Market Makers, Asymmetric Information and Price Information

by

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MARKET MAKERS, ASYMMETRIC INFORMATION AND PRICE FORMATION
ABSTRACT

Recognizing that the market maker holds private information about order flow, I generalize the 1985 Kyle model. This generalization allows consideration of the special case where, with some probability, noise traders do not exist in the market but trade still occurs. The properties of such a market are fully explored in both single and multiperiod settings. The primary findings of this research are that when market makers possess private information about order flow: prices converge more rapidly to the asset’s liquidation value; the informed agent’s private information is revealed faster; and the market need not be as deep as a market where the information about order flow is public knowledge. Further, I find that in a market where market makers hold private information about order flow, making that information public may actually increase the profits of the informed agent without increasing the amount of private information impounded into price.
1. INTRODUCTION

Market makers and specialists perform a variety of specialized tasks in financial marketplaces. They represent, at present, the best public source of information about asset price as they participate actively in the market and make trades to maintain a balance in order flow. Many papers recognize that these individuals are at risk in this activity since they may trade with agents who possess better information (eg. Milgrom and Stokely (1982); Kyle (1984); Glosten and Milgrom (1985); Easley and O'Hara (1987); Hughson (1988); Admati and Pfleiderer (1989)), however few papers recognize that these individuals may, in fact, have private information themselves (Gould and Verrecchia (1985); Rock (1990); Gennette and Leland (1990)).

Following Kyle (1985) the model in this paper involves a single risky asset which is traded between three types of economic agents: a single informed agent who has private knowledge of the ex post liquidation value of the asset; uninformed noise traders who trade randomly; and competitive market makers who set the market clearing price conditioned upon aggregate demand. I generalize the Kyle model by recognizing that the market makers possess private information about the order flow. Specifically, I assume that the market makers know the variance of the noise traders' demand while the informed agent must estimate this variance. This captures the notion that the informed agent knows more about the underlying asset value while the market makers know more about the condition of the market.

This study has strong public policy implications both for the organization of securities markets and for the institutions operating within them. The primary findings of this research are that when market makers possess private information about order flow: prices converge more rapidly to the asset’s liquidation value; the informed agent’s private information is revealed faster; and the market need not be as deep as when the information about order flow is public knowledge. Further, the results suggest that for electronic exchanges, where there are no market makers and the order flow is common knowledge, there must be an increase in the depth of the market if the informativeness of price is to be preserved.

The general version of the model is presented in Section 2 with two possible levels of noise trader volatility. The informed agent is assumed to have a probability measure over the volatilities while the market makers know which volatility has actually been realized in the market. Section 3 contains two examples which apply the model to a market in which noise traders may not exist. The first example establishes the framework and serves to further develop intuition through a direct application of the model. This example is interesting because we are able to develop expressions for the average market depth and the average level of price informativeness, allowing a direct comparison to the Kyle (1985) results. The second example extends the first

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1 Rock develops a model for the bid-ask spread when the specialist has private information contained in the limit order book. Because the specialist can choose to take all of the smaller (uninformed) trades and only a residual portion of larger (informed) trades, this information provides the specialist with market power. Gould and Verrecchia model a monopolistic specialist who fixes price to maximize expected surplus. Gennette and Leland have supply informed agents, but no market makers. This paper uses an entirely different approach.
to a multiperiod setting and investigates price behavior in a dynamic setting. In the final section I summarize the major findings and briefly discuss applications of the model to multiple markets and sunshine trading. All proofs are collected in the Appendix.

2. A GENERAL MODEL

In this section, I develop a single period model with asymmetric information about both the underlying value of the asset and the level of noise trading present in the market. The informed trader\(^2\) is assumed to have private information about the payoff of the asset, while the market makers are assumed to have private information about the condition of the market.

Trading takes place in two steps. First, the informed trader and the liquidity traders simultaneously submit market orders to buy or sell the asset. At the time the order is submitted, the informed trader’s information set consists of the payoff of the risky asset and a probability distribution over market conditions. The informed agent does not observe the market price of the asset or the demand of the liquidity traders before choosing the quantity to trade but does take into account the fact that trading will affect the resulting price. In the second step, the market makers set a price and trade the necessary quantity to make the market clear. In choosing their actions, the market makers rely on their private information about the condition of the market, the total demand of the informed and liquidity traders, and their knowledge of the probability of different market conditions.

Specifically, assume that the \textit{ex post} liquidation value of the risky asset is denoted as \(v\) and is normally distributed with mean \(p_o\) and variance \(\Sigma_o\). The demand of the noise traders, \(\zeta\), is assumed to be conditionally normal with mean 0 and variance \(\Sigma_{\zeta}\) where \(s=1\) when the market is in State 1 and \(s=2\) when the market is State 2.\(^3\) State 1 is the low volatility state and State 2 is the high volatility state (without loss of generality we can assume that \(\Sigma_2=\eta^2\Sigma_1\) where \(\eta>1\)). The random variables \(v\) and \(\zeta\) are assumed to be independent. The distribution of \(v\) is common knowledge, but the actual realization of \(v=v\) is known only to the risk neutral, informed trader, who observes \(v\) noiselessly and costlessly. Similarly, the conditional distributions of \(\zeta\) are common knowledge but the actual state of the market is observed only by the market makers. State 1 occurs with probability \(z\) and State 2 occurs with probability \((1-z)\) where \(z\) is independent of the other random variables and is assumed to be common knowledge.

\(^2\)This paper considers a single informed trader. It is possible to generalize the model to one in which the choice to become informed is endogenous, in which case there could be multiple informed traders.

\(^3\)Two states serve to keep the algebra simple, the model can be extended to countably many states.
The assumption that the market makers have private information about the state of the market (i.e. whether high or low noise trader volatility has been realized) is what distinguishes this work from other papers based on Kyle (1985). It is possible to attribute the two levels of volatility to whether or not discretionary liquidity traders are present in the market. Assuming a "normal" level of noise trading, when discretionary liquidity traders enter the market the variance of demand associated with noise trading increases. The assumption that the presence of the discretionary liquidity traders is observed by only the market makers explicitly captures the fact that observation of the order flow provides market makers with market power.

In the first step of trading, the exogenous values of s, ν=v and ζ=u are realized and the informed trader chooses to trade the quantity of the risky asset that maximizes expected utility, taking into account both the value of v and the effect that the trade will have on the market clearing price. This maximizing quantity is denoted as x, and the profits of the informed trader, Π, are given by Π=(ν-p)x where p is the market clearing price. In the second step, the market makers observe the collective market demand y=x+u and the value of s. The market makers then establish the competitive market price p and trade the quantity that clears the market.

**Definition 1 (Equilibrium)** An equilibrium is defined by a pair (x,p) such that:

a) **Profit Maximization:** For any quantity x̄ ≠ x and for all ν=v,

\[ E[\Pi(x, p) | v=v, Pr(\Sigma_1) = z, Pr(\Sigma_2) = (1-z)] ≥ E[\Pi(x̄, p) | v=v, Pr(\Sigma_1) = z, Pr(\Sigma_2) = (1-z)] \]

b) **Market Efficiency:** The market price p satisfies

\[ p(x, p) = E[v | y=x+u, z, I_{\Sigma_1}, I_{\Sigma_2}] \]

where:

\[ I_{\Sigma_1} = \begin{cases} 1 & \text{if } \Sigma = \Sigma_1 \\ 0 & \text{otherwise} \end{cases} \quad I_{\Sigma_2} = \begin{cases} 1 & \text{if } \Sigma = \Sigma_2 \\ 0 & \text{otherwise} \end{cases} \]

This equilibrium is essentially a Nash equilibrium in which the informed trader and the market makers respectively choose actions x and p that are optimal given their information sets at the time those actions are chosen. The profit maximization condition ensures that the informed agent trades rationally, taking

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4 Admati and Pfleiderer (1988) define discretionary liquidity traders as those agents who can choose the timing of their trades subject to the constraint of trading a certain number of shares by a given time.

5 Under the usual assumption that the demand of individual noise traders is uncorrelated.

6 One can think of the NYSE where the limit order book provides this private information to the specialist.

7 Technically, for this equilibrium to be considered Nash, the market makers must also maximize an objective function. Admati and Pfleiderer (1988) suggest that the objective function could be defined as minus the squared deviation of the market price from the realized asset value. Since we have assumed that there are competitive market makers in this model, the results will be the same.
into account both the effect of the trade on the price and the rule that the market makers use to set the price. The conditioning reflects the fact that the informed agent must trade based on a probabilistic assessment about the volatility of the noise traders' demand. The market efficiency condition captures the notion that the price should reflect all available information (semi-strong form efficiency), including the market makers' private information about the state of nature.

**Proposition 1 (Existence)** For the market described above, there exists a unique linear equilibrium defined by:

\[
x(v) = \beta(v - p_0) \\
p(y) = p_0 + \lambda y
\]

where the constants \(\beta\) and \(\lambda\) are defined as:

\[
\beta = \left[ \frac{2}{(2z-1)(\Sigma_2 - \Sigma_1) + [(2z-1)^2(\Sigma_2 - \Sigma_1)^2 + 4\Sigma_1 \Sigma_2]^{\frac{1}{2}}} \right]^{\frac{1}{2}} \left( \frac{\Sigma_1 \Sigma_2}{\Sigma_0} \right)^{\frac{1}{2}}
\]

and:

\[
\lambda = \left[ \frac{Q^{\frac{1}{2}}}{Q + 1} \left( \frac{\Sigma_0}{\Sigma_1} \right)^{\frac{1}{2}} \right] \quad \text{if} \quad \Sigma = \Sigma_1
\]

\[
\lambda = \left[ \frac{\eta Q^{\frac{1}{2}}}{Q + \eta^2} \left( \frac{\Sigma_0}{\Sigma_2} \right)^{\frac{1}{2}} \right] \quad \text{if} \quad \Sigma = \Sigma_2
\]

where:

\[
Q = \frac{2\eta^2}{(2z-1)(\eta^2 - 1) + [(2z-1)^2(\eta^2 - 1)^2 + 4\eta^2]^{\frac{1}{2}}}
\]

The proof of the proposition is given in the Appendix, however a brief discussion of the solution technique will help to make the remainder of the paper clearer. The basic idea is that the "active" market participants (the informed trader and the market makers) must solve for each other's optimal response given their own information set. The informed trader does not know the realized level of the noise traders' volatility and must therefore develop an estimate of the market makers' response function based upon the probability of each state occurring. Similarly, the market makers do not know the realized value of the asset and must therefore develop an optimal response based upon their knowledge of the distribution of \(v\), the true state of the market and the value of \(z\). The optimal responses of the agents represent best replies to each other.

The solution of the informed agent's optimal action is the most difficult. In models of this type the assumptions of risk neutrality and normal distributions provide a linear model in a straightforward manner through the use of the projection theorem. Introduction of uncertainty about the volatility of the noise traders'
demand transforms the usual optimization problem for the informed agent into an optimization problem with a partially revealed state space. The distribution function is no longer normal but a convolution of normals. Nevertheless, through proper ordering, it is possible to apply the projection theorem conditionally, which results in the linear demand function given above. The market makers' problem is a straightforward application of the projection theorem since they can solve the informed trader's optimization problem and they observe the true state of the market.

For comparison, it is useful to introduce the following result:

**Lemma 1 (Kyle Model)** The Kyle (1985) model is a special case of the model presented above. If $\Sigma_1 = \Sigma_2 = \Sigma$ (i.e., if $\eta = 1$), then the expressions for $\beta$ and $\lambda$ given in Proposition 1 are the same as those in Theorem 1 of Kyle (1985) for all $z$:

$$\beta = \left(\frac{\Sigma}{\Sigma_0}\right)^{\frac{1}{2}}$$

$$\lambda = \left(\frac{1}{2}\right) \left(\frac{\Sigma}{\Sigma_0}\right)^{-\frac{1}{2}}$$

Notice that the informed agent's demand given by Proposition 1 is, as we would expect, a function of the probability $z$, the two possible levels of noise trader variance $\Sigma_1$ and $\Sigma_2$, the variance of the asset $\Sigma_0$ and the signal $v$. By inspection, the informed agent's demand is increasing in $|v-p_0|$. Simply put, the agent buys relatively more when a high payoff is realized and sells relatively more when a low payoff is realized. It is easy to show that as $\Sigma_0$ increases there is a *ceteris paribus* decrease in the demand of the informed agent, however the intuition is somewhat subtle and will be discussed when we consider the response function of the market makers. Increasing either $\Sigma_1$ or $\Sigma_2$, or both, increases the demand of the informed trader. This is the standard result that the informed agent is able to "hide" behind the noise traders.

What is not so obvious is how the agent's response ($\beta$) to a signal compares with the Kyle result. If we condition on $\Sigma_1$ occurring, it is possible to show that the informed agent will have traded more aggressively than in the Kyle model where the volatility of the noise traders is common knowledge. Conversely, if we condition on $\Sigma_2$ occurring we will find that the agent has traded more conservatively. In each case, the demand of the informed agent is suboptimal when compared to the common knowledge case. There is a simple intuition behind this result. In the high volatility state the agent's demand is (in the Kyle model) greater than in the low volatility state since it is possible to hide more demand behind the noise traders. When the informed agent is uncertain about the condition of the market, demand must lie between the demands of the common knowledge model, high demand must be tempered by the possibility that the low volatility state occurs and vice versa.

The price established by the market makers is a function of the total order flow $y = x + u$, the volatility of the asset value $\Sigma_0$, the realization of the state of the market $\Sigma_1$ or $\Sigma_2$, the probabilities of those states, and the relative sizes of the two possible noise trader variances $\eta$. As total order flow increases -- price increases, capturing the usual notion that price increases with demand. The inverse of the market makers' sensitivity to
order flow, \(1/\lambda\), measures the "depth" of the market, the order flow necessary to change prices by one dollar. Market depth is an element of market liquidity that reflects the sensitivity of the market makers to innovations in order flow. It measures the ability of the market to absorb demand without a large price response. Other elements of market liquidity include tightness, the cost of turning over a position in a short period of time, and resiliency, the speed with which prices converge to the asset's liquidation value. Since these last two elements of market liquidity cannot be captured in the context of the single period model considered here, further discussion of these concepts is postponed until Section 3B.

Note that market depth increases as the volatility of the noise traders' demand increases and decreases as the volatility of the asset's liquidation value increases. The intuition behind the first effect is that an increase in the volatility of the noise traders' demand implies, ceteris paribus, a higher probability of the market makers observing a high positive or high negative demand that does not contain any information. Since the market makers observe only the sum of the noise traders' and informed trader's demands, total demand does not necessarily convey information about the liquidation value of the asset. The market makers are, therefore, relatively less sensitive to demand than they would have been given a lower volatility of noise trading. This implies that the market has relatively more depth when the high volatility state is realized than when the low volatility state is realized. The intuition behind the effect of the asset's volatility is similar. An increase in the volatility of the asset's liquidation value implies, ceteris paribus, a higher probability of observing a high positive or high negative demand that contains information. The market makers will therefore be relatively more sensitive to demand when the volatility of the asset is high than when the volatility is low. This helps to explain the behavior of the informed agent discussed above. When the volatility of the asset's payoff is increased, the market makers increase their sensitivity to order flow -- the informed agent's optimal response will be to decrease demand.

The expression for \(\lambda\) given in Proposition 1 consists of a ratio of standard deviations multiplied by a term that has a maximum of \(\frac{1}{2}\) when \(\eta = 1\) and is decreasing in \(\eta\) (recall that \(\eta > 1\) by definition). This implies that for anything other than the trivial case of \(\eta = 1\), the market described above is deeper than the market in which the volatility of the noise traders is common knowledge. So, whether the low volatility state or the high volatility state occurs, the market is deeper than it would have been under common knowledge. This may seem counter-intuitive, but the result depends on two effects. When the low volatility state is realized, the market makers know that the informed agent has traded too aggressively (ie. informed demand is higher than it would be under common knowledge about the state of the market). To satisfy the market efficiency condition they must therefore "discount" the order flow and be less sensitive -- resulting in a "deeper" market. If, however, the high level of noise trading is realized, the behavior of the market makers is not as clear. We must recognize that the market makers are essentially adjusting the market price based on deviations of the order flow away from zero. The larger the deviation, the more the price is adjusted away from the mean of the asset's
distribution. Since the informed agent is trading less aggressively than in the common knowledge case, large deviations in order flow are more likely to have arisen from the noise traders. Thus the market maker will be less sensitive to those deviations and the market will be deeper. The primary results about market depth are summarized in the following lemma.

Lemma 2 (Market Depth) Whether the low volatility or high volatility market condition is realized, the market is deeper than the common knowledge market. Moreover, market depth is increasing in $\eta^2$, the ratio of $\Sigma_2$ to $\Sigma_1$, and the market is deeper in the high volatility case than in the low volatility case.

Next we turn our attention to how much of the informed trader's information is impounded into the price. Our measure of price informativeness is the conditional variance of the asset’s payoff given the market price. If price is perfectly informative, the conditional variance is zero and if the price is uninformative the conditional variance is $\Sigma_o$ (as a reference point, in the common knowledge (Kyle) model the price informativeness is $\frac{1}{\eta^2}\Sigma_o$).

Lemma 3 (Informativeness of Price) When the low variance of noise trader demand is realized the price is more informative than in the common knowledge case. When the high variance is realized the price is less informative. Price informativeness is given in each case by:

$$\text{Var}[v | p, z] = \begin{cases} \frac{1}{1+Q} \Sigma_o & \text{if } \Sigma = \Sigma_1 \\ \frac{1}{1+Q} \eta^2 \Sigma_2 & \text{if } \Sigma = \Sigma_2 \end{cases}$$

where $Q$ is defined as in Proposition 1.

Note that the informativeness of price does not depend upon the actual levels of possible noise trading, but on their relative levels through the value of $\eta$. As $\eta$ increases, the informativeness of the low volatility realization increases and the informativeness of the high volatility realization decreases. This is a direct result of the optimal behavior of the informed agent. As $\eta$ increases the agent trades more, but at a decreasing rate. When the low volatility market is realized, the informed agent has traded more aggressively than what would have been optimal in the common knowledge model which reveals more of the private information about the asset's value. When the high volatility market is realized, the informed agent has traded too little and, as a consequence, has revealed less of the private information.

Finally, we consider the expected profit to the informed trader conditioned upon the state of the market. Since, as noted in the discussion above, the informed agent's action is always suboptimal when

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8In fact, price informativeness is a function of only the informed agent’s actions.
compared to the common knowledge market, it is not unreasonable to expect that profits would be less in the asymmetric market considered here.

**Lemma 4 (Expected Profits)** The informed agent's conditional expected profits are lower than in the common knowledge market. Conditional expected profits are given in each case by:

\[
E[\Pi | \Sigma = \Sigma_1] = \frac{Q^{\gamma}}{Q + 1} \left( \frac{\Sigma_1}{\Sigma_0} \right)^{1/2} (v - p_o)^2
\]

and:

\[
E[\Pi | \Sigma = \Sigma_2] = \frac{\eta Q^{\gamma}}{Q + \eta^2} \left( \frac{\Sigma_2}{\Sigma_0} \right)^{1/2} (v - p_o)^2
\]

Expected profits are decreasing in \( \eta \) and profits are greater when \( \Sigma_2 \) is realized than when \( \Sigma_1 \) is realized.

The intuition behind the relationship of expected profits and \( \eta \) is also straightforward. As \( \eta \) increases, the informed agent's actions in each case move farther and farther away from the optimal actions which would be chosen in the common knowledge case. As those actions become "less" optimal, expected profits become correspondingly lower.

This section of the paper presents a general model of a single period market in which the private information of the market makers is recognized. The single period model developed by Kyle (1985) is shown to be a special case within this model. Recognition of the market power held by the market makers has direct application to problems like the one considered by Admati and Pfleiderer (1988) where there are "regular" and "discretionary" liquidity traders and to the study of market systems where the order book is made public. We have seen that it may be too simple to assume that the informed agent knows the true market condition at all times.

In the remainder of the paper I consider a very special subcase, a market where there is some probability that noise traders do not exist. One should keep in mind that this pushes the model to a corner solution which provides some surprising results. The main advantage to this approach is that the algebra becomes much simpler and we are able to make more direct comparisons to the Kyle (1985) model in both a single and a multiperiod setting.
3. A Simple Example

In this section, I solve for the equilibrium in a market in which the informed agent is uncertain about the presence of noise traders. This differs from the Kyle (1985) model of informed trading in which the informed agent knows with certainty that noise traders are present. The example illustrates the main conclusions of the general model presented in Section 2 and allows us to obtain additional results. In the first part, I solve the problem in a single period setting and in Part B extend the solution to a multiperiod setting.

3A Single Period

Let us maintain the framework developed in Section 2 and assume that \( (1-z) \) is the probability that there are no noise traders in the market and that \( z \) is the probability that there are noise traders with a variance of \( \Sigma \) (i.e. \( \Sigma_1 = \Sigma \) and \( \Sigma_2 = 0 \)). Whether or not noise traders are realized in the market is known only to the market makers. All other assumptions remain the same as in Section 2.

**Proposition 2 (Existence)** There exists a unique linear equilibrium for the market in which noise traders exist with probability \( z \) where \( z \) is common knowledge. The equilibrium actions are given by:

\[
\begin{align*}
\pi(\nu) &= \beta(\nu - p_o) \\
p(y) &= p_o + \lambda y
\end{align*}
\]

where:

\[
\beta = (2z-1) \frac{1}{2} \left( \frac{\Sigma}{\Sigma_o} \right) \frac{1}{2} \quad \text{if } z > \frac{1}{2}
\]

\[
\beta = 0 \quad \text{if } z \leq \frac{1}{2}
\]

and:

\[
\lambda = (2z-1) \frac{1}{2} \left( \frac{\Sigma}{\Sigma_o} \right) \frac{1}{2}
\]

without noise traders

\[
\lambda = \left( \frac{2z-1}{4z^2} \right) \frac{1}{2} \left( \frac{\Sigma}{\Sigma_o} \right) \frac{1}{2}
\]

with noise traders

where \( z \) lies in the interval \((\frac{1}{2}, 1]\) and, as before, \( \pi \) is the demand of the informed agent and \( p \) is the market clearing price.
Perhaps the most striking feature about this proposition is that a threshold prior probability of \( z > \frac{1}{2} \) is necessary for the informed agent to trade. In other words, the informed agent must believe that there is a better than even chance of noise traders in the market before choosing to participate. The result is very counter-intuitive. One would expect that if the informed agent held a small prior that there were noise traders, the optimal action would be to trade a small amount. This logic neglects, however, the optimal action of the market makers.

Consider what happens when noise traders are not realized in the market. The market makers know that all the trade that they observe comes from the informed agent. Further, from our expression for \( \beta \), they know that the informed agent is trading less aggressively than would have been optimal if noise trading had been realized. This implies that the market makers must "scale up" the observed trade to fully capture the information content of the informed agent's demand. The smaller the probability of noise trading \( (z) \), the more the market makers scale up the demand; the larger the probability the less demand is scaled. When \( z = \frac{1}{2}, \) the informed agent's expected profits are equal to zero, so the informed agent is indifferent between trading and not trading. When the probability is greater than \( \frac{1}{2} \), expected profits will be positive and the agent will trade. Below \( \frac{1}{2} \) the market maker's optimal response (when noise traders are not realized and order flow is observed) is to set price at infinity, in which case the informed agent does not expect to make a profit from trading and does not participate in the market. I should be careful to point out that this threshold probability arises only because we are attempting to study a market in which noise trading may not arise. If there were always some level (even arbitrarily small) of noise trading then the informed agent would participate in the market to some extent regardless of the probability.

The intuition and results about market depth are the same as in Section 2, at least for the case when noise trading occurs in the market -- the market is deeper than in the common knowledge market. When there are no noise traders in the market, the Kyle model breaks down and no trading takes place. Therefore we have no meaningful basis for comparison in the no noise trading case. But we can see that there is at least some depth to the market when noise traders are not realized (and \( z > \frac{1}{2} \)), although the market is deeper when there are noise traders than when there are not. What is more interesting is to contrast the average or expected depth of this market with the common knowledge market. When we compare a Kyle market in which the volatility

---

9 The informed agent trades if the probability is greater than \( \frac{1}{2} \), but less aggressively than what would be optimal if noise traders were a certainty.

10 The assumption is that if the agent is indifferent between trading and not trading that no trade occurs. Although the effect of transactions costs are not explicitly considered in this model, any cost associated with the act of trading would remove the indifference and the agent will not trade.

11 Intuition suggests that if we were dealing with a risk averse agent this threshold probability would be higher, and the more risk averse the agent the higher the threshold.
of noise traders is the same as the expected volatility of noise trading in the asymmetric market, we find that the Kyle market is deeper.

**Lemma 5 (Average Market Depth)** For the market where noise traders may not exist, the market is, on average, not as deep as the market where noise traders exist with certainty.  

The average market depth is given by:

$$\frac{1}{\lambda} = 2(2z-1)^{\frac{1}{2}} \left( \frac{\Sigma}{\Sigma_o} \right)^{\frac{1}{2}}$$

where $z$ lies in the interval $(\frac{1}{2}, 1]$.

Now consider what happens to price informativeness. Intuition would suggest that when there are no noise traders in the market, price should be fully informative. Moreover, we would expect, based on the results in Section 2, that when noise traders are present in the market, prices are less informative than in the common knowledge case. This intuition proves to be correct, but again, it is more interesting to consider the average or expected level of price informativeness.

**Lemma 6 (Price Informativeness)** When there are no noise traders present in the market, the market price is perfectly informative. When noise traders are present, price is less informative than the certainty market. Average price informativeness is given by $\frac{1}{2} \Sigma_o$, which is the same level of informativeness that would be observed in the certainty market.

Combining Lemmas 5 and 6 gives the following important result:

**Proposition 3 (Depth and Informativeness)** Because of their private information about order flow, market makers are able to achieve the same level of price informativeness as in the certainty market but with less depth.

Proposition 3 implies that if we start with a market structure in which market makers possess private information about order flow, making that information public will actually increase the expected profits of the informed agent without increasing the information content of price. To close this example, we turn our attention to the conditional expected profits for the informed agent.

**Lemma 7 (Expected Profits)** When noise traders are not realized, the informed agent’s conditional expected profits are 0. When noise traders are realized, conditional expected profits are given by:

$$E[\Pi | \Sigma] = \left( \frac{(2z-1)^{\frac{1}{2}}}{2z} \right) \left( \frac{\Sigma}{\Sigma_o} \right)^{\frac{1}{2}} (\Sigma-p_o)^2$$

---

12 Another way to state Lemma 6 is to say that the market makers are, on average, more sensitive to volume in the uncertain case than they are in the certainty case.
which are less than the certainty market profits. Unconditional expected profits are given by:

\[ E[\Pi | \Sigma] = \frac{1}{2} (2z-1)^{\frac{1}{2}} \left( \frac{\Sigma}{\Sigma_o} \right)^{\frac{1}{2}} (v-p_o)^2 \]

which again are less than in the certainty market.

3C Multi-period

I now show how the previous example can be extended to a multi-period setting. In addition to the technical demonstration, a multi-period setting allows us to address the elements of liquidity introduced earlier -- market tightness and resiliency.

Let there be T meetings of the market, indexed by \( t = 1, 2, 3, \ldots \), where in each meeting of the market the same two-step trading mechanism introduced in Section 2 takes place. Noise traders, who exist with probability \( z \), have demand \( \zeta_t \), which is i.i.d. normal with mean zero and variance \( \Sigma_u \). The liquidation value of the asset, \( v \), is still assumed to be independent and normally distributed with mean \( p_o \) and variance \( \Sigma_o \). The realization \( v = v \) is known only to the informed agent, and whether noise traders have been realized in any given meeting of the market is known only to the market makers. The probability of noise traders is common knowledge.

The demand of the informed agent in meeting \( t \) is denoted as \( x_t \), and the informed agent's cumulative position is given by \( \sum_{t=1}^{T} x_t \). Let \( p_t \) denote the market clearing price in the \( t \)th meeting of the market. The random future profits of the informed agent at meeting \( t \) are given by: \( \Pi_t = \sum_{t=1}^{T} (v-p_t)x_t \).

At each meeting of the market, the informed agent's information set consists of the liquidation value of the asset \( v \), the probability of noise traders \( z \), and the past sequence of prices \( \overline{p}_{t-1} \). Similarly, the market makers' information set consists of the distribution of \( v \), the past sequence of prices, and their private knowledge about whether noise traders are present.

**Definition 2** *(Sequential Equilibrium)* A sequential equilibrium is defined as a pair \((\overline{x}, \overline{p})\) such that:

\[ \overline{x} = \left[ x_1, x_2, \ldots, x_T \right] \]

\[ \overline{p} = \left[ p_1, p_2, \ldots, p_T \right] \]

\[ \Pi_t = \sum_{t=1}^{T} (v-p_t)x_t \]

\[ \overline{p}_{t-1} \]

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\(^{13}\) I use a bar over a value to denote the vector of past values, for instance \( \overline{p} = [p_1, p_2, p_3, \ldots, p_T] \).
a) Profit Maximization: For all $t=1,...,T$ and for all $x'_t$ and $x_t$ such that

$$
\frac{x'_t}{x_t} = \lambda_{t-1},
$$

$$
E[\Pi_t(x_t,p_t) | v=z, \bar{p}_{t-1}, Pr\{\Sigma\}=z] \geq E[\Pi_t(x'_t,p_t) | v=z, \bar{p}_{t-1}, Pr\{\Sigma\}=z]
$$

b) Market Efficiency For all $t=1,...,T$, the market price satisfies

$$
p_t(x_t,p) = E[v | x_t+u_t, z \in [0,1], I_z]
$$

where:

$$
I_z = \begin{cases} 
1 & \text{if noise traders exist} \\
0 & \text{otherwise} 
\end{cases}
$$

A recursive linear equilibrium (Kyle (1985)) is an equilibrium in which there are constants $\lambda_1,...,\lambda_t$ such that for all $t$: $p_t = p_{t-1} + \lambda_t (x_t + u_t)$.

The profit maximization condition ensures that the informed agent behaves rationally at each point in time, taking into account the effect of trade on both the current price and future prices. At each meeting of the market, the informed agent trades the amount that maximizes expected profits over the remaining time, recognizing that noise traders may not be realized in the current or in future meetings of the market. Similarly, the market efficiency condition ensures that all available information is reflected in the market price at each point in time. This condition implies that the sequence of prices follows a martingale whose volatility at any point reflects the amount of the informed agent's private information that is contained in the price.

**Proposition 4 (Existence)** For the multiperiod market in which noise traders exist in each meeting with probability $z$, there exists a unique subgame perfect, recursive linear equilibrium where the equilibrium is defined at times $t=1,...,T$ by:

$$
x_t = \beta_t (v - p_{t-1})
$$

and:

$$
p_t = \lambda_{tN} (x_t + u_t) \quad \text{if noise traders}
$$

$$
p_t = \lambda_{t0} x_t \quad \text{otherwise}
$$

where:

$$
\beta_t = \frac{1 - 2z \alpha_t \lambda_{tN}}{2(E[\lambda_t | z, p_{t-1}] - z \alpha_t \lambda_{tN})^2}
$$

$$
E[\lambda_t | z, p_{t-1}] = z \lambda_{tN} + (1-z) \lambda_{t0}
$$

$$
\lambda_{tN} = \frac{\beta_t \Sigma_{t-1}}{\beta_t^2 \Sigma_{t-1} + \Sigma_{x}}
$$

$$
\beta_t = \frac{1}{\beta_t}
$$

13
\[ \Sigma_{t+1} = (1 - \lambda_{tN} \beta_t) \Sigma_{t-1} \quad \Sigma_{t0} = 0 \]

\[ \delta_{t-1} = z(\sigma, \lambda_{tN} \Sigma_a + \delta_t) \]

\[ \alpha_{t-1} = \beta_t (1 - \beta_t E[\lambda_t | z_{t+1}]) + z \alpha_t (1 - \lambda_{tN} \beta_t)^2 \]

subject to:

\[ E[\lambda_t | z_{t+1}] - z \alpha_t \lambda_{tN} \geq 0 \quad \forall \ t \]

and:

\[ \alpha_t = 0 \quad \delta_t = 0 \]

**Lemma 8 (Kyle Model)** The Kyle multiperiod market is a special case of the model presented above. If the probability of noise traders occurring in each meeting of the market is unity, the expressions given in Proposition 4 are the same as those given in Theorem 2 of Kyle (1985).

The solution technique for Proposition 4 is essentially the same as the recursive solution given in Kyle (1985) adjusted for the fact that in any meeting of the market noise traders may not obtain. When noise traders are not realized in the market the informed agent makes zero profit for that and all future meetings. This is not surprising since we expect that without noise traders all of the informed agent's private information about \( \nu \) is perfectly revealed. This means that the informed agent must perform a delicate balancing act. On one hand, the fact that the market meets a number times encourages the informed agent to spread trade out optimally over the meetings so as to not "give" away too much of the private information. But on the other hand, the fact that the market may end at any particular moment encourages the informed agent to trade as much as possible since there may not be another chance.

One way to think about the problem is as a series of Kyle-type models. In the Kyle model, the informed agent knows with certainty that the market will not end until time \( T \) (by end, I mean that private information will be revealed at time \( T \) and the informed agent can expect to trade and make profits until that time). In our model, the informed agent forms an expectation over the time remaining in the market, and trades as in the Kyle model with an ending time equivalent to the expected stopping time. It is easy to show that the expected stopping time in this case is, at each meeting \( t \), the minimum of \( (T-t) \) and \( 1/(1-z) \). Each time that the informed agent trades, the trade will be optimal for a Kyle model where the number of trades remaining is given by the expected stopping time. This implies that for high values of \( z \) (where \( 1/(1-z) \) will be very large) the informed agent's optimal action will be almost the same as the optimal action in the Kyle model. When \( 1/(1-z) \) is less than \( (T-t) \), the agent will trade more aggressively than in the Kyle model because the effect associated with the market ending early begins to outweigh the desire to spread out the trade.

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\[ ^{14} \text{This is, of course, also a result of the Kyle model.} \]
The solution to the multiperiod problem is difficult to interpret, but some general observations can be made. Expected future profits for time $t+1$ and forward at meeting $t$ are given by $\delta_{t,t}$ which is decreasing in the probability of no noise traders. Therefore as noise traders become less likely, expected future profits become smaller. This is a direct result of the increased probability of the market ending prior to reaching $T$. Similarly, when the informed agent submits the demand for the $t^{th}$ meeting of the market, the expected profit for that meeting is reflected (in part) by the parameter $\alpha_{t-1}$, which is also decreasing in the probability of no noise traders. This reflects the fact that noise traders may not be realized and that the demand submitted will be fully revealing. When noise traders are not realized in the market, all of the informed agent's private information is perfectly revealed, the market price is $v$ and current and future profits to informed agent are zero.

As discussed above, since the market may end at any point, the informed agent trades more aggressively than in the Kyle model. The market makers' optimal action is clear. When noise traders are present in the meeting of the market, the market makers are more sensitive to demand than in the Kyle model and the market is therefore not as deep. When noise traders are not present, the market makers are perfectly responsive and the market is even shallower. In the case with noise traders present, more of the informed agent's information is revealed than in the Kyle model and in the case without noise traders all of the private information is revealed. This implies that the results of Proposition 3 hold in the multiperiod model. Because of the asymmetric information about order flow, the market makers can extract the same level of price informativeness with less depth than is required in the Kyle model.

In Section 2, the concepts of market tightness and resiliency were introduced. Market resiliency refers to the speed with which prices converge to the asset's liquidation value. For the market described above, prices converge faster than in the Kyle multiperiod model. When noise traders are present more information is revealed since the informed agent is trading more aggressively and when noise traders are not realized, all of the informed agent's private information is revealed. Tightness refers to the cost associated with turning over a position in a short period time. Ignoring the jump in price that must occur when noise traders are not realized, the market described above is also tighter than the Kyle multiperiod model. Since prices are moving faster toward the asset's liquidation value, for the same number of periods, the incremental price changes must be smaller than in the Kyle model. We can summarize our finding about liquidity in the following lemma.

Lemma 9 (Market Liquidity) In the multiperiod market, where the market makers possess private information about the existence of noise traders in any meeting of the market, the market is: (1) not as deep as, (2) tighter than and (3) more resilient than the market where the existence of noise traders is common knowledge.
4 SUMMARY AND CONCLUSIONS

I develop a model in this paper which recognizes that the market makers possess private information about order flow in the market; thereby generalizing the Kyle (1985) model. The informed agent possesses private information about the liquidation value of the single risky asset and takes into account the effect that trade has on market price. Competitive market makers fix the price for the asset given the net demand of the informed agent and liquidity traders and their private information about the volatility of liquidity trading realized in the market.

In Section 2, the general model is developed in a single period framework with two possible levels of noise trader volatility. The informed agent and market makers are assumed to have as common knowledge the probability measure over the two possible volatilities. Market makers know whether high volatility or low volatility noise trading is realized, while the informed agent knows the ex post liquidation value of the risky asset. I find that the market makers are less sensitive than they would be in a market where the volatility of noise traders is common knowledge and the market is therefore deeper than in the common knowledge case. If the high volatility market is realized then the market price will be less informative than the common knowledge price and if the low volatility market is realized the price will be more informative. In both cases, the expected profits of the informed agent are less than they would be in a market where the volatility of noise trading is common knowledge.

The most interesting results are demonstrated in the two examples discussed in Section 3. In the first example, I apply the model from Section 2 to a market in which noise traders may not exist. I find, somewhat surprisingly, that noise traders must exist with a probability greater than \( \frac{1}{2} \) or the informed agent will not participate in the market. The existence of a threshold probability arises from the consideration of the optimal action of the market makers and the assumption of risk neutrality. When noise traders are not realized in the market, the price is fully informative, and when noise traders are realized, the price is less informative than in the common knowledge case because the informed agent trades less aggressively. In this framework, I develop expressions for the average market depth and average level of price informativeness. These results are collected in Proposition 3, which shows that because of their private information about order flow, market makers are able to extract the same level of price informativeness as in the common knowledge market but with less depth. This proposition holds significant implications for the (re)design of markets. In a market where market makers hold private information about order flow, making that information public may actually increase the profits of the informed agent without increasing the amount of private information impounded into price.

The second example extends the first to a multiperiod setting with similar results. When market makers possess private information about order flow, the price sequence converges to the underlying asset.
value faster than in the common knowledge market, both because the informed agent trades more aggressively and because price becomes perfectly informative when no noise traders are realized. Further, the market is tighter for the asymmetric case when compared to the common knowledge case -- that is that the incremental price changes are smaller so the cost of turning over a position quickly is smaller. All of these results are obtained with less market depth, thereby demonstrating that the conclusions drawn from the single period example are robust.

The two examples serve to illustrate that asymmetric markets may not be all "bad" (see Scitovsky (1990)). The private information of market makers about demand allows them to ensure that the informed agent's information is revealed "faster" than if the information about demand were made public. Since faster revelation of the asset value benefits the liquidity traders, asymmetric information about order flow generates a positive externality.

The model and examples can also provide insight into other issues associated with market structure/design. For instance, the model confirms the results of Chowdhry and Nanda (1990), who find that when an asset trades at multiple locations, the market with the highest level of "normal" noise trading becomes the dominant marketplace. From the model presented above, we can see that there are two ways in which a market can become dominant, the first being the same as that in Chowdhry and Nanda. Given two markets, the market with the highest expected noise trader volatility will be selected by the informed agent since this will maximize expected profits. The second is more subtle. Given two markets, one in which the level of noise trading is uncertain and one in which the level of noise trading is common knowledge, the informed agent will choose the latter. In both cases, we can see that whichever market the informed agent chooses there will be less private information revealed in price.

We can also consider the effect of "sunshine" trading. Admati and Pfleiderer (1990) demonstrate, in an entirely different framework, that when certain traders preannounce their trades there can be aggregate welfare gains in the market. We can obtain a similar result from this model. Consider a market in which a subgroup of the liquidity traders can choose to notify the market makers of their demand. Aggregate demand will then be \( y = x + u + a \), where \( x \) is the demand of the informed agent, \( u \) is the demand of "regular" noise traders and liquidity traders that do not choose to announce and \( a \) is the demand of the liquidity traders who choose to announce. This fits into neatly into the model where the volatility of noise traders is uncertain and generates the benefits discussed above: tighter markets, more informative prices and faster convergence to the underlying value. In such a market, the liquidity traders who can preannounce become "active" participants in the market since with a randomized strategy of announce/don't announce they can affect market price.

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15 Recall that competition keeps the market makers' expected profits zero, so the "benefit" to the noise traders is that they lose less.

16 Notice that this is slightly different from the usual notion of sunshine trading since the order book is closed and only the market makers are notified of the upcoming trade.
APPENDIX

All necessary proofs are collected here. Because of the length of some of the proofs only the major steps are shown, the interested reader should have no difficulty with the intervening algebra.

**Proposition 1:** Begin by assuming a linear solution

\[ x(v) = \alpha + \beta v \]
\[ p(y) = \mu + \lambda y = \mu + \lambda (x + u) \]

the informed agent maximizes expected profit conditioned upon the expected behavior of the market makers where \( z = \Pr[\Sigma = \Sigma_1] \) and \( (1-z) = \Pr[\Sigma = \Sigma_2] \).

\[
E[\Pi | v=v, z] = E[(v-p)x | v=v, z] \\
= (v - E[p | z])x \\
= (v - E[\mu + \lambda (x + \zeta | z)])x \\
= vx - E[\mu | z]x - E[\lambda | z]x^2 - E[\lambda \zeta | z]x
\]

notice that \( \mu \) and \( \lambda \) are set by the market makers so the informed agent must estimate; \( v = v \) and \( x \) are known to the informed agent. For now we will assume that \( E[\lambda \zeta | z] = 0 \); this will be verified latter.

Maximizing expected profits, the first order condition is

\[ v - E[\mu | z] - 2E[\lambda | z]x = 0 \]

which implies

\[ x = \frac{-E[\mu | z]}{2E[\lambda | z]} + \frac{1}{2E[\lambda | z]} v \]

therefore

\[ \alpha = \frac{-E[\mu | z]}{2E[\lambda | z]} \quad \beta = \frac{1}{2E[\lambda | z]} \]

Now the informed agent must estimate what the market makers do

\[ p = E[v | y=x+u, z] \]
\[ = E[\mu + \lambda y | y, z] \]
\[ = E[\mu | z] + E[\lambda | z]y \]

Under the assumptions of normality

\[
f(y | \Sigma) = \frac{1}{(2\pi(\beta^2 \Sigma + \Sigma))^{1/4}} \exp \left[ -\frac{(y-\alpha - \beta p)^2}{2(\beta^2 \Sigma + \Sigma)} \right]
\]

and
\[ f(v) = \frac{1}{(2\pi\sigma)^{d/2}} \exp \left[ -\frac{(v-p_0)^2}{2\sigma^2} \right] \]

therefore

\[ f(y,v | \Sigma) = \frac{1}{2\pi|\Omega|^{1/2}} \exp \left[ -\frac{1}{2} \left( \begin{array}{c} y-\alpha - \beta p_0 \\ v-p_0 \end{array} \right)^T \Omega^{-1} \left( \begin{array}{c} y-\alpha - \beta p_0 \\ v-p_0 \end{array} \right) \right] \]

where

\[ \Omega = \begin{pmatrix} \beta^2\Sigma_0 + \Sigma & \beta \Sigma_0 \\ \beta \Sigma_0 & \Sigma_0 \end{pmatrix} \]

therefore we may write

\[ f(v | \Sigma, y) = \frac{1}{\sqrt{2\pi(\Sigma_0 + \beta^2\Sigma_0)}} \exp \left[ -\frac{1}{2} \left( \begin{array}{c} y-p_0 - \frac{\beta \Sigma_0}{\beta^2\Sigma_0 + \Sigma} (y-\alpha - \beta p_0) \\ \Sigma_0 - \frac{(\beta \Sigma_0)^2}{\beta^2\Sigma_0 + \Sigma} \end{array} \right)^2 \right] \]

By Bayes' theorem and our assumptions of independence, \( f(v, \Sigma | y, z) = f(v | \Sigma, y) f(\Sigma) \); we may therefore write

\[ f(v, \Sigma | y, z) = \left[ z \delta(\Sigma - \Sigma_1) + (1-z) \delta(\Sigma - \Sigma_2) \right] N(\gamma(\Sigma), \Gamma(\Sigma)) \]

where

\[ \gamma(\Sigma) = p_0 + \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma} (y-\alpha - \beta p_0) \]

\[ \Gamma(\Sigma) = \Sigma_0 - \frac{\beta^2 \Sigma_0^2}{\beta^2 \Sigma_0 + \Sigma} \]

and where \( \delta() \) is the delta function such that \( \delta(r=0) = 1 \) and \( \delta(r \neq 0) = 0 \).

The informed agent's estimate of the market maker's action is

\[
p = E[v | y, z] = p_0 + \frac{z \beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma_1} + \frac{(1-z) \beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma_2} (y-\alpha - \beta p_0) \]

\[
= p_0 + \frac{z \beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma_1} + \frac{(1-z) \beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma_2} (a + \beta p_0) + \frac{z \beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma_1} + \frac{(1-z) \beta \Sigma_0}{\beta^2 \Sigma_0 + \Sigma_2} y \]

\[
= E[\mu | z] + E[\lambda | z] y \]

We now have a system of four equations which the informed agent must solve
\[ \alpha = \frac{-E[\mu | z]}{2E[\lambda | z]} \quad \beta = \frac{1}{2E[\lambda | z]} \]

\[ E[\lambda | z] = \frac{z \beta \Sigma_o}{\beta^2 \Sigma_o + \Sigma_1} + \frac{(1-z) \beta \Sigma_o}{\beta^2 \Sigma_o + \Sigma_2} \]

\[ E[\mu | z] = p_o - E[\lambda | z](\alpha + \beta p_o) \]

Using the second order condition \( E[\lambda | z] > 0 \) for a maximum, it is relatively straightforward to show that the unique solution to this system is

\[ E[\mu | z] = p_o \quad \alpha = \frac{-p_o}{2E[\lambda | z]} \quad E[\lambda | z] = \frac{1}{2\beta} \]

\[ \beta = \left[ \frac{2}{(2z-1)(\Sigma_2 - \Sigma_1) + (2z-1)^2(\Sigma_2 - \Sigma_1)^2 + 4\Sigma_1 \Sigma_2} \right]^{\frac{1}{4}} \left( \frac{\Sigma_o}{\Sigma} \right)^{\frac{1}{4}} \]

which provides the solution for the informed agent once we note that \( \alpha + \beta p_o = 0 \) and verify our earlier assumption by \( E[\lambda \zeta | z] = E[\lambda \Sigma]E[\zeta | \Sigma | z] = 0 \).

To solve the market maker's problem, recall that they observe which level of noise trading is realized in the market. Assume first that the realized level of noise trader volatility is \( \Sigma_1 \) (i.e., \( \Sigma = \Sigma_1 \)).

The market makers' problem is

\[ p = E[v | y, \Sigma_1, z] = \mu + \lambda y \]

since \( (y | z, \Sigma_1) = \alpha + \beta v + v \), under the normality assumptions

\[ (y | z, \Sigma_1) \sim N(\alpha + \beta p_o, \beta^2 \Sigma_o + \Sigma_1) \]

\[ \begin{pmatrix} y \\ \Sigma_1 \end{pmatrix} \sim N \begin{pmatrix} p_o \\ \Sigma_o \end{pmatrix} \begin{pmatrix} \beta \Sigma_o \\ \beta \Sigma_o + \Sigma_1 \end{pmatrix} \]

therefore we may write

\[ f(v | \Sigma_1, y, z) = N(\gamma(z), \Gamma(z)) \]

where

\[ \gamma(z) = p_o + \left( \frac{\beta \Sigma_o}{\beta^2 \Sigma_o + \Sigma_1} \right)(y - \alpha - \beta p_o) \]

and

\[ \Gamma(z) = \Sigma_o - \frac{\beta^2 \Sigma_o^2}{\beta^2 \Sigma_o + \Sigma_1} \]

since \( \alpha + \beta p_o = 0 \) and \( \mu = p_o \) for all \( z \), we have
\[ \lambda = \frac{\beta \Sigma_o}{\beta^2 \Sigma_o + \Sigma_1} \]

Since the market maker's can solve the informed agent's maximization problem, they know that

\[ \beta(z) = \left( \frac{2}{(2z-1)(\Sigma_2 - \Sigma_1) + [(2z-1)^2(\Sigma_2 - \Sigma_1)^2 + 4\Sigma_1 \Sigma_2]^2} \right)^{\frac{1}{16}} \left( \frac{\Sigma_1 \Sigma_2}{\Sigma_o} \right) \]

Now recall that we have assumed \( \Sigma_2 = \eta \Sigma_1 \) and that \( \Sigma_1 \) has been realized, the expression for \( \beta \) may be simplified for the case of \( \Sigma = \Sigma_1 \) to

\[ \beta(z) = Q^{1/4}(\Sigma, \eta) \left( \frac{\Sigma_1}{\Sigma_o} \right)^{1/4} \]

where

\[ Q(z, \eta) = \frac{2\eta^2}{(2z-1)(\eta^2 - 1) + [(2z-1)^2(\eta^2 - 1)^2 + 4\eta^2]^2} \]

Substituting for \( \beta \) we can write

\[ \lambda = \frac{Q^{1/4}(\Sigma, \eta)^{1/4}}{Q^{1/4}(\Sigma, \eta) + 1} = \left[ \frac{Q^{1/4}}{Q + 1} \left( \frac{\Sigma_o}{\Sigma_1} \right)^{1/4} \right] \]

For the case where \( \Sigma_2 \) is realized, we get

\[ \lambda = \left[ \frac{\eta Q^{1/4}}{Q + \eta^2} \left( \frac{\Sigma_o}{\Sigma_2} \right)^{1/4} \right] \]

This completes the proof of Proposition 1.

**Lemma 1:** Simple substitution of \( \Sigma_1 = \Sigma_2 = \Sigma \) into the expression for \( \beta \) provides

\[ \beta = \left( \frac{\Sigma}{\Sigma_o} \right)^{1/4} \]

Now consider the expression for \( \lambda \); in particular

\[ Q(z, \eta) = \frac{2\eta^2}{(2z-1)(\eta^2 - 1) + [(2z-1)^2(\eta^2 - 1)^2 + 4\eta^2]^2} \]

Setting \( \eta = 1 \), we find that \( Q = 1 \) for all \( z \) and therefore

\[ \lambda = \frac{1}{2} \left( \frac{\Sigma_o}{\Sigma} \right)^{1/4} \]

**Lemma 2:** First, we show that \( \eta^2 > 1 \) implies that \( Q > 1 \).

Assume the converse, that \( \eta^2 > 1 \) but \( Q < 1 \).
\[ \frac{2\eta^2}{(2z-1)(\eta^2-1)+[(2z-1)^2(\eta^2-1)^2+4\eta^2]^\frac{1}{4}} < 1 \]

\[ (2z-1)(\eta^2-1)+[(2z-1)^2(\eta^2-1)^2+4\eta^2]^\frac{1}{4} > 2\eta^2 \]

\[ 4\eta^2 > 4\eta^2 - 4\eta^2 (2z-1)(\eta^2-1) \]

which implies that \( 1 > \eta^2 \) for all \( z \) less that 1 (for \( z = 1 \), \( \eta^2 = 1 \)) which is a contradiction. This implies that \( Q > \eta^2 > 1 \) or \( \eta^2 > Q > 1 \) (\( Q = \eta^2 \) only when \( \eta = 1 \) as shown in Lemma 1). Assume that \( Q > \eta^2 > 1 \)

\[ \frac{2\eta^2}{(2z-1)(\eta^2-1)+[(2z-1)^2(\eta^2-1)^2+4\eta^2]^\frac{1}{4}} > \eta^2 \]

\[ (2z-1)(\eta^2-1)+[(2z-1)^2(\eta^2-1)^2+4\eta^2]^\frac{1}{4} < 2 \]

\[ 4\eta^2 < 4 - 4(2z-1)(\eta^2-1) \]

which implies that \( \eta^2 < 1 \) which is a contradiction. If we assume that \( \eta^2 > Q \) we can verify that \( \eta^2 > 1 \), indicating that \( \eta^2 > Q > 1 \) is the proper ordering.

Now it is possible to show that

\[ \frac{Q}{Q+1} \leq \frac{1}{2} \]

and

\[ \frac{\eta Q}{Q+\eta^2} \leq \frac{1}{2} \]

both of which are decreasing in \( \eta \) and where equality holds only when \( \eta = 1 \). Further, it is easy to show that when \( \Sigma_2 \) is realized \( \lambda \) is smaller than when \( \Sigma_1 \) is realized. Simple comparison with the Kyle result verifies the lemma.

**Lemma 3:** Using normality and assuming \( \Sigma = \Sigma_1 \)

\[ \nu \sim N(p_{\sigma}, \Sigma_{\sigma}) \]
\[ p \sim N(p_{\sigma}, \lambda^2 \beta^2 \Sigma_{\sigma} + \lambda^2 \Sigma_1) \]

therefore

\[ \text{Var}[\nu | p] = \Sigma_{\sigma} - \frac{(\lambda \beta \Sigma_{\sigma})^2}{\lambda \beta^2 \Sigma_{\sigma} + \lambda^2 \Sigma_1} \]

\[ = \begin{bmatrix} \Sigma_1 \\ \beta^2 \Sigma_{\sigma} + \Sigma_1 \end{bmatrix} \Sigma_{\sigma} \]

\[ = \begin{bmatrix} 1 \\ 1+Q \end{bmatrix} \Sigma_{\sigma} \]

when \( \Sigma = \Sigma_1 \) the derivation is similar. Since \( \eta > 1 \) the conditional variance is higher (for any \( 0 < z < 1 \)) when \( \Sigma = \Sigma_2 \) (i.e. when high volatility of noise traders is realized), higher conditional variance implies that price is less informative.
Lemma 4: Consider first the case $\Sigma = \Sigma_1$. Conditional expected profits can be found

$$
E[\Pi | \Sigma = \Sigma_1] = (\beta - \lambda_1 \beta^2)(v-p_o)^2
$$

$$
= \left( Q^{\lambda} \left( \frac{\Sigma_1}{\Sigma_o} \right)^{\lambda} - \lambda_1 \mu \left( \frac{\Sigma_1}{\Sigma_o} \right) \right) (v-p_o)^2
$$

$$
= \left[ \frac{Q^{\lambda}}{Q+1} \right]^2 \left( \frac{\Sigma_1}{\Sigma_o} \right)^{\lambda} (v-p_o)^2
$$

where $\lambda_1$ is the market makers' optimal action when $\Sigma = \Sigma_1$ as defined in Proposition 1 and where $Q$ is as defined earlier. Similarly, when $\Sigma = \Sigma_2$

$$
E[\Pi | \Sigma = \Sigma_2] = (\beta - \lambda_2 \beta^2)(v-p_o)^2
$$

$$
= \left[ \frac{Q^{\lambda}}{Q+\eta^2} \right]^2 \left( \frac{\Sigma_2}{\Sigma_o} \right)^{\lambda} (v-p_o)^2
$$

As show in Lemma 2, the expressions in square brackets are less than $\frac{1}{2}$ in each case. Therefore the informed trader makes less profit than in the certainty case regardless of which level of volatility is realized. It is easy to show that the informed agent makes more profits (in the expectations sense) when the higher level of noise trading is realized than when the lower level is realized.

Proposition 2: The solution technique is essentially the same as that used in Proposition 1, therefore only the key changes are developed here.

The informed agent must make an estimate of $\lambda$ given the prior that noise traders with variance $\Sigma$ occur with probability $z$

$$
E[\lambda | z] = \frac{(1-z)\beta \Sigma_o - z \beta \Sigma_o}{\beta^2 \Sigma_o + \Sigma} = \frac{1}{2 \beta}
$$

solving for $\beta$

$$
\beta = (2z-1)^{\lambda} \left( \frac{\Sigma}{\Sigma_o} \right)^{\lambda} \in (\frac{1}{2}, 1]
$$

$$
\beta = 0 \quad z \in [0, \frac{1}{2}]
$$

since $\beta$ must be real and expected profits are 0 when $z = \frac{1}{2}$.

Solving for the market makers' actions, if no noise traders are realized

$$
\lambda_o = \frac{1}{\beta} = \frac{1}{(2z-1)^{\lambda}} \left( \frac{\Sigma_o}{\Sigma} \right)^{\lambda} \in (\frac{1}{2}, 1]
$$

If noise traders are realized
\[ \lambda_N = \frac{\beta \Sigma_o}{\beta^2 \Sigma_o + \Sigma} = \left( \frac{(2z-1)^{\frac{1}{\alpha}}}{2z} \left( \frac{\Sigma_o}{\Sigma} \right) \right)^{\frac{1}{\alpha}} \mathbb{I}(\mathbb{I},1) \]

**Lemma 5:** Average market depth is found by considering the average of the market makers' optimal actions

\[
\bar{\lambda} = z \lambda_N^{1-z} (1-z) \lambda_0
\]

\[
= z \frac{(2z-1)^{\frac{1}{\alpha}}}{2z} \left( \frac{\Sigma_o}{\Sigma} \right)^{\frac{1}{\alpha}} + (1-z) \frac{1}{(2z-1)^{\frac{1}{\alpha}}} \left( \frac{\Sigma_o}{\Sigma} \right)^{\frac{1}{\alpha}}
\]

\[
= \frac{1}{(2z-1)^{\frac{1}{\alpha}}} \left[ \frac{1}{2} \left( \frac{\Sigma_o}{\Sigma} \right)^{\frac{1}{\alpha}} \right]
\]

where the last term in brackets is the same as in Kyle (1985). Strictly speaking however, we should compare this average value to a Kyle market with the same expected noise trader variance; in such a market:

\[
\lambda = \frac{1}{2} \left( \frac{z \Sigma}{\Sigma_o} \right)^{\frac{1}{2}} = \frac{1}{z^{\frac{1}{\alpha}}} \left[ \frac{1}{2} \left( \frac{\Sigma}{\Sigma_o} \right)^{\frac{1}{2}} \right]
\]

Since \( z^{\frac{1}{\alpha}} > (2z-1)^{\frac{1}{\alpha}} \), market makers are, on average, more sensitive than in the Kyle market, and therefore the market has less depth.

**Lemma 6:** Without noise traders in the market the market makers know that observed demand is only the demand of the informed agent; therefore price is (define \( \lambda_0 \) as the market makers' response when noise traders are not realized)

\[ p = p_o + \lambda_0 x = p_o + \lambda_o (\nu - p_o) \]

and

\[ p = N(p_o, \lambda_0^2 \beta^2 \Sigma_o) \]

\[ \left( \begin{array}{c} \nu \\ \rho \end{array} \right) 
\sim N \left( \begin{array}{c} p_o \\ p_o \end{array} \right), \left( \begin{array}{cc} \Sigma_o & \lambda_0 \beta \Sigma_o \\ \lambda_0 \beta \Sigma_o & \lambda_0^2 \beta^2 \Sigma_o \end{array} \right) \]

therefore price informativeness is given by

\[ \text{Var} [\nu | p, z] = \Sigma_o - \frac{(\lambda_0 \beta \Sigma_o)^2}{\lambda_0^2 \beta^2 \Sigma_o} = \Sigma_o - \Sigma_o = 0 \]

and price is perfectly informative. With noise traders the joint distribution is given by

\[ \left( \begin{array}{c} \nu \\ \rho \end{array} \right) \sim N \left( \begin{array}{c} p_o \\ p_o \end{array} \right), \left( \begin{array}{cc} \Sigma_o & \lambda_N \beta \Sigma_o \\ \lambda_N \beta \Sigma_o & \lambda_N^2 \beta^2 \Sigma_o + \lambda_N \Sigma \end{array} \right) \]

where \( \lambda_N \) is the optimal action of the market makers when noise traders are realized. In this case price informativeness is
\[
\text{Var}[v|p,z] = \Sigma_o \frac{(\lambda_N \beta^2 \Sigma)^2}{\lambda_N^2 \beta^2 \Sigma_o + \lambda_N^2 \Sigma} = \left[1 - \frac{\beta^2}{\beta^2 + \Sigma \Sigma_o}\right] \Sigma_o
\]

\[
= \frac{\Sigma}{(2z-1)\left(\frac{\Sigma}{\Sigma_o}\right) \Sigma_o + \Sigma} \Sigma_o
\]

\[
= \frac{1}{2z} \Sigma_o
\]

since \(1/2z > \frac{1}{2}\) price is less informative than in the Kyle (1985) model.

Average price informativeness is

\[
z\left(\frac{1}{2z} \Sigma_o\right) + (1-z) 0 = \frac{1}{2} \Sigma_o
\]

which is the same as in Kyle (1985).

**Proposition 3:** Follows directly from Lemmas 5 and 6.

**Lemma 7:** For the no noise case, expected profits are

\[
E[\Pi|\Sigma=0] = \Pi_0
\]

\[
= (\beta - \lambda_0 \beta^2) (v - \rho_o)^2
\]

\[
= (\beta - \beta) (v - \rho_o)^2
\]

\[
= 0
\]

For the noise case

\[
E[\Pi|\Sigma] = \Pi_N
\]

\[
= (\beta - \lambda_N \beta^2) (v - \rho_o)^2
\]

\[
= \left[\frac{(2z-1)^{\frac{1}{z}}}{2z} \left(\frac{\Sigma}{\Sigma_o}\right) \right]^{\frac{1}{z}} (v - \rho_o)^2
\]

Unconditional expected profits are

\[
E[\Pi] = z \Pi_N + (1-z) \Pi_0
\]

\[
= [z(\beta - \lambda_N \beta^2) + (1-z)(\beta - \lambda_0 \beta^2)](v - \rho_o)^2
\]

\[
= \frac{1}{2} (2z-1)^{\frac{1}{z}} \left(\frac{\Sigma}{\Sigma_o}\right)^{\frac{1}{z}} (v - \rho_o)^2
\]

which is less than in the Kyle market but is still positive.

**Proposition 4:** Expected profits at any time \(1 \leq t+1 \leq T\) for the informed agent are

\[
E[\Pi_{t+1} | \tilde{p}, v, z] = \alpha_t (v - p)^2 + \delta_t
\]

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where the first term on the right hand side is the expected profit for the trade at t+1 and the second term is the expected future profits. (Recall that a bar over a value denotes the vector of the past values.) Recursively

$$
\Pi_t = (v-p_t)x_t + \Pi_{t+1}
$$

Denote $p_{t,0}$ as the price at time $t$ if noise traders are realized and $p_{t,0}$ as the price at time $t$ if noise traders are not realized. Let $h_t$ be a function of past order flow (i.e., order flow prior to time $t$) and assume that the pricing function is linear in each case

$$
p_{t,N} = p_{t-1} + \lambda_{t,N} x_t + h_t
$$

$$
p_{t,0} = p_{t-1} + \lambda_{t,0} x_t + h_t
$$

Expected profits for the informed agent can now be found

$$
E[\Pi_t | \bar{p}_{t-1}, v, z] = E[zE[\Pi_{t,N} + (1-z)\Pi_{t,0} | \bar{p}_{t-1}, v, z]]
$$

$$
= E[z((v-p_{t,N} - \lambda_{t,N} x_t + \Pi_{t,1,N}) + (1-z)((v-p_{t,0} x_t + \Pi_{t,1,0}) | \bar{p}_{t-1}, v, z)]
$$

assuming $\Pi_{t+1,0}=0$ and $h_t=0$ (we shall confirm this later) we may write the above expression as

$$
E[z((v-p_{t-1} - \lambda_{t,N} x_t + \Pi_{t,1,N}) x_t + \alpha_t (v-p_{t-1} - \lambda_{t,N} x_t + \Pi_{t,1,N})^2 + \delta_t)
$$

$$
+ (1-z)((v-p_{t-1} - \lambda_{t,0} x_t) x_t | \bar{p}_{t-1}, v, z)]
$$

the first order condition is

$$
x_t = \frac{1 - 2z \alpha_t \lambda_{t,N}}{2z \lambda_{t,N} + (1-z) \lambda_{t,0} - z \alpha_t \lambda_{t,N}^2} (v-p_{t-1})
$$

which implies

$$
\beta_t = \frac{1 - 2z \alpha_t \lambda_{t,N}}{2z \lambda_{t,N} + (1-z) \lambda_{t,0} - z \alpha_t \lambda_{t,N}^2}
$$

When no noise traders are realized the market makers' optimal action is simply

$$
\lambda_{t,0} = \frac{1}{\beta_t} = \frac{2z \lambda_{t,N} + (1-z) \lambda_{t,0} - z \alpha_t \lambda_{t,N}^2}{1 - 2z \alpha_t \lambda_{t,N}}
$$

To find $\lambda_{t,N}$

$$
E[v | (x_t, u_t), p_{t-1}, z] = p_t + v - N(p_{t-1}, v, z),
$$

$$
(x_t + \zeta_t) - N(0, \Sigma_{t-1} + \Sigma_u)
$$

$$
E[v | (x_t, u_t), p_{t-1}, z] = p_{t-1} + \frac{\beta_t \Sigma_{t-1}}{\beta_t^2 \Sigma_{t-1} + \Sigma_u} (x_t + u_t)
$$

therefore
\[ \lambda_{t,N} = \frac{\beta_t \Sigma_{t-1}}{\beta_t^2 \Sigma_{t-1} + \Sigma_u} \]

To find the variance when noise traders are realized

\[ Var[v | (x_t + u_t)p_{t-1}x] = \Sigma_{t,N} = \Sigma_{t-1} - \frac{\beta_t^2 \Sigma_{t-1}^2}{\beta_t^2 \Sigma_{t-1} + \Sigma_u} \]

\[ = \frac{\Sigma_u \Sigma_{t-1}}{\beta_t^2 \Sigma_{t-1} + \Sigma_u} \]

\[ = \left[ 1 - \frac{\beta_t^2 \Sigma_{t-1}}{\beta_t^2 \Sigma_{t-1} + \Sigma_u} \right] \Sigma_{t-1} \]

\[ = [1 - \lambda_{t,N} \beta_t] \Sigma_{t-1} \]

For no noise traders

\[ Var[v | x_t p_{t-1}x] = \Sigma_{t,0} = \Sigma_{t-1} - \frac{\beta_t^2 \Sigma_{t-1}^2}{\beta_t^2 \Sigma_{t-1}} \]

\[ = \Sigma_{t-1} - \Sigma_{t-1} = 0 \]

Note that in the no noise case

\[ E[v | x_t p_{t-1}x] = p_t = p_{t-1} + \frac{1}{\beta_t} x_t \]

\[ = p_{t-1} + \frac{1}{\beta_t} (v - p_{t-1}) \]

\[ = v \]

therefore price is fully revealing and \( \Pi_{t,0} = 0 \) as assumed earlier.

We can also verify our assumption about \( h_t \). Market efficiency implies

\[ E[\Delta p_t | (x_t + u_t)p_{t-1}x] = 0 \]

\[ 0 = E[z \lambda_{t,N} (\beta_t (v - p_{t-1} - h_t) + u_t) + (1 - z) \lambda_{t,N} \beta_t (v - p_{t-1} - h_t) + h_t (x_{t-1} + u_{t-1})p_{t-1}x] \]

\[ = z \lambda_{t,N} \beta_t h_t - (1 - z) h_t + h_t \]

since \( z > \frac{1}{2} > 0 \) this implies

\[ (1 - \lambda_{t,N} \beta_t) h_t = 0 \]

recall that

\[ \lambda_{t,N} = \frac{\beta_t \Sigma_{t-1}}{\beta_t^2 \Sigma_{t-1} + \Sigma_u} \]

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therefore
\[
\beta_i \lambda_{t,N} = \frac{\beta_i^2 \Sigma_{t-1}}{\beta_i^2 \Sigma_{t-1} + \Sigma_u} \\
1-\beta_i \lambda_{t,N} = \frac{\Sigma_u}{\beta_i^2 \Sigma_{t-1} + \Sigma_u} \neq 0
\]
which implies that \( h_t = 0 \) as assumed.

All that remains is to derive the expressions for \( \alpha_t \) and \( \delta_t \). Recall
\[
E[\Pi_t | p_{t-1}, v, z] = z((v-p_{t-1} - \lambda_{t,N} x_t) x_t + \alpha_i (v-p_{t-1} - \lambda_{t,N} x_t)^2 + \alpha_i \lambda_{t,N}^2 \Sigma_u + \delta_t) + (1-z)(v-p_{t-1} - \lambda_{t,N} x_t)
\]
also
\[
E[\Pi_t | p_{t-1}, v, z] = \alpha_{t-1} (v-p_{t-1})^2 + \delta_{t-1}
\]
equating these relations, substituting \( x_t = \beta_i (v-p_{t-1}) \) we get for \( \delta_{t-1} \)
\[
\delta_{t-1} = z(\alpha_i \lambda_{t,N}^2 \Sigma_u + \delta_t)
\]
and for \( \alpha_{t-1} \)
\[
\alpha_{t-1} = \beta_i (1-\beta_i (z \lambda_{t,N}^2 + (1-z) \lambda_{t,N}^2)) + z \alpha_i (1-\lambda_{t,N} \beta_i)^2
\]
subject to \( \alpha_t = 0 \) and \( \delta_t = 0 \).

*Lemma 8:* The result follows directly by setting \( z=1 \).

*Lemma 9:* Proof is contained in text.
BIBLIOGRAPHY


