-r)t} X(t)) dt + X(0)W_0 + \int_0^T X(t)y(t)e^{-rt} dt.$$

Next, we minimize the right-hand-side of the above equation with respect to $X$, i.e.,

$$\min_{X \in \mathcal{D}} \left[ \int_0^T e^{-\delta t} \bar{u}(e^{(\delta-r)t} X(t)) dt + X(0)W_0 + \int_0^T X(t)y(t)e^{-rt} dt \right], \quad (D)$$

where $\mathcal{D}$ denotes the set of non-negative and decreasing functions. We call the above minimization problem the dual problem of $(P)$, and consequently $(P)$ the primal problem. This dual problem is similar to the one studied in He and Pearson (1989). Before solving the dual problem $(D)$, we show in the next subsection that the solution to $(D)$, if it exists, completely determines the solution to $(P)$.

### 2.3 Dual Optimality and Implicit Arrow-Debreu Prices

We now establish the duality between $(P)$ and $(D)$. Define

$$\mathcal{L}(X) \equiv \int_0^T e^{-\delta t} \bar{u}(e^{(\delta-r)t} X(t)) dt + X(0)W_0 + \int_0^T X(t)y(t)e^{-rt} dt.$$

The following theorem establishes the connection between $(P)$ and $(D)$.

**Theorem 1** If $X^*(t)$ is a solution to the dual problem $(D)$, then

$$c^*(t) = f(e^{(\delta-r)t} X^*(t))$$

is a solution to the primal problem $(P)$.

**Proof.** We first show that $c^*$ is feasible, i.e., it satisfies the non-negative wealth constraints. Consider a decreasing function $X^* \equiv X^* + \epsilon 1_{[0,r]} \in \mathcal{D}$, where $\epsilon > 0$ and $r \in (0,T]$. Since $\mathcal{L}(X^*) \geq \mathcal{L}(X^*),$

$$\lim_{\epsilon \downarrow 0} \sup \frac{\mathcal{L}(X^*) - \mathcal{L}(X^*)}{\epsilon} \geq 0.$$

This implies that

$$\lim_{\epsilon \downarrow 0} \sup \int_0^T e^{-\delta s} \frac{\bar{u}(e^{(\delta-r)s} X^*(s)) - \bar{u}(e^{(\delta-r)s} X^*(s))}{\epsilon} ds + \int_0^T y(s)e^{-rs} ds + W_0 \geq 0.$$
Since \( \tilde{u}(e^{(s-r)s}X^*(s)) \leq \tilde{u}(e^{(s-r)s}X^*(s)) \) and \( \tilde{u}' = -f \), Fatou's Lemma implies that
\[
\int_0^T c^*(s)e^{-rs}ds \leq W_0 + \int_0^T y(s)e^{-rs}ds,
\]
for all \( r \in (0,T] \). Thus, \( c^* \) is feasible.

We now show that \( c^* \) is also optimal. For any feasible consumption policy \( c \), we have
\[
\int_0^T \left( \int_0^T c(s)e^{-rs}dX^*(r) \right) dX^*(r) \geq \int_0^T \left( W_0 + \int_0^T y(s)e^{-rs}ds \right) dX^*(r).
\]
Integrating by parts and using (1) with \( r = T \) yields
\[
\int_0^T X^*(s)c(s)e^{-rs}ds \leq X^*(0)W_0 + \int_0^T X^*(s)y(s)e^{-rs}ds + X^*(T) \left( \int_0^T c(s)e^{-rs}ds - W_0 - \int_0^T y(s)e^{-rs}ds \right)
\]
\[
\leq X^*(0)W_0 + \int_0^T X^*(s)y(s)e^{-rs}ds.
\]
(2)

We further show that an equality relationship holds for \( c^* \). To demonstrate this, we consider \( X^\epsilon = X^*(1 + \epsilon) \in \mathcal{D} \) for small \( \epsilon \). Since \( \mathcal{L}(X^\epsilon) \geq \mathcal{L}(X^*) \), we have
\[
\limsup_{\epsilon \to 0} \frac{\mathcal{L}(X^\epsilon) - \mathcal{L}(X^*)}{\epsilon} \geq 0, \quad \liminf_{\epsilon \to 0} \frac{\mathcal{L}(X^\epsilon) - \mathcal{L}(X^*)}{\epsilon} \leq 0.
\]
Invoking Fatou's Lemma, we obtain
\[
\int_0^T X^*(s)c^*(s)e^{-rs}ds = X^*(0)W_0 + \int_0^T X^*(s)y(s)e^{-rs}ds.
\]
According to the Lagrangian method, \( c^* \) is in fact the solution to the program
\[
\max_{c \geq 0} \int_0^T e^{-\delta t}u(c(t))dt
\]
\[
\text{s.t. } \int_0^T X^*(t)c(t)e^{-rt}dt \leq X^*(0)W_0 + \int_0^T X^*(t)y(t)e^{-rt}dt.
\]
Since the maximum utility in (P) is bounded above by the maximum utility of the above program, \( c^* \) must also be the solution to (P).

Note that (2) holds for \((c,X)\), where \( c \) is a feasible consumption policy and \( X \in \mathcal{D} \). Therefore, \( X \) can be interpreted as an implicit system of Arrow-Debreu state prices under which the present value of a feasible consumption policy is less than or equal to the initial wealth plus the present value of the future labor income. The optimal solution to the dual problem, \( X^* \), comprises an implicit system of Arrow-Debreu state prices according to which the individual does not want to
choose a consumption policy that would violate the borrowing constraints. Moreover, if $X^*$ were the Arrow-Debreu state prices and the individual were allowed to sell his labor income at these prices, then the individual’s optimal consumption policy based on $X^*$ would be identical to the solution of $(P)$.

We further claim that the dual problem involves choosing a system of implicit Arrow-Debreu state prices that minimizes the individual’s maximum attainable utility level. Consider the optimal solution of the utility maximization problem under any price system $X \in \mathcal{P}$,

$$
\max_{c \geq 0} \int_0^T e^{-\delta t} u(c(t)) dt
$$

s.t. $\int_0^T X(t)c(t)e^{-rt} dt \leq X(0)W_0 + \int_0^T X(t)y(t)e^{-rt} dt$.

Assume that the solution to the above problem exists. Let $\text{Val}(X)$ denote the maximum utility level of this problem. According to the Lagrangian method, the optimal solution to the above problem is

$$
c(t) = f(he^{(\delta-r)t}X(t)),
$$

for some constant $h > 0$ such that

$$
\int_0^T X(t)f(he^{(\delta-r)t}X(t))e^{-rt} dt = X(0)W_0 + \int_0^T X(t)y(t)e^{-rt} dt.
$$

Since $\mathcal{L}(X^*) \leq \mathcal{L}(hX)$ and $\mathcal{U}(z) = u(f(x)) - x f(x)$, we have

$$
\text{Val}(X^*) = \int_0^T e^{-\delta t} u(c^*(t)) dt = \mathcal{L}(X^*) \leq \mathcal{L}(hX)
$$

$$
= \int_0^T e^{-\delta t} u(c(t)) dt - \int_0^T hX(t)c(t)e^{-rt} dt + hX(0)W_0 + \int_0^T hX(t)y(t)e^{-rt} dt
$$

$$
= \int_0^T e^{-\delta t} u(c(t)) = \text{Val}(X).
$$

Thus, the dual problem chooses a system of implicit Arrow-Debreu state prices that minimizes the individual’s maximum attainable utility level. In this next subsection we shall provide a characterization of the solution to the dual problem. The existence proofs for the primal and dual problems are deferred to Section 3.3.

### 2.4 Characterization

In order to characterize the optimal solution of the dual problem, we employ the dynamic programming method. Assume for the moment that the decreasing function $X(t)$ is absolutely continuous.
with respect to $t$, meaning that there exists a measurable function $\psi(t) \geq 0$ such that $\frac{dX(t)}{dt} = -\psi(t)$. Define
\[
J(x, t) = \inf_{(X(t), X(t)) = x} \int_t^T \left( e^{-\delta s} \bar{u}(e^{(\delta - \rho)s} X(s)) + X(s) y(s)e^{-r s} \right) ds
\]
to be the dual indirect utility function. Since $\bar{u}$ is strictly convex, $J$ is also strictly convex in $x$. It is easy to verify that $J$ is decreasing in $z$. If we assume that $J$ is continuously differentiable with respect to both $x$ and $t$, then the Bellman Optimality Principle implies that
\[
\min_{\psi \geq 0} \left\{ e^{-\delta t} \bar{u}(e^{(\delta - r)t} x) + x y(t)e^{-r t} - J_x \psi(t) + J_t \right\} = 0,
\]
see Fleming and Rishel (1975). We obtain the following characterization for $\psi$:
\[
\begin{align*}
\psi^* &\in [0, +\infty], & \text{if } J_x = 0, \\
\psi^* &\equiv 0, & \text{if } J_x < 0.
\end{align*}
\]
Since $J_x \leq 0$, the Bellman equation suggests that at the optimum $X^*$ can decrease at any rate when $J_x = 0$, but must stay constant when $J_x < 0$. Thus, $X^*$ may have jumps when $J_x = 0$. We henceforth define the no-jump region $\Omega_1$ and the jump region $\Omega_2$ as
\[
\Omega_1 = \{(x, t) : J_x(x, t) < 0\}, \quad \Omega_2 = \{(x, t) : J_x(x, t) = 0\},
\]
and define the critical boundary $\gamma$ to be the curve that separates $\Omega_1$ and $\Omega_2$, i.e., $\text{cl}(\Omega_1) \cap \text{cl}(\Omega_2)$, see Figure 1. At any time $t$, if $X^*$ starts at $x$ where $(x, t) \in \Omega_2$, then $X^*$ will exhibit an immediate jump from $x$ to $\gamma(t)$, since $J(x, t) = J(\gamma(t), t)$. If $X^*$ starts at $x$ where $(x, t) \in \Omega_1$, then $X^*$ will move horizontally until it hits $\gamma$. Once it hits the critical boundary, $X^*$ may move downwards along the critical boundary, but remain inside the closure of $\Omega_1$. We will see below that the critical boundary helps us determine at what time the borrowing constraints become binding.

Since $J_x \leq 0$, the Bellman equation becomes
\[
\min \left\{ e^{-\delta t} \bar{u}(e^{(\delta - r)t} x) + x y(t)e^{-r t} + J_t, \ -J_x \right\} = 0,
\]
a first order partial differential equation with a free boundary. If $J$ is the dual indirect utility function and if $J$ is differentiable, then $J$ must satisfy the above Bellman equation.

Conversely, we prove in the following verification theorem that if there exists a solution to the Bellman equation, it must be the dual indirect utility function. Moreover, the optimal consumption policy can be determined completely from the dual indirect utility function. The proof for this theorem is deferred to Section 3.4, where we prove a similar theorem for the uncertainty case.
Theorem 2 (Verification) Let \( J(x,t) \) be a solution to the Bellman differential equation
\[
\min \left\{ e^{-\delta t} \bar{u}(e^{(\delta-r)t}x) + x y(t) e^{-rt} + J_t, -J_x \right\} = 0, \quad (x,t) \in \mathbb{R}_+ \times [0,T]
\]
together with the boundary condition
\[
J(x,T) = 0, \quad x \in \mathbb{R}_+
\]
such that \( J \) is convex in \( x \) and continuously differentiable w.r.t. both \( x \) and \( t \), up to the second order. Moreover, there exists \( X^* \in \mathcal{D} \) with a continuous path such that, for all \( t \in [0,T] \),
\[
\begin{align*}
&i) \quad e^{-\delta t} \bar{u}(e^{(\delta-r)t}X^*(t)) + X^*(t)y(t)e^{-rt} + J_t(X^*(t),t) = 0; \\
&ii) \int_t^{t+\epsilon} J_x(X^*(s),s) dX^*(s) = 0 \text{ for all } \epsilon > 0; \\
&iii) J_x(X^*(0),0) + W_0 = 0; \\
&iv) \quad \lim_{T \to \infty} \left[ \int_0^T e^{-\delta t} \bar{u}(e^{(\delta-r)t}X^*(t)) dt + \int_0^T X^*(t)y(t) dt + J(X^*(T),T) \right] \\
&= \int_0^\infty e^{-\delta t} \bar{u}(e^{(\delta-r)t}X^*(t)) dt + \int_0^\infty X^*(t)y(t) dt.
\end{align*}
\]
Then, \( J \) is the dual indirect utility function, and \( X^* \) is the optimal solution to the dual problem. The optimal consumption and wealth functions are determined by
\[
c^*(t) = f(e^{(\delta-r)t}X^*(t)), \quad W^*(t) = -e^{rt} J_x(X^*(t),t).
\]

Remark 1 Since \( J_x(\gamma(0),0) = 0 \) and \( W_0 > 0 \), \( X^*(0) \) must be strictly less than \( \gamma(0) \). Therefore, \( X^* \) starts at the interior of \( \Omega_1 \), and remains inside the closure of \( \Omega_1 \).

The optimal control problem considered here is known as impulsive control or singular control in the control literature, see Bensoussan and Lions (1984) and Harrison (1985). The sample path of the control variable could exhibit both sudden jumps and continuous changes. Standard dynamic programming can be applied to such control problems. Typically, the optimal solution involves solving a partial differential equation with a free boundary. The papers by Davis and Norman (1988) and Hindy and Huang (1989) also involve singular control. When the starting point of the control variable is given a priori, the optimal sample path may have an immediate jump if the starting point is inside the jump region. No jumps will occur thereafter. In the above problem, the initial value of \( X^* \) is a part of the solution. Since \( X^* \) starts at the no-jump region, it has a continuous sample path and stays inside the closure of the no-jump region.

We now show that the economic effect of the borrowing constraints is to shorten an individual's planning horizon. The individual divides his lifetime horizon \([0,T]\) into several sub-periods. In
some of the sub-periods, the individual is able to "sell" his labor income in the security market and solves the consumption-saving problem as if there were no borrowing constraints. In other sub-periods, the individual expects his income to rise and chooses to consume his labor income as it is received.

**Proposition 1** Suppose that $X^*$ is the solution to (D) and that $J$ is the dual indirect utility function. Define

$$
\begin{align*}
\tau_1 &= \inf\{t > 0 : X^*(t) < X^*(0)\} \\
\tau_2 &= \inf\{t > \tau_1 : X^*(t) \neq \gamma(t)\} \\
\tau_3 &= \inf\{t > \tau_2 : X^*(t) < X^*(\tau_2)\} \\
\tau_4 &= \inf\{t > \tau_3 : X^*(t) \neq \gamma(t)\} \quad \cdots
\end{align*}
$$

Then,

i) the individual's first planning horizon is $[0, \tau_1]$, i.e., if the individual were allowed to sell his labor income in the security market, the optimal consumption policy on $[0, \tau_1]$ would not violate the borrowing constraints, whereas the optimal consumption policy on $[0, \tilde{t})$, for any $\tilde{t} > \tau_1$, would violate the borrowing constraints;

ii) on $[\tau_2, \tau_3)$, $W^*(t) \equiv 0$ and $c^*(t) \equiv y(t)$, i.e., the individual consumes the labor income as it is received, while maintaining his wealth at zero level.

The optimal consumption policies on $[\tau_2, \tau_3)$ and $[\tau_3, \tau_4)$ can be analyzed similarly.

**Proof.** Clearly, $W^*(t) = -e^{rt}J_x(X^*(t), t) \geq 0$ for $t \in [0, \tau_1)$ and $W^*(\tau_1) = 0$. We first claim that the optimal consumption policy $c^*$ from 0 to $\tau_1$ is exactly the same as the solution to the following program:

$$
\begin{align*}
\max_{c \geq 0} & \quad \int_0^{\tau_1} e^{-\delta t} u(c(t))dt \\
\text{s.t.} & \quad \int_0^{\tau_1} c(t)e^{-rt}dt \leq W_0 + \int_0^{\tau_1} y(t)e^{-rt}dt,
\end{align*}
$$

with $X^*(0)$ being the Lagrangian multiplier of this program. This follows directly from the fact that $c^*(t) = f(e^{(\delta-r)s}X^*(0))$ for $t \in [0, \tau_1)$. If the planning horizon were $[0, \tau_1)$ and if there were no borrowing constraints, then the individual would solve exactly the same program.

Next, we note that $X(t) = \gamma(t)$ on $[\tau_1, \tau_2)$ by construction. Since $J_x(X^*(t), t) \equiv 0$, we have

$$
\int_0^t c^*(s)e^{-rs}ds = W_0 + \int_0^t y(s)e^{-rs}dt
$$
for \( t \in [r_1, r_2] \). We deduce that \( c^*(t) = y(t) \) on \([r_1, r_2]\). That is, after running into zero wealth, the individual consumes his labor income as it is received.

Finally, we demonstrate that the optimal consumption policy would violate the borrowing constraints if the effective planning horizon were \([0, \tilde{t}]\). Consider the optimal wealth function \( W \) for the problems on \([0, \tilde{t}]\). Suppose that \( W \) does not violate the borrowing constraints, then \( W(s) \geq 0 \) if \( s < \tilde{t} \) and \( W(\tilde{t}) = 0 \). Since \( W^*(\tilde{t}) \geq 0 \) and \( W^*(t) = 0 \) for \( t \in [r_1, r_2] \), we can find a \( t_0 \in (r_1, \tilde{t}) \) such that \( W^*(t_0) = W(t_0) \). Since both wealth functions and consumption functions are optimal, they must be identical on \([0, t_0]\). The optimal consumption function without the borrowing constraints on \([0, \tilde{t}]\) has a form \( f(he^{(\delta-r)t}) \). Hence,

\[
c^*(t) = f(e^{(\delta-r)t}X^*(t)) = f(he^{(\delta-r)t}) \quad \forall t \in [0, \tilde{t}].
\]

This implies that \( X^*(t) \equiv h \) for all \( t \in [0, \tilde{t}] \), which contradicts the definition of \( r_1 \). The proof is now complete.  

Remark 2 On \([r_1, r_2]\), we have \( c^*(t) = f(e^{(\delta-r)t}X^*(t)) = y(t) \), where \( X^* \) is decreasing. This can happen only when \( e^{-(\delta-r)t}u'_+ (y(t)) \) is decreasing. When \( \delta \leq r \), \( y \) has to be increasing.

In summary, Theorem 2 and Proposition 1 completely characterize the individual's consumption and saving policies. The borrowing constraints divides the individual's initial planning horizon into several sub-periods. In the first period, the individual saves part of his initial wealth and labor income to smooth his consumption, whereas in the second period, the individual consumes all of his labor income and maintains a zero wealth level.

### 2.5 Examples: CRRA Utility Functions

For illustrative purposes, we consider in this section some examples for which closed form solutions exist. We assume the CRRA class utility functions, \( u(x) = \frac{x^\alpha}{\alpha} \), where \( \alpha < 1, \alpha \neq 0 \). We also assume that \( T \) is finite and that \( y(t) > 0 \) is differentiable with respect to \( t \).

**Example I:** \( e^{\frac{r-t}{\alpha}}y(t) \) is increasing in \( t \).

We claim that the critical boundary is \( \gamma(t) = e^{(r-\delta)t}y(t)^{\alpha-1} \), which is a decreasing function of \( t \), see Figure 2. Note first that the Bellman equation on the no-jump region \( \Omega_1 \) is

\[
-e^{\delta t} \frac{x^3}{b} + y(t)e^{-rt} + J_t = 0,
\]
where \( b = \frac{a}{\alpha - 1} \) and \( \theta = -\delta + b(\delta - r) \). Hence, the solution to this partial differential equation (PDE) on \( \Omega_1 \) has a form

\[
J^1(x, t) = \begin{cases} 
\frac{1}{b} (e^{\theta T} - e^{\theta t}) \frac{x^b}{b} + \left( \int_t^T y(s)e^{-rs}ds \right) x + A(x), & \text{if } x > \gamma(T) \\
\frac{1}{b} (e^{\theta T} - e^{\theta t}) \frac{x^b}{b} + \left( \int_t^T y(s)e^{-rs}ds \right) x, & \text{if } x \leq \gamma(T) 
\end{cases}
\]

where \( A(x) \) is a differentiable function of \( x \) such that \( A(\gamma(T)) = 0 \). By the continuity of \( J \) at the critical boundary, we have

\[
J^1_x(\gamma(t), t) = \frac{1}{\theta} (e^{\theta T} - e^{\theta t}) \gamma(t)^{b-1} + \int_t^T y(s)e^{-rs}ds + A'(\gamma(t)) = 0.
\]

This allows us to back out the functional form of \( A(\cdot) \). For each \((x, t) \in \Omega_1\), define the inverse function of \( \gamma \) as \( t(x) = \inf\{0 \leq t \leq T : \gamma(t) \leq x\} \), and define

\[
A(x) \equiv -\int_{\gamma(T)}^{x} \left( \frac{1}{\theta} (e^{\theta t} - e^{\theta \eta}) \eta^{b-1} + \int_{t(\eta)}^{T} y(s)e^{-rs}ds \right) d\eta.
\]

We now claim that the dual indirect utility function \( J(x, t) \) is

\[
J(x, t) = \begin{cases} 
J^1(x, t), & \text{if } (x, t) \in \Omega_1 \\
\frac{1}{b} (e^{\theta t} - e^{\theta T}) \frac{x^b}{b} + \left( \int_t^T y(s)e^{-rs}ds \right) \gamma(t) + A(\gamma(t)), & \text{if } (x, t) \in \Omega_2 
\end{cases}
\]

To verify that \( J \) is the dual indirect utility function, we need to check that \( J \) is convex in \( x \), and

1) On \( \Omega_1 \), \( J_x < 0 \) and

\[-e^{\theta t} \frac{x^b}{b} + xy(t)e^{-rt} + J_t = 0;\]

2) On \( \Omega_2 \), \( J_x = 0 \) and

\[-e^{\theta t} \frac{x^b}{b} + xy(t)e^{-rt} + J_t > 0;\]

3) On \( \text{cl}(\Omega_1) \cap \text{cl}(\Omega_2) \), \( J_x = -e^{\theta t} \frac{x^b}{b} + xy(t)e^{-rt} + J_t = 0.\)

Indeed, we have on \( \Omega_1 \) that

\[
J_x(x, t) = \int_t^{t(\gamma)} (-e^{\theta s} x^{b-1} + y(s)e^{-rs}) ds.
\]

\[= \int_t^{t(\gamma)} e^{\theta s} (y(s)^{b-1} - x^{b-1}) ds.\]

\[\text{Intuitively, when } x \leq \gamma(T), X^* \text{ stays constant at } x, \text{ hence } J^1 \text{ is equal to the integration with } X^* \equiv x. \text{ When } x > \gamma(T), X^* \text{ might be decreasing, therefore } J^1 \text{ needs to be adjusted by } A(x).\]
Hence, \( J_{zz} = (1 - b) \int^t_x e^s x^{b-2} ds > 0 \), i.e., \( J \) is convex in \( x \). Since \( x < \gamma(t) \) for \( s \in [t, t(x)] \) and \( b - 1 < 0 \), we have \( J_x < 0 \). Furthermore, \( J_x = 0 \) if and only if \( x = \gamma(t) \), i.e., \((x, t)\) is on the critical boundary with \( t(\gamma(t)) = t \).

On \( \Omega_2 \), we have that

\[
J_t = e^{bt} \gamma(t)^b t - y(t)e^{-rt} \gamma(t) + \left( \frac{1}{b} (e^{bt} - e^{bt}) \gamma(t)^{b-1} + \int_t^T y(s)e^{-rs} ds + A'(\gamma(t)) \right) \frac{d\gamma(t)}{dt}.
\]

Hence,

\[
-e^{bt} x^b_t + xy(t)e^{-rt} + J_t = -e^{bt} x^b_t + xy(t)e^{-rt} + e^{bt} \gamma(t)^b - \gamma(t)y(t)e^{-rt}.
\]

The right-hand-side of the above equation is non-negative if and only if \( x \geq \gamma(t) \) and is zero if and only if \( x = \gamma(t) \). We conclude that \( J \) is indeed the dual indirect utility function.

To find the optimal starting point \( X^*(0) \), we solve the following equation for \( X^*(0) \):

\[
J_x(X^*(0), 0) + W_0 = 0
\]

or equivalently

\[
\int^t_{\gamma(0)} \left( -e^s X^*(0))^{b-1} + y(s)e^{-rs} \right) ds + W_0 = 0.
\]

Since \( J_x(\gamma(0), 0) = 0 \), we expect \( X^*(0) < \gamma(0) \). There are two possibilities.

a) \( X^*(0) > \gamma(T) \). In this case, the borrowing constraints will be binding at the first time, \( r_1 \), when \( X^* \) hits the critical boundary, and \( X^* \) decreases after \( r_1 \). Hence, the individual is able to “sell” his labor income between \([0, r_1]\) in the security market. At time \( r_1 \), the individual’s wealth reaches zero, and the individual consumes all of his labor income from then on.

b) \( X^*(0) \leq \gamma(T) \). In this case, the borrowing constraints will not be binding, and \( X^* \) is constant on \([0, T]\). The individual is able to “sell” all of his labor income in the security markets at time 0, and makes his optimal consumption and saving decisions as if there were no borrowing constraints at all.

In summary, if the labor income grows quickly enough so that \( e^{(\mathbb{R} - s)(\mathbb{R} - l)}y(t) \) is increasing, the borrowing constraints may or may not be binding depending upon the size of the initial wealth. If the initial wealth is small, then the individual can “sell” his labor income only for the initial part of the time horizon. If the initial wealth is large, then the individual can “sell” his labor income for the entire time horizon.
Example II: $e^{(\frac{x-\epsilon}{\alpha-1})t}y(t)$ is decreasing in $t$.

We show below that the critical boundary is

$$\gamma(t) = \left[ \frac{\int_t^T y(s)e^{-rs}ds}{\int_t^T e^{\theta s}ds} \right]^{\frac{1}{\alpha - 1}}.$$ 

We claim that $\gamma(t)$ lies above $e^{(r-\delta)t}y(t)^{\alpha-1}$ and is increasing in $t$, see Figure 3. This is because

$$\int_t^T y(s)e^{-rs}ds = \int_t^T y(s)e^{(\frac{x-\epsilon}{\alpha-1})s}e^{\theta s}ds \leq e^{\frac{x-\epsilon}{\alpha-1}t}y(t)\int_t^T e^{\theta s}ds.$$ 

Hence,

$$\gamma(t) \geq \left[ \frac{e^{\frac{x-\epsilon}{\alpha-1}t}y(t)\int_t^T e^{\theta s}ds}{\int_t^T e^{\theta s}ds} \right]^{\frac{1}{\alpha - 1}} = e^{(r-\delta)t}y(t)^{\alpha-1}.$$ 

A direct computation shows that $\gamma(t) \geq e^{(r-\delta)t}y(t)^{\alpha-1}$ implies that $\frac{d\gamma(t)}{dt} \geq 0$, i.e., $\gamma(t)$ is increasing in $t$.

The dual indirect utility function is

$$J(x,t) = \begin{cases} \frac{1}{\theta}(e^{st} - e^{\theta t})x^b + \left(\int_t^T y(s)e^{-rs}ds\right)z, & \text{if } (x,t) \in \Omega_1 \\ \frac{1}{\theta}(e^{st} - e^{\theta t})x^b + \left(\int_t^T y(s)e^{-rs}ds\right)\gamma(t), & \text{if } (x,t) \in \Omega_2 \end{cases}$$

Clearly, $J$ is convex in $x$. It is easily checked that on $\Omega_1$ we have $J_x(x,t) = -(\int_t^T e^{\theta s}ds)x^{b-1} + \int_t^T y(s)e^{-rs}ds < 0$, since $x < \gamma(t)$, and $J_x = 0$ if and only if $x = \gamma(t)$. On $\Omega_2$, we have

$$-e^{st}x^b + xy(t)e^{-rt} + J_z = -e^{st}x^b + xy(t)e^{-rt} + e^{st}\gamma(t)^b - \gamma(t)y(t)e^{-rt}.$$ 

Since $\gamma(t) \geq e^{(r-\delta)t}y(t)^{\alpha-1}$, the right-hand-side of the above equation is positive as long as $x > \gamma(t)$ and is zero if and only if $x = \gamma(t)$. This completes our verification that $J$ is the dual indirect utility function. Since $\gamma$ is increasing, $X^*$ is constant.

We conclude that when labor income decreases quickly enough so that $e^{(\frac{x-\epsilon}{\alpha-1})t}y(t)$ is decreasing, the individual is able to "sell" his future labor income for the entire time horizon at time $0$ and makes his consumption and saving decisions as if there were no borrowing constraints at all.

Example III: $e^{-\frac{x-\epsilon}{\alpha}t}y(t)$ is strictly decreasing on $t \in [0, \frac{T}{2})$ and strictly increasing on $(\frac{T}{2}, T]$.

We claim that the critical boundary $\gamma$ is increasing on $[0, \frac{T}{2}]$ and decreasing on $[\frac{T}{2}, T]$. The proof of this claim is lengthy, and is therefore omitted. The optimal policies in this case are similar to that of Example I.
Example IV: $e^{\frac{k-1}{2}y(t)}$ is strictly increasing in $t$ for $t \in [0, \frac{T}{2})$ and strictly decreasing for $t \in (\frac{T}{2}, T]$. 

We claim that the critical boundary $\gamma$ is decreasing on $[0, \frac{T}{2})$ and increasing on $[\frac{T}{2}, T]$. The proof of this claim is also lengthy, and is therefore omitted. Since $\gamma$ is of V shape, $X^*$ stays constant at the beginning, moves downwards along $\gamma$ after it hits the boundary, and then stays constant after it reaches the lowest point of the boundary. This implies that the individual divides the time span $[0, T]$ into three sub-periods. In the first and the last periods, the individual is capable of “selling” his labor income in the security market, while in the second period, he consumes all of the labor income as it is received.

3 The Uncertainty Case

In this section we extend the previous analysis with labor income and borrowing constraints to the uncertainty case. We establish a similar duality theorem between the individual’s consumption-portfolio problem and a dual minimization problem. As in the certainty case, the dual problem allows us to establish the existence as well as the characterization of the solution to the individual’s problem.

3.1 The Economic Setting

For simplicity, we assume that there are only two assets available for trading.\(^6\) The first asset is a risky one, with price $S(t)$ at time $t$ being described by the stochastic differential equation

$$S(t) = S(0) + \int_0^t \mu(S(s), s)ds + \int_0^t \sigma(S(s), s)dw(s), \quad t \in [0, T],$$

(3)

where $w$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, $\mu(x,t)$ and $\sigma(x,t)$ are continuous in both $x$ and $t$, and

$$\int_0^t |\mu(S(s), s)| ds < \infty, \quad P - a.s., \quad \forall t \in [0, T],$$

$$\int_0^t |\sigma(S(s), s)|^2 ds < \infty, \quad P - a.s., \quad \forall t \in [0, T],$$

such that the two integrals in (3) are well defined.\(^7\) The second asset is a “locally” riskless one, with price $B(t)$ defined as

$$B(t) = e^{\int_0^t r(S(s), s)ds},$$

\(^6\)We can easily generalize our model to allow multiple assets.

\(^7\)We allow the possibility that $T = +\infty$. An element $\omega \in \Omega$ is a state of nature which completely describes the uncertain realization from time $0$ to time $T$. The sigma-field $\mathcal{F}$ is the collection of distinguishable events at time $T$, and $P$ is the probability. We model the intertemporal resolution of uncertainty by an increasing and right-continuous family of sub-sigma field of $\mathcal{F}$, or a filtration $\mathcal{F} = \{\mathcal{F}_t; t \in [0, T]\}$, where $\mathcal{F}_t = \bigcap_{t < s \leq T} \mathcal{F}_s$ and $\mathcal{F}_t$ is generated by $\{w(s); 0 \leq s \leq t\}$, and $\mathcal{F} \equiv \bigvee_{0 \leq s \leq T} \mathcal{F}_s$. Under this definition, the probability space $(\Omega, \mathcal{F}, P)$ is well defined even
where \( r(x, t) \geq 0 \), \( r \) is continuous in \( x \) and \( t \), and \( \int_0^t r(S(t), t) dt < \infty \), \( P - a.s. \). Similar to the certainty case, the individual receives an initial endowment \( W_0 > 0 \) at time 0 as well as a stream of non-negative labor income \( y(t) \) at time \( t \) from 0 to \( T \). We assume that \( y(t) = Y(S(t), t) \), where \( Y(x, t) \) is a continuous function of \( x \) and \( t \). Although the labor income evolves stochastically, it does not constitute an additional source of uncertainty.

At any point in time the individual can invest his wealth in both assets. In particular, the individual can short one asset and long the other asset. For example, he can purchase the risky asset by borrowing at the "locally" riskless rate. The borrowing constraints require that at any point in time the individual’s wealth (the value of his portfolio) be non-negative, i.e., the individual cannot run into debt.

The individual's preferences are represented by the expected utility of a time-additive utility function, \( u(x, t) \), for consumption rate \( c(t) \) at time \( t \). For simplicity, we assume zero utility for final wealth. Hence, the individual solves the following dynamic consumption-portfolio program:

\[
\max_{c \geq 0} E \left[ \int_0^T u(c(t), t) dt \right] \tag{*}
\]

s.t. \( dW(t) = \alpha(t) dB(t) + (y(t) - c(t)) dt + \theta(t) dS(t), \quad W(0) \geq 0, \)

where \((\alpha(t), \theta(t))\) is the number of shares invested in the riskless and the risky assets, respectively, and \(W(t) = \alpha(t) B(t) + \theta(t) S(t)\). We impose a natural informational constraint that \((c, \alpha, \theta)\) are predictable with respect to \(\mathbb{F}^{8}\), i.e., they are "almost" left-continuous. It is also known that with \(W(t) \geq 0\) there can be no arbitrage opportunities; cf. Huang and Pagès (1990).

If there were no labor income, the individual's dynamic consumption-portfolio problem could be equivalently transformed into a static utility maximization problem, see Cox and Huang (1987),

\[
\max_{c \geq 0} E \left[ \int_0^T u(c(t), t) dt \right] \\
\text{s.t. } E \left[ \int_0^T \xi(t) \frac{c(t)}{B(t)} dt \right] \leq W_0,
\]

where \(\xi(t, \omega) \equiv \exp \left\{ \int_0^t \kappa(s) dw(s) - \frac{1}{2} \int_0^t |\kappa(s)|^2 ds \right\} \) with \(\kappa(t) = (r(t) S(t) - \mu(t))/\sigma(t)\). \(\xi(\omega)\) is interpreted as the implicit Arrow-Debreu price for one unit of consumption good at time \(t\) and state \(\omega\). We assume that \(|\kappa(t)|\) is uniformly bounded.

---

8A process \(X\) is predictable with respect to \(\mathbb{F}\) if it is measurable with respect to the \(\sigma\)-field on \(\Omega \times \mathbb{R}_+\) generated by all left-continuous \(\mathbb{F}\)-adapted processes.

9Although Cox and Huang's (1987) analysis requires that \(T\) be finite, their approach can be extended to \(T = \infty\), see Huang and Pagès (1990).
If there were no borrowing constraints or if the individual were able to "sell" his labor income in the security market, then the individual would solve the static utility maximization problem

$$\max_{c \geq 0} E \left[ \int_0^T u(c(t), t) dt \right]$$

s.t. \( E \left[ \int_0^T \xi(t) \frac{c(t)}{B(t)} dt \right] \leq W_0' \)

where \( W_0' = W_0 + E \left[ \int_0^T \xi(t) \frac{y(t)}{B(t)} dt \right] \). When labor income and borrowing constraints are present, we need to impose more restrictions on the set of feasible consumption policies. Integrating the dynamics of the wealth process, we obtain that

$$\int_0^t c(s) ds + W(t) = W_0 + \int_0^t y(s) ds + \int_0^t \alpha(s) dB(s) + \int_0^t \theta(s) d\mathcal{S}(t).$$

Applying Itô's lemma,

$$\int_0^t \xi(s) \frac{c(s)}{B(s)} ds + \xi(t) \frac{W(t)}{B(t)} = W_0 + \int_0^t \xi(s) \frac{y(s)}{B(s)} ds + \int_0^t \psi(t) dw(t), \quad (4)$$

where \( \psi(t) = \xi(t)(\theta(t)\sigma(t) + W(t)\kappa(t))/B(t) \). Hence, the non-negative wealth constraints become

$$\int_0^t \xi(s) \frac{c(s)}{B(s)} ds \leq W_0 + \int_0^t \xi(s) \frac{y(s)}{B(s)} ds + \int_0^t \psi(t) dw(t).$$

Comparing the above constraints with those in the certainty case, we conjecture that problem \((\ast')\) is equivalent to the following static maximization problem

$$\max_{c \geq 0} E \left[ \int_0^T u(c(t), t) dt \right] \quad (P')$$

s.t. \( \sup_{r \in \mathcal{T}} E \left[ \int_0^r \xi(t) \frac{c(t) - y(t)}{B(t)} dt \right] \leq W_0, \)

where \( \mathcal{T} \) denotes the set of stopping times.\(^{10}\) The following lemma establishes our conjecture. First, we make an assumption.

**Assumption 2**

$$E \left[ \int_0^T \xi(t) \frac{y(t)}{B(t)} dt \right] < \infty,$$

i.e., the present value of the future labor income is finite.

\(^{10}\)A random variable \( r \) is called a stopping time if for any \( t \geq 0 \), the events \( \{ r \leq t \} \) are in \( \mathcal{T} \).
Lemma 1 Suppose that \( c \geq 0 \) satisfies the condition
\[
\sup_{t \in [0, T]} E \left[ \int_0^T \xi(t) \frac{c(t) - y(t)}{B(t)} \, dt \right] \leq W_0.
\]
Then, there exists a non-negative wealth process \( W \) and a pair of \((\alpha, \theta)\) such that
\[
dW(t) = \alpha(t) \, dB(t) + (y(t) - c(t)) \, dt + \theta(t) \, dS(t).
\]

Proof. The proof follows closely the proofs of Theorem 6.4, Karatzas (1988). We allow the possibility that \( T \) might be infinite, as in Karatzas. Define \( Q(t) = \int_0^t \xi(s) \frac{d(s) - y(s)}{B(s)} \, ds \). We note, by assumption, that
\[
E \int_0^T \xi(t) \frac{c(t)}{B(t)} \, dt \leq W_0 + E \int_0^T \xi(t) \frac{y(t)}{B(t)} \, dt < \infty.
\]
Hence, \( Q \) is of class \( \mathcal{D}[0, T] \), i.e., the collection \( \{X_r, r \in T\} \) is uniformly integrable. This allows us to obtain the Snell envelope for \( Q \), i.e., the smallest supermartingale which majorizes \( Q \), cf. Dellacherie and Meyer (1982), Appendix I, 22-23. Let \( V \) be the Snell envelope of \( Q \), then,
\[
V(0) = \sup_{r \in T} E[Q(r)],
\]
and \( V(T) = Q(T) \).\(^\dagger\) The Doob-Meyer Decomposition Theorem allows us to write
\[
V(t) = V(0) + M(t) - A(t)
\]
where \( M \) is a uniformly integrable martingale under \( P \) with \( M(0) = 0 \) and \( A \) is an increasing process with \( A(0) = 0 \). The martingale representation theorem then allows us to find a \( \psi \) with \( \int_0^{T_n} |\psi(s)|^2 \, ds < \infty \) and \( T_n \uparrow \infty \), \( P \)-almost surely, such that
\[
M(t) = \int_0^t \psi(s) \, dw(s).
\]

Now define
\[
W(t) = \frac{B(t)}{\xi(t)} [W_0 - V(0) + V(t) - Q(t) + A(t)] = \frac{B(t)}{\xi(t)} \left[ W_0 + \int_0^t \psi(s) \, dw(s) - Q(t) \right].
\]

Clearly, \( W \) is non-negative. We obtain that
\[
\int_0^t \xi(s) \frac{c(s)}{B(s)} \, ds + \xi(t) \frac{W(t)}{B(t)} = W_0 + \int_0^t \xi(s) \frac{y(s)}{B(s)} \, ds + \int_0^t \psi(t) \, dw(t).
\]
Based on (4), we have
\[
dW(t) = \alpha(t) \, dB(t) + (y(t) - c(t)) \, dt + \theta(t) \, dS(t),
\]
where \( \theta(t) = (B(t) \psi(t)/\xi(t) - W(t) \kappa(t))/\sigma(t) \).

\(^\dagger\)We define \( Q(\infty, \omega) = \lim \sup_{t \to \infty} Q(t, \omega), \omega \in \Omega \).
3.2 An Example

In Example II of Section 2.5, we saw that if labor income $y(t)$ is constant over time and $r > \delta$, then the borrowing constraints will never be binding. In this section, we demonstrate that the borrowing constraints do matter under uncertainty, even when the labor income is constant.

We assume for this example that $\mu(t) = \mu S$, $\sigma(t) = \sigma S$, $u(x, t) = e^{-\delta t} \frac{x^\alpha}{\alpha}$, $y(t) = Y$ and $T = 1$. If there were no borrowing constraints, the individual would solve the following problem:

$$\max_{c_{t \geq 0}} E \left[ \int_0^1 e^{-\delta t} \frac{c(t)^\alpha}{\alpha} dt \right]$$

s.t. $E \left[ \int_0^1 \xi(t)c(t)e^{-rt} dt \right] = \tilde{W}$,

where $\tilde{W} = W_0 + Y$. The optimal solution to the above problem is

$$c^*(t) = \lambda_0 \xi(t)^{\frac{1}{\alpha - 1}} e^{\frac{r - \delta}{\alpha - 1} t},$$

where $\lambda_0$ is such that the constraint is satisfied. The optimal wealth process at time $t$ is

$$W(t) = E \left[ \int_t^1 \frac{\xi(s)}{\xi(t)} c^*(s)e^{-r(1-s)} ds | S(t) \right] - E \left[ \int_t^1 \frac{\xi(s)}{\xi(t)} Ye^{-r(s-t)} ds | S(t) \right]$$

$$= \lambda_0 E \left[ \int_t^1 \frac{\xi(s)}{\xi(t)} \frac{-1}{\alpha - 1} e^{-r(s-t) + \frac{r - \delta}{\alpha - 1} s} ds | S(t) \right] - Y \frac{1 - e^{-r(1-s)}}{r}$$

$$= \lambda_0 A(t) \xi(t)^{\frac{1}{\alpha - 1}} - Y \frac{1 - e^{-r(1-s)}}{r},$$

where

$$A(t) = E \left[ \int_t^1 \left( \frac{\xi(s)}{\xi(t)} \right)^{\frac{\alpha - 1}{\alpha - 1}} e^{-r(s-t) + \frac{r - \delta}{\alpha - 1} s} ds \right]$$

is a deterministic function of $t$. Since $Y > 0$, the wealth process $W(t)$ can go below zero if the state price $\xi(t)$ is too low, which means that this consumption policy $c^*$ violates the borrowing constraints. If $Y$ were equal to zero, then $W(t)$ would always be non-negative.

3.3 Dual Optimality and Existence

We now consider the dual problem for the consumption-portfolio problem ($P'$). We define

$$\tilde{u}(y, t) = \max_{z \geq 0} \{u(x, t) - xy \},$$

$$f(y, t) = \inf \{ x \geq 0 : u_+(x, t) \leq y \}.$$
Based on the duality theory established earlier, we conjecture that the dual minimization problem under uncertainty has the following form:

$$
\min_{X \in \mathcal{D}'} E \left[ \int_0^T \tilde{u}(X(t)\xi(t)/B(t),t)dt + \int_0^T X(t)\xi(t)\frac{y(t)}{B(t)}dt \right] + X(0)W_0,
$$

(D')

where \( \mathcal{D}' \) denotes the set of non-negative, decreasing, and predictable processes. For future use, we define

$$\mathcal{L}'(X) \equiv E \left[ \int_0^T \tilde{u}(X(t)\xi(t)/B(t),t)dt + \int_0^T X(t)\xi(t)\frac{y(t)}{B(t)}dt \right] + X(0)W_0.$$

Dual Optimality

As in the certainty case, we establish a duality between the original consumption-portfolio problem \((P')\) and the dual minimization problem \((D')\). Specifically, we show that the solution to the dual problem \((D')\), if it exists, completely determines the solution to \((P')\).

**Theorem 3** If \( X^* \) is a solution to the dual problem \((D')\), then

$$c^*(t) = f(X^*(t)\xi(t)/B(t),t)$$

is a solution to \((P')\).

**PROOF.** See Appendix. \( \square \)

**Remark 3** It follows from the proof that \( E[X^*(T)\xi(T)W^*(T)] = 0 \), where \( X^*(T) \geq 0 \) and \( W^*(T) \geq 0 \). Hence, \( W^*(T) = 0 \), whenever \( X^*(T)\xi(T) > 0 \).

The economic interpretation of \( X^* \) is similar to that in the certainty case. The process \( \{X^*(t)\xi(t)\} \) forms an implicit system of Arrow-Debreu prices (after proper normalization) or shadow prices for this particular individual. If \( X^*\xi \) were the Arrow-Debreu state prices and the individual were allowed to sell his labor income at these prices, then the individual’s optimal consumption and investment decisions would be identical to those of the original problem with borrowing constraints.
Existence

We now establish the existence of solutions to the primal consumption-portfolio problem \((P')\) and to the dual minimization problem \((D')\). According to the duality theorem between the primal and the dual problems, the existence of a solution to the dual problem implies the existence of a solution to the primal problem. We therefore establish the existence of a solution only to the dual problem.

**Theorem 4** Suppose that there exist constants \(A, A', B > 0, B' > 0, 0 < \alpha < 1, \alpha' < 0, \delta > 0, \) and \(\delta' > 0\) such that

\[
e^{-\delta t}(A' - B'x^{\alpha'}) \leq u(x, t) \leq e^{-\delta t}(A + Bx^\alpha).
\]

Suppose further that

\[
E \left[ \int_0^T e^{-\frac{\delta}{1-\alpha} t} \left( \frac{B(t)}{\xi(t)} \right)^{\frac{\alpha}{1-\alpha}} dt \right] < \infty, \quad E \left[ \int_0^T e^{-\frac{\delta'}{1-\alpha'} t} \left( \frac{B(t)}{\xi(t)} \right)^{\frac{\alpha'}{1-\alpha'}} dt \right] < \infty.
\]

Then there exists a solution to \((D')\).

**Proof.** We first assume that \(T < \infty\). Define

\[
J(x) = \inf_{X \in \mathcal{P}, X(0) = x} H(X),
\]

where

\[
H(X) = E \left[ \int_0^T \tilde{u}(X(t)\xi(t)/B(t), t) dt + \int_0^T X(t)\xi(t) \frac{y(t)}{B(t)} dt \right].
\]

We claim that the infimum in the definition of the \(J\) function exists and is attained. Consider a constant policy, \(X(t) \equiv x\). Since \(u(x, t) \leq e^{-\delta t}(A + Bx^\alpha)\), we have \(\tilde{u}(x, t) \leq Ae^{-\delta t} + Ce^{-\frac{\delta}{1-\alpha} t} x^{\alpha-1}\), where \(C = B(1 - \alpha)(\alpha B)^{\frac{\alpha}{1-\alpha}} > 0\). Hence,

\[
J(x) \leq E \left[ A \int_0^T e^{-\delta t} dt + C x^{\alpha-1} \int_0^T e^{-\frac{\delta}{1-\alpha} t} \left( \frac{B(t)}{\xi(t)} \right)^{\frac{\alpha}{1-\alpha}} dt + x \int_0^T \xi(t) \frac{y(t)}{B(t)} dt \right] < \infty,
\]

by assumption. Now, let \(X_n\) be a minimizing sequence. Since \(X_n \geq 0\) is bounded above by \(x\), it is weakly compact in \(L^1(dP \times dt)\) and the weak limit is also in \(\mathcal{P}'\). We can easily verify that \(H(X)\) is convex in \(X\) (since \(\tilde{u}\) is convex). We further argue that \(H\) is lower semi-continuous in the \(L^1\)-norm. This is because

\[
\tilde{u}(x, t) \geq A'e^{-\delta t} - C'e^{-\frac{\delta'}{1-\alpha'} t} x^{\frac{\alpha'}{1-\alpha'}-1},
\]
where $C' = B'(1 - \alpha')(\alpha'B')^{\alpha' - 1} > 0$, hence

$$
\bar{u}(X(t)\xi(t)/B(t), t) \geq A'e^{-\delta't} - C'x^{\frac{\alpha'}{\alpha'-1}}e^{-\frac{\delta'}{1-\alpha'}t}\left(\frac{B(t)}{\xi(t)}\right)^{\frac{\alpha'}{1-\alpha'}}.
$$

The right-hand side of the above inequality is integrable, hence Fatou's Lemma implies that

$$
H(X) \leq \liminf_{n \to \infty} H(X_n),
$$

if $X_n \to X$ in $L^1$-norm. Theorem 2.6.1, Balakrishnan (1981) allows us to conclude that there exists $X^*$, the weak limit of $\{X_n\}$, such that $J(x) = H(X^*)$. Since $X^*$ is also an almost-sure limit of a sequence, formed by some finite convex combination of $X_n$, cf. Dunford and Schwartz (1958), $X^*$ must be decreasing. Let $X^*$ be the modification of $X^*$ such that $X^*$ is left-continuous, then $X^* \in D$ and $J(x) = H(X^*) = H(X^*)$.

Next, we can easily verify that $J(x)$ is strictly convex, decreasing, and lower semi-continuous on $\mathbb{R}_+$, since

$$
J(x) \geq \inf_X \left[ \int_0^T \left( e^{-\delta't}A' - X(t)\frac{\alpha'}{\alpha'-1}C' e^{-\frac{\delta'}{1-\alpha'}t}\left(\frac{B(t)}{\xi(t)}\right)^{\frac{\alpha'}{1-\alpha'}} \right) dt + \int_0^T X(t)\xi(t)\frac{y(t)}{B(t)} dt \right]
$$

$$
= \int_0^T \left( e^{-\delta't}A' - x^{\frac{\alpha'}{\alpha'-1}}C' e^{-\frac{\delta'}{1-\alpha'}t}\left(\frac{B(t)}{\xi(t)}\right)^{\frac{\alpha'}{1-\alpha'}} \right) dt
$$

for some constants $F^*, G^* > 0$. We have, $J(x) + xW_0 \to +\infty$ as $x \to \infty$. It is now easy to see that there exists a unique minimum to the program

$$
\min_{x \geq 0} J(x) + xW_0.
$$

We conclude that the dual problem has a solution when $T < \infty$. The proof for $T = \infty$ can be done in exactly the same way, if we define a new measure, $d\mu = e^{-t}dt$, and a new sample space, $\Omega \times [0, \infty)$, and endow this sample space with the product measure $dP \times d\mu$.

3.4 Characterization

We assume throughout this subsection that there exists a solution to the dual problem. To characterize the solution of the dual problem, we apply the stochastic dynamic programming method. Define a new state variable $Z(t) = X(t)\xi(t)/B(t)$, which satisfies the stochastic differential equation

$$
\frac{dZ(t)}{Z(t)} = \frac{dX(t)}{X(t)} - r(S(t), t)dt + \kappa(S(t), t)d\omega(t).
$$

(5)
Since $X$ will be a feedback control of $S$ and $Z$ at the optimum, we consider only those $X$ that are feedback controls of $Z$, $S$. Hence, $(Z, S)$ forms a Markov process. We therefore define the dual indirect utility function as follows:

$$J(z, S, t) = \inf_{X \in \mathcal{D}_1, Z(t) = z} \mathbb{E} \left[ \int_t^T (\hat{u}(Z(s), s) + Z(s)Y(S(s), s)) \, ds \mid Z(t) = z, S(t) = S \right],$$

where $\mathcal{D}_1$ denotes the set of decreasing, predictable processes which are feedback controls of $S$ and $Z$. Since $\hat{u}$ is strictly convex, it is easily checked that $J$ is also strictly convex in $z$. If $J$ is continuously differentiable with respect to $(z, S, t)$ and $dX(t) = -X(t)\psi(t)dt$, then the Bellman equation becomes

$$
\min_{\psi \geq 0} \left\{ \hat{u}(z, t) + zY(S, t) + J_z - \psi(z)zJ_z - r(S, t)zJ_z + \mu(S, t)J_S \\
+ \frac{1}{2} \kappa^2(S, t)z^2J_{zz} + \frac{1}{2} \sigma^2(S, t)J_{SS} + \kappa(S, t)\sigma(S, t)zJ_{zz} \right\} = 0.
$$

We obtain the following characterization for $\psi$:

$$
\psi^* \in [0, +\infty], \quad \text{if } J_z = 0 \\
\psi^* = 0, \quad \text{if } J_z < 0
$$

The Bellman equation suggests that the optimal path $Z^*$ might have jumps in the region $\{J_z = 0\}$. We denote the no-jump region by $\Omega_1'$ and the jump region by $\Omega_2'$ (see Figure 4):

$$
\Omega_1' = \{(z, t) : J_z(z, S, t) < 0\}, \quad \Omega_2' = \{(z, t) : J_z(z, S, t) = 0\},
$$

and define the critical boundary $\gamma' = \gamma'(S, t)$ to be the surface that separates $\Omega_1'$ and $\Omega_2'$, i.e., $\text{cl}(\Omega_1') \cap \text{cl}(\Omega_2')$. The critical boundary cuts the space $\{z \geq 0\}$ into halves, with the upper half being the jump region and the lower half being the no-jump region. At any time $t$, if $Z$ starts inside the jump region, then it will jump downwards to the critical boundary $\gamma'(S(t), t)$. If $Z$ starts inside the no-jump region, then locally it will be equal to $\xi(t)/B(t)$ multiplied by a constant.

Conversely, if $J$ is a solution to the Bellman equation, then, under some regularity conditions, we can show that the $J$ is indeed the dual indirect utility function and the individual’s optimal consumption and portfolio policies can be characterized by $J$.

Theorem 5 (Verification) Let $J(z, S, t)$ be the solution to the Bellman differential equation

$$
\min \left\{ \hat{u}(z, t) + zY(S, t) + A_t(J), -J_z \right\} = 0, \quad (z, S, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T]
$$

where

$$
A_t(J) = J_t - rzJ_z + \mu JS + \frac{1}{2} \kappa^2 z^2J_{zz} + \frac{1}{2} \sigma^2 J_{SS} + \kappa \sigma zJ_{zz}
$$

25
together with the boundary condition

\[ J(z, S, T) = 0, \quad (z, S) \in \mathbb{R}_+ \times \mathbb{R} \]

such that \( J \) is convex in \( z \) and continuously differentiable w.r.t. \( z, S, \) and \( t, \) up to the third order. Suppose further that \( Z^*(t) = X^*(t) Z(t)/B(t), \) where \( X^* \in \mathcal{D}_t \) and \( X^* \) has a continuous sample path such that for all \( t \in [0, T], \)

i) \( \tilde{u}(Z^*(t), t) + Z^*(t) Y(S(t), t) + A(t) \left( Z^*(t), S(t), t \right) = 0; \)

ii) \( \int_1^{t+\epsilon} \frac{Z^*(t)}{X^*(t)} J_x(Z^*(t), S(t), t) dX^*(t) = 0, \) for all \( \epsilon > 0; \)

iii) the stochastic integral,

\[ \int_0^t [\sigma J_S(Z^*(s), S(s), s) + k Z^*(s) J_x(Z^*(s), S(s), s)] dw(s) \]

is a martingale under \( P; \)

iv) \( J_z(Z^*(0), S(0), 0) + W_0 = 0; \)

v) if \( T = \infty, \) then

\[
\lim_{r \to +\infty} E \left[ \int_0^r \left( \tilde{u}(Z^*(s), s) + Z^*(s) Y(S(s), s) \right) ds + J(Z^*(r), S(r), r) \right] = E \left[ \int_0^\infty \left( \tilde{u}(Z^*(s), s) + Z^*(s) Y(S(s), s) \right) ds \right].
\]

Then, \( J \) is the dual indirect utility function and \( Z^* \) is the optimal solution to the dual problem. The optimal consumption and portfolio policies are determined as follows,

\[ c^*(t) = f(Z^*(t), t), \quad W^*(t) = -J_z(Z^*(t), S(t), t), \]

\[ \theta^*(t) = \frac{\mu(t) - r(t) S(t)}{\sigma^2(t)} Z^*(t) J_{xx}(t) - S(t) J_{xS}(t). \]

PROOF. See Appendix.

Since \( J_x'\gamma'(S(0), 0), S(0), 0) = 0, \) the optimal sample path must start inside the no-jump region and stay inside the closure of the no-jump region all the time. The optimal solution of the dual problem is a reflecting diffusion process inside the no-jump region. We now show that the effect of borrowing constraints is also to shorten an individual’s planning horizon. Unlike the certainty case, duration of the time in which the individual is capable of “selling” his future labor income is now a random variable and should be characterized by a stopping time. The following proposition shows that the individual’s planning horizon can be characterized by the local times of a one-dimensional diffusion process at the critical boundary \( \gamma' = \gamma(S, t). \)
Proposition 2 Let \( Z^* \) be the solution to the dual problem and \( J \) be the dual indirect utility function. Define

\[
  r_1 = \inf \{ t > 0 : J_s(Z(t), S(t), t) = 0 \}, \\
  r_2 = \inf \{ t > r_1 : J_s(Z(t), S(t), t) = 0 \}, \ldots
\]

Then, \( W(r_1) = 0 \), and the individual's optimal consumption policies with borrowing constraints on \([0, r_1)\) is the same as that without the borrowing constraints in which the individual has a planning horizon \([0, r_1)\) and is capable of "selling" his labor income on \([0, r_1)\) in the security market. The optimal policies on \([r_1, r_2), \ldots\), etc., can be analyzed similarly.

**Proof.** Following Theorem 5, \( W^*(r_1) = -J_s(Z^*(r_1), S(r_1), r_1) = 0 \). Hence,

\[
  E \left[ \int_0^{r_1} X^*(t) \xi(t) \frac{c^*(t)}{B(t)} dt \right] = W_0 + E \left[ \int_0^{r_1} X^*(t) \xi(t) \frac{y(t)}{B(t)} dt \right],
\]

where \( c^*(t) = f(X^*(t) \xi(t)/B(t), t) \). Since \( X^* \) is constant on \([0, r_1)\), the Lagrangian method implies that \( c^* \) is a solution to the following program:

\[
  \max_{c \geq 0} E \left[ \int_0^{r_1} u(c(t), t) dt \right] \\
  \text{s.t.} E \left[ \int_0^{r_1} \xi(t) \frac{c(t)}{B(t)} dt \right] \leq W_0 + E \left[ \int_0^{r_1} \xi(t) \frac{y(t)}{B(t)} dt \right],
\]

where \( X^*(0) \) is the Lagrangian multiplier. \( \Box \)

Proposition 2 implies that the individual's optimal policies can be characterized by the local times of a diffusion process at the free boundary. The individual is capable of "selling" his future labor income only up to time \( r_1 \). At time \( r_1 \), the individual runs into zero wealth. Unlike the certainty case, the individual will not consume all of his labor income after time \( r_1 \). Instead, the individual constructs another time horizon in which he is capable of "selling" his labor income in the security market and optimally allocates his consumption across time. The second effective planning horizon is also stochastic and can be characterized by a new stopping time. The individual will again run into zero wealth at the end of the second period. In almost all states of nature, the individual's wealth will run into zero an infinite number of times.

3.5 An Example: Continued

We now continue our previous example in Section 3.2. For simplicity, we assume that \( T = \infty \) and \( \alpha < 0 \). This requires \( \delta > \alpha r + \frac{\sigma^2}{2(1-\alpha)} \) to ensure the existence of a solution. The indirect dual utility
function can now be written as
\[
J(z,t) = \inf_{x \in D_1} E \left[ \int_t^\infty e^{-\delta s} (\tilde{u}(Z(s)) + Z(s)Y) \, ds \mid Z(t) = z \right],
\]
where \(Z(t) = e^{(\delta - r)t} X(t) \xi(t)\). Clearly, \(J\) can be written as \(J(z,t) = e^{-\delta t} \tilde{J}(z)\) for some \(\tilde{J}\). The Bellman equation becomes
\[
\min \left\{ -\frac{z^b}{b} + zY - \delta \tilde{J} + (\delta - r)z \tilde{J}_z + \frac{\kappa^2}{2} z^2 \tilde{J}_{zz}, -\tilde{J}_z \right\} = 0,
\]
where \(b = \frac{\alpha}{\alpha - 1}\). We claim that the critical boundary is a horizontal line in the \((t,z)\)-space, \(\gamma(t) = \gamma > 0\), and \(\tilde{J}\) has the form
\[
\tilde{J}(z) = \begin{cases} 
-Az^\beta + \frac{Y}{r} z - \frac{1}{\Delta} \gamma^b, & \text{if } z \leq \gamma \\
-A\gamma^\beta + \frac{Y}{r} \gamma - \frac{1}{\Delta} \gamma^b, & \text{if } z > \gamma
\end{cases}
\]
where \(\Delta = (\delta - \alpha r - \frac{\kappa^2}{2} \alpha)/(1 - \alpha) > 0\), and \(\beta\) is the larger root of the equation
\[
\frac{\kappa^2}{2} x^2 + (\delta - r - \frac{\kappa^2}{2}) x - \delta = 0.
\]
It is easy to verify that \(\beta > 1\). The parameters \(A\) and \(\gamma\) are such that
\[
\tilde{J}_z(\gamma) = 0,
\]
\[
-\delta \tilde{J}(\gamma) - \frac{\gamma^b}{b} + \gamma Y = 0.
\]
We have
\[
-A\beta \gamma^{\beta - 1} + \frac{Y}{r} - \frac{1}{\Delta} \gamma^{\beta - 1} = 0,
\]
\[
-\delta \left( -A\gamma^\beta + \frac{Y}{r} \gamma - \frac{1}{\Delta} \gamma^b \right) - \frac{\gamma^b}{b} + \gamma Y = 0.
\]
In the Appendix, we show that the above system of equations has a unique positive solution. Moreover, \(J\) is convex, and
\[
\tilde{J}_z < 0, \quad \forall z < \gamma
\]
\[
-\delta \tilde{J}(z) - \frac{z^b}{b} + zY \geq 0, \quad \forall z \geq \gamma
\]
This completes our verification that \(e^{-\delta t} \tilde{J}\) is the dual indirect utility function, and \(\gamma(t) = \gamma\) defines the critical boundary that divides the jump and no-jump regions.\textsuperscript{12} Hence, the optimal

\textsuperscript{12}Since \(X^*(t) \leq X^*(0)\) for all \(t > 0\), we can easily verify the regularity conditions of Theorem 5 for \(J\).
solution \( X^* \) must be such that \( X^*(t) \leq e^{-(\delta - r)t} \gamma / \xi(t) \) for all \( t \). Moreover, \( X^* \) decreases only when \( X^*(t) = e^{-(\delta - r)t} \gamma / \xi(t) \). This implies that

\[
X^*(t) = \min \left( X_0, \inf_{0 \leq s \leq t} \left\{ e^{-(\delta - r)s} \frac{\gamma}{\xi(s)} \right\} \right),
\]

where \( X_0 < \gamma \) is the starting point for \( X^* \). The optimal sample path of \( X^* \) is singular in the sense that it remains constant at almost all the times, and the set of points at which \( X^* \) is strictly decreasing has Lebesgue measure zero. Following Proposition 2, we know that the optimal wealth process hits zero infinitely many times, but the set of all such times has Lebesgue measure zero.

4 Concluding Remarks

We have employed a duality approach to study an individual's optimal intertemporal consumption and portfolio decisions when labor income and borrowing constraints are present. When an individual has limited opportunities to borrowing against future labor income and cannot totally insure the risk of income fluctuations, his decision horizon is shortened. In the certainty case, the individual divides his lifetime planning horizon into several sub-periods. In some of the periods, the individual is able to "sell" the labor income in the security market and solve an unconstrained problem for those periods. In other periods, the individual consumes all of his labor income as it is received. In the uncertainty case, the individual also divides his lifetime planning horizon into several sub-periods characterized by the local times of the Arrow-Debreu state price process at the critical boundary. In all of the periods, the individual is able to "sell" the labor income in the security market and solve an unconstrained problem for those periods. The individual's wealth may run into zero infinitely many times.

Although the main assumption of our analysis is that the labor income does not itself constitute an additional source of uncertainty, we conjecture that the qualitative aspects of the characterizations of the optimal consumption and portfolio policies might not depend crucially upon this assumption, as far as the effect of borrowing constraints is concerned. We hope to extend our analysis to include general stochastic labor income in future research.
References


Appendix

PROOF OF THEOREM 3: The proof for this theorem is similar to the proof for Theorem 1. We first show that \( c^* \) is feasible. For any \( r \in \mathcal{T} \), we consider the decreasing function \( X^r = X^* + \epsilon 1_{(0, r)} \in \mathcal{D}' \), where \( \epsilon > 0 \). Since \( \mathcal{L}'(X^r) \geq \mathcal{L}'(X^*) \), we obtain that

\[
\limsup_{\epsilon \downarrow 0} \frac{\mathcal{L}'(X^r) - \mathcal{L}'(X^*)}{\epsilon} \geq 0.
\]

This implies that

\[
\limsup_{\epsilon \downarrow 0} \mathbb{E} \left[ \int_0^T \frac{\bar{u}(X^r(t)\xi(t)/B(t), t) - \bar{u}(X^*(t)\xi(t)/B(t), t)}{\epsilon} \, dt \right] + \mathbb{E} \left[ \int_0^r \xi(t) \frac{y(t)}{B(t)} \, dt \right] + W_0 \geq 0.
\]

Since \( \bar{u}(X^r(t)\xi(t)/B(t), t) \leq \bar{u}(X^*(t)\xi(t)/B(t), t) \), Fatou's Lemma implies that

\[
\mathbb{E} \left[ \int_0^r \xi(t) \frac{c^*(t)}{B(t)} \, dt \right] \leq W_0 + \mathbb{E} \left[ \int_0^r \xi(t) \frac{y(t)}{B(t)} \, dt \right],
\]

for all \( r \in \mathcal{T} \). We conclude from Lemma 1 that \( c^* \) is feasible.

We now show that \( c^* \) is also optimal. For any feasible consumption policy \( c \), we can find a \( \psi \) such that

\[
\int_0^t \xi(s) \frac{c(s)}{B(s)} \, ds + \xi(t) \frac{W(t)}{B(t)} = W_0 + \int_0^t \xi(s) \frac{y(s)}{B(s)} \, ds + \int_0^t \psi(s) \, dw(s).
\]

Since \( W(t) \geq 0 \) P-a.s., we have

\[
\int_0^T \left( \int_0^t \xi(s) \frac{c(s)}{B(s)} \, ds \right) \, dX^*(t) \geq \int_0^T \left( W_0 + \int_0^t \xi(s) \frac{y(s)}{B(s)} \, ds + \int_0^r \psi(s) \, dw(s) \right) \, dX^*(t).
\]

Since \( X \) is of bounded variation and the integrands on both sides of the above inequality have continuous sample paths, we can apply integration by parts

\[
\mathbb{E} \left[ \int_0^T X^*(s) \xi(s) \frac{c(s)}{B(s)} \, ds \right] \leq X^*(0)W_0 + \mathbb{E} \left[ \int_0^T X^*(s) \xi(s) \frac{y(s)}{B(s)} \, ds \right] + \mathbb{E} \left[ X^*(T) \left( \int_0^T \xi(s) \frac{c(s)}{B(s)} \, ds - W_0 - \int_0^T \xi(s) \frac{y(s)}{B(s)} \, ds + \int_0^T \psi(s) \, dw(s) \right) \right] \leq X^*(0)W_0 + \mathbb{E} \left[ \int_0^T X^*(s) \xi(s) \frac{y(s)}{B(s)} \, ds \right],
\]

where we have used the fact that \( W(T) \geq 0 \) for all \( t \). This shows that (6) must be satisfied for all of the feasible consumption policies. We show below that in fact (6) holds with equality for \( c^* \). To see that, we consider \( \tilde{X}^r = X^*(1 + \epsilon) \in \mathcal{D}' \) for small \( \epsilon \). Since \( \mathcal{L}'(\tilde{X}^r) \geq \mathcal{L}'(X^*) \), we have

\[
\limsup_{\epsilon \downarrow 0} \frac{\mathcal{L}'(\tilde{X}^r) - \mathcal{L}'(X^*)}{\epsilon} \geq 0, \quad \liminf_{\epsilon \downarrow 0} \frac{\mathcal{L}'(\tilde{X}^r) - \mathcal{L}'(X^*)}{\epsilon} \leq 0.
\]
Invoking Fatou's lemma, we obtain
\[
E \left[ \int_0^T X^*(s) \xi(s) \frac{c^*(s)}{B(s)} ds \right] = X^*(0)W_0 + E \left[ \int_0^T X^*(s) \xi(s) \frac{y(s)}{B(s)} ds \right].
\]

According to the Lagrangian method, \(c^*\) is in fact the solution to the program
\[
\max_{c \geq 0} E \left[ \int_0^T u(c(t), t) dt \right]
\]
\[
\text{s.t. } E \left[ \int_0^T X^*(t) \xi(t) \frac{c(t)}{B(t)} dt \right] \leq X^*(0)W_0 + E \left[ \int_0^T X^*(t) \xi(t) \frac{y(t)}{B(t)} dt \right].
\]

Since the maximum utility of \((P')\) is bounded above by the maximum utility of the above program, \(c^*\) must also be the solution to \((P')\).

**Proof of Theorem 5.** We prove this theorem for \(T < \infty\). The proof for \(T = \infty\) can be done similarly by passing a limit. We first show that
\[
J(Z^*(t), S(t), t) = E \left[ \int_t^T (\bar{u}(Z^*(s), s) + Z^*(s)Y(S(s), s)) ds \mid Z^*(t), S(t) \right]. \tag{7}
\]

Applying the generalized Itô's lemma (see VIII.27, Dellacherie and Meyer(1982)), we have
\[
\int_t^T (\bar{u}(Z^*(s), s) + Z^*(s)Y(S(s), s)) ds + J(Z^*(T), S(T), T)
\]
\[
= \int_t^T (\bar{u}(Z^*(s), s) + Z^*(s)Y(S(s), s)) ds + J(Z^*(t), S(t), t) + \int_t^T A_s(J) ds
\]
\[
= J(Z^*(t), S(t), t) + \int_t^T \kappa(s)Z^*(s)J_z(s)dw(s) + \int_t^T \sigma(s)J_z(s)dw(s),
\]
where we have used the fact that \(J\) satisfies the Bellman equation and that \(J_{zz}dX^* = 0\). Since the stochastic integral on the right-hand side of the above equation is a martingale under \(P\), (7) follows by taking the conditional expectation.

To prove that \(J\) is the dual indirect utility function, it suffices to show that \(c^*(t) = f(Z^*(t), t)\) is the optimal solution to \((P')\). Define \(W^*(t) = -J_x(Z^*(t), S(t), t)\). Since \(\bar{u}(z, t) + zY(S(t), t) + A_t(J) = 0\), we have
\[
-f(z, t) + Y(S, t) + \frac{\partial A_t(J)}{\partial z} = 0. \tag{8}
\]

Applying Itô's Lemma, we have
\[
d \left[ \int_0^t c^*(s) ds - \int_0^t Y(S(s), s) ds - J_x(Z^*(t), S(t), t) \right]
\]

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\[
= c^*(t)dt - Y(S(t),t)dt - J_{zt}(t)dt - J_{zz}(t)dZ^*(t) - J_{zs}(t)dS(t) \\
- \frac{1}{2}(Z^*(t))^2\kappa^2\sigma^2 J_{zzz}(t)dt - \frac{1}{2}\kappa\sigma Z^*(t)J_{zss}(t)dt - \sigma^2 J_{ss}(t)dS(t) \\
= (-rJ_z(t) + \kappa^2 Z^*(t)J_{zz}(t) + \kappa\sigma J_{zs}(t))dt + (-\kappa Z^*(t)J_{zz} - \sigma J_{zs})dw(t) \\
= r(-J_z(t) + \frac{\kappa}{\sigma}S(t)Z^*(t)J_{zz} + \sigma(t)J_{zs}(t))dt + (-\frac{\kappa}{\sigma} Z^*(t)J_{zz} - J_{zs}(t))dS(t),
\]

where the second equality follows from (8) and from the fact that \( J_{zz}dZ^*(t) = 0 \). We deduce from the definition of \( \theta \) that
\[
\int_0^t c^*(s)ds + W^*(t) = W_0 + \int_0^t \alpha(s)dB(s) + \int_0^t \theta(s)dS(s),
\]
where \( \alpha(t) = (W^*(t) - \theta(t)S(t))/B(t) \). Since \( W^*(t) \geq 0 \) for all \( t \), we conclude that \( c^* \) is a feasible consumption policies. It is now easy to see that \( c^* \) is also optimal. This is because \( c^* \) is the solution of the following program:
\[
\begin{align*}
\max_{c \geq 0} & \quad E \left[ \int_0^T u(c(t),t)dt \right] \\
\text{s.t.} & \quad E \left[ \int_0^T Z^*(t)c(t)dt \right] \leq W_0,
\end{align*}
\]
see Proof of Theorem 3.

SUPPLEMENT TO SECTION 3.5:

Claim I: The system of equations
\[
\begin{align*}
-xz\beta z^{b-1} + \frac{Y}{r} - \frac{1}{\Delta}z^{b-1} &= 0, \\
-xz\beta z^{b-1} + \frac{\beta Y}{r} - \frac{\beta z^{b-1}}{\Delta b} + \frac{\beta z^{b-1}}{\delta b} - \frac{\beta Y}{\delta} &= 0.
\end{align*}
\]
has a unique positive solution for \((x, z)\), denoted by \((A, \gamma)\).

Clearly, a necessary and sufficient condition for the above equations to have a positive solution is that the following equations
\[
\left( \frac{\beta}{\Delta} - \frac{\beta}{\delta} \right) Y - \left( \frac{\beta}{\Delta} - \frac{\beta}{\delta} \right) \frac{z^{b-1}}{b} = \frac{Y}{r} - \frac{1}{\Delta}z^{b-1}
\]
have a unique positive solution \( \gamma \) and that both sides of the equation are positive when evaluated at \( \gamma \). We observe that
\[
\begin{align*}
\frac{1}{r} - \left( \frac{\beta}{\Delta} - \frac{\beta}{\delta} \right) &= \frac{\kappa^2 \beta (\beta - 1)}{2r\delta} > 0, \\
\frac{1}{\Delta} - \left( \frac{1}{\Delta} - \frac{1}{\delta} \right) \frac{\beta}{b} &= \frac{\kappa^2 \beta (\beta - b)}{2\Delta \delta} > 0.
\end{align*}
\]
Therefore, (9) has a unique positive solution \( \gamma \), see Figure 5(a). To show that \( \frac{Y}{r} - \frac{1}{\Delta} \gamma^{b-1} > 0 \), we consider the difference between the left-hand side and the right-hand side of (9) when evaluated at \( \left( \frac{\Delta Y}{r} \right)^{\frac{1}{b-1}} \),

\[
LHS - RHS = \beta \left( \frac{1}{r} - \frac{1}{\delta} - \frac{1}{rb} + \frac{\Delta}{rb\delta} \right) = \frac{\kappa^2 \beta (1-b)}{2r\delta} > 0
\]

Hence, \( \gamma > \left( \frac{\Delta Y}{r} \right)^{\frac{1}{b-1}} \), implying that \( \frac{Y}{r} - \frac{1}{\Delta} \gamma^{b-1} > 0 \).  

Claim II: \(-\delta J(z) + Y z - \frac{z}{\delta} \geq 0\) for all \( z \geq \gamma \).

It is sufficient to show that \( \gamma > Y^{\frac{1}{b-1}} \). We consider the difference between the LHS and the RHS of (9) when evaluated at \( Y^{\frac{1}{b-1}} \):

\[
LHS - RHS = \left( \frac{\beta}{r} - \frac{\beta}{\delta} - \frac{\beta}{b\Delta} + \frac{\beta}{b\delta} - \frac{1}{r} + \frac{1}{\Delta} \right) Y = \frac{\kappa^2 \beta}{2\delta} \left( \frac{b}{\Delta} - \frac{\beta - 1}{r} \right) Y > 0.
\]

see Figure 5(b). Thus, \( \gamma > Y^{\frac{1}{b-1}} \).  

Claim III: \(-A \beta z^{b-1} + \frac{Y}{r} - \frac{1}{\Delta} z^{b-1} < 0\) for \( z < \gamma \), and \( J \) is convex in \( z \).

Consider the function \( f(z) = -A \beta z^{b-1} + \frac{Y}{r} - \frac{1}{\Delta} z^{b-1} \). Since \( \beta > 1 \) and \( b < 1 \), \( f \) has a unique maximum \( z^* > 0 \). Moreover, \( f \) is increasing for \( z < z^* \) and decreasing for \( z > z^* \), see Figure 5(c).

Since \( f(\gamma) = 0 \), it suffices to show that \( \gamma = z^* \). At \( z = \gamma \), we have

\[
\gamma f'(\gamma) = - (\beta - 1) A \gamma^{b-1} + \frac{1-b}{\Delta} \gamma^{b-1} = - (\beta - 1) \left( \frac{Y}{r} - \frac{1}{\Delta} \gamma^{b-1} \right) + \frac{1-b}{\Delta} \gamma^{b-1} = - (\beta - 1) \frac{Y}{r} + \frac{\beta - b}{\Delta} \gamma^{b-1}.
\]

Since \( \frac{\kappa^2 \beta}{2\delta r} \beta (\beta - 1) = - \left( \frac{\beta}{r} - \frac{\beta}{\delta} - \frac{1}{r} \right) \),

Multiplying \( \gamma f'(\gamma) \) by \( \frac{\kappa^2 \beta}{2\delta} \), we obtain

\[
\frac{\beta \kappa^2}{2\delta} \gamma f'(\gamma) = \left( \frac{\beta}{\Delta} - \frac{\beta}{\delta} - \frac{b}{\Delta} \right) \gamma^{b-1} + \frac{\beta - b}{\Delta} \frac{\beta \kappa^2}{2\delta} \gamma^{b-1} = 0,
\]

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where we have used the facts that $\Delta = \delta(1 - \beta) + rb + \frac{\alpha b}{2}(1 - b)$ and $\frac{\alpha b}{2} + \beta(\delta - r - \frac{\sigma^2}{2}) - \delta = 0$

to simplify the expression. We conclude that $z^* = \gamma$, and $f$ is strictly increasing for $z < \gamma$.  \[ \blacksquare \]
\[
f(x) = \frac{k^2}{2} x^2 + (\sigma - \frac{k^2}{2}) x - \sigma
\]

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5 (a)

Figure 5 (b)

Figure 5 (c)

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