Continuously Rebalanced Investment Strategies

by

Mark Rubinstein

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Abstract

This paper examines the question of how long an investor needs to be prepared to wait before the probability becomes high that an all stock portfolio will outperform an all bond portfolio. Following on an article published in the Financial Analysts Journal which only contained some numerical results, this article supplies the rather elegant mathematical arguments behind them as well as further numerical results including results on continually rebalanced stock-bond portfolios. In addition, it provides a simple proof of the well-known "capital-growth theorem" which says that the probability that the logarithmic utility strategy will outperform any other continuously rebalanced strategy approaches 1 as time approaches infinity. Logarithmic utility strategies are quite important in financial economics because they provide help in deciding how much of a portfolio to allocate between safe and risky assets. The paper also offers new evidence that while the capital-growth theorem is true, to be 95% sure of beating an all cash strategy will require 208 years, and to be 95% sure of beating an all stock strategy will require 4,700 years -- much much longer than one might have guessed from reading the literature on this subject.

An article by Martin Liebowitz and William Krasker, entitled "The Persistence of Risk: Stocks versus Bonds over the Long Term," shows that an investor should be prepared to wait a surprisingly long time before the probability becomes high that a stock portfolio will outperform a bond portfolio.

Under the conditions envisioned in that article, it is possible to derive a simple expression for this probability that investors may find useful. Here is the theorem:

Assume that all available assets collectively follow a stationary random walk in continuous time. Let $X$ and $Y$ be the values after elapsed time $t > 0$ from following two strategies (with equal initial total investment), each being the result of continually rebalancing a portfolio to maintain constant proportions in the available assets. Then:

$$\text{prob}(X > Y) = N\left[\frac{(\mu_x - \mu_y)\sqrt{t}}{\sigma_x \sigma_y + \rho \sigma_x \sigma_y}\right]$$

where $X$ and $Y$ are jointly lognormally distributed with

$$\mu_x \sqrt{t} = E(\ln X), \quad \mu_y \sqrt{t} = E(\ln Y),$$
$$\sigma_x \sqrt{t} = \text{std}(\ln X), \quad \sigma_y \sqrt{t} = \text{std}(\ln Y),$$
$$\rho = \text{correlation}(\ln X, \ln Y),$$
and

$N( )$ is the standard normal distribution function.

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1 Mark Rubinstein is a professor of finance at the University of California at Berkeley. This paper was motivated by an article published in the Financial Analysts Journal by Martin Liebowitz and William Krasker entitled "The Persistence of Risk: Stocks versus Bonds over the Long Term," November-December, 1988.

2 We also need the technical assumption that the return variances of the individual assets and their paired return covariances are finite numbers.
Proof: \( \text{prob}(X > Y) = \text{prob}(\ln X > \ln Y) = \text{prob}(\ln X - \ln Y > 0) \). It is well known that under the conditions stated, a continuously rebalanced portfolio will be lognormally distributed over any finite time interval. Thus \( X \) and \( Y \) are jointly lognormally distributed. Therefore, \( \ln X \) and \( \ln Y \) are jointly normally distributed and their difference is also normally distributed. The probability that the normally distributed random variable \( z = \ln X - \ln Y \) is greater than 0 is \( \text{N}(\mu/\sigma) \), where \( (\mu, \sigma) \) are the mean and standard deviation of \( z \). Therefore, \( \text{prob}(X > Y) = \text{N}(\mu/\sigma) \). The result follows since \( \mu = \mu_x - \mu_y \) and \( \sigma^2 = \sigma_x^2 - 2\rho\sigma_x\sigma_y + \sigma_y^2 \).

Notice that we can also write the result as:

\[
\text{prob}(X > Y) = \text{N}(a\sqrt{t}), \quad \text{where} \quad a = (\mu_x - \mu_y)/[\sigma_x^2 - 2\rho\sigma_x\sigma_y + \sigma_y^2] \tag{4}
\]

As the horizon lengthens, other things equal, \( a \) remains unchanged and this probability increases as a function of \( \sqrt{t} \). Note also that the sign of \( a\sqrt{t} \) is the same as the sign of \( \mu_x - \mu_y \).

Lebowitz and Krasker compare an all stock portfolio \( X \) with an all bond portfolio \( Y \). They assume that \( \mu_x - \mu_y = .025, \sigma_x = .18, \sigma_y = .10 \) and \( \rho = .4. \) In that case, \( a = .148 \) and we can derive the following table:

<table>
<thead>
<tr>
<th>( t ) (years)</th>
<th>( \text{prob}(X &gt; Y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>.681</td>
</tr>
<tr>
<td>20</td>
<td>.747</td>
</tr>
<tr>
<td>30</td>
<td>.792</td>
</tr>
<tr>
<td>40</td>
<td>.826</td>
</tr>
<tr>
<td>50</td>
<td>.853</td>
</tr>
<tr>
<td>123</td>
<td>.950</td>
</tr>
</tbody>
</table>

For example, after 20 years, the probability that the stock portfolio will outperform the bond portfolio is about 75%. Or, phrased another way, the probability that the stock portfolio will underperform the bond portfolio is about 25%. It will take 123 years to reduce this probability of underperformance to less than 5%. This table is little changed if the bond portfolio were riskless \( (\sigma_y = 0) \), since then \( a = .137 \).

In this first application of our theorem, the portfolios were automatically rebalanced between stock and bonds since they were continuously 100% invested in one or the other. Let us now consider a wider class of strategies where portfolios are continuously rebalanced back to a constant proportion possibly intermediate between 0% and 100%.

To take a simple example, suppose we restrict ourselves to portfolios with returns \( r_x \) and \( r_y \) each involving continuous rebalancing between two assets. In particular, suppose that one asset is risky with return \( r_x \) and the other is riskless with return \( r \). Let \( \alpha > 0 \) (\( \beta > 0 \)) be the proportion of our total investment that we invest in the risky asset for strategy X (Y). It can be shown that:\(^4\)

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\(^3\) This logarithmic differential is derived from assumed expected arithmetic returns of .13 and .09 for \( X \) and \( Y \) using the formulas:

\[
\mu_x = (\ln 1.13) - \frac{1}{2} \times .18^2 \quad \text{and} \quad \mu_y = (\ln 1.09) - \frac{1}{2} \times .10^2.
\]

\(^4\) See John Cox and Hayne Leland, "On Dynamic Investment Strategies," paper presented to the American Finance Association, New York, December 1981. In a more general case, where rebalancing takes place between two risky and one riskless asset:

\[
\ln r_x = \alpha_x \ln r_x + \alpha_y \ln r_y + (1-\alpha_x-\alpha_y)\ln r + [\beta \alpha_x (1-\alpha_x)\sigma_x^2 - \alpha_x \alpha_y \rho \sigma_x \sigma_y + \beta \alpha_y (1-\alpha_y)\sigma_y^2]
\]
\[ \ln r_x = \alpha \ln r_1 + (1-\alpha) \ln r + \frac{1}{2} \sigma (1-\alpha) \sigma^2 \]
\[ \ln r_y = \beta \ln r_1 + (1-\beta) \ln r + \frac{1}{2} \beta (1-\beta) \sigma^2 \]

where \( \sigma (= \text{std}(\ln r)) \) is the (logarithmic) volatility of the risky asset and for later reference \( \mu (= E(\ln r)) \) is the expected (logarithmic) return of the risky asset. Taking expectations, we have:

\[ \mu_x = \alpha \mu + (1-\alpha) \ln r + \frac{1}{2} \alpha (1-\alpha) \sigma^2, \quad \sigma_x = \alpha \sigma, \]
\[ \mu_y = \beta \mu + (1-\beta) \ln r + \frac{1}{2} \beta (1-\beta) \sigma^2, \quad \sigma_y = \beta \sigma, \]

and \( \rho = 1 \).

Substituting these expressions into our earlier result for \( \text{prob}(X > Y) \), we can derive:

\[ \text{prob}(X > Y) = N\{ (\sqrt{t}/\sigma)[\mu - (\ln r) + \frac{1}{2} \alpha^2 (1-\alpha-\beta)] \text{sgn}(\alpha-\beta) \} \]

That is, given the market parameters \( (\mu, \sigma, r) \) and the target proportions \( (\alpha, \beta) \) of two portfolio strategies, this formula can be used to show how the probability that strategy \( X \) will outperform strategy \( Y \) depends on the length of time these strategies are pursued.

Thinking of \( Y \) as a benchmark strategy, one case is of particular interest: the benchmark of all cash \( (\beta = 0) \). Ask now, for various target proportions \( \alpha \), how long will it take to have a probability of at least .95 of outperforming the all cash benchmark? Continuing to use the Liebowitz-Krasker estimates \( (\mu - \ln r = .025 \) and \( \sigma = .18) \):

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( t ) (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>80.8</td>
</tr>
<tr>
<td>1</td>
<td>142.1</td>
</tr>
<tr>
<td>1.5</td>
<td>313.0</td>
</tr>
</tbody>
</table>

Suppose we want to choose a strategy which has a probability greater than \( \frac{1}{2} \) of beating any other (continuously rebalanced) strategy we might set against it. The best such strategy can easily be inferred from the expression for \( \text{prob}(X > Y) \). By this criterion, if strategy \( X \) is to beat strategy \( Y \), then the argument of \( N \) must be positive. In turn, the argument will by positive if and only if \( \mu_x > \mu_y \). This implies that the best strategy by our criterion is the one which maximizes \( \mu_x \). This is none other than the strategy which maximizes the expected logarithmic return (logarithmic utility).

To use the logarithmic utility strategy as a benchmark, we would choose \( \beta \) to maximize \( \mu_y \). A little calculus shows this to be:

\[ \beta^* = \frac{1}{2} + (\mu - \ln r)/\sigma^2 \]

Staying with the Liebowitz-Krasker estimates \( (\mu - \ln r = .025 \) and \( \sigma = .18) \), the logarithmic utility investor

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where \( (r_1, r_2) \) are the two risky asset returns, \( (\sigma_1, \sigma_2) \) are the two risky asset volatilities, and \( (\alpha_1, \alpha_2) \) are the target proportions invested in the two risky assets.

---

\(^5\) This solution requires that the expected rate of return of the risky asset be greater than the interest rate.
would choose $\beta^*=1.267$.

With this benchmark, substituting for $\beta^*$ for $\beta$ into the above expression for $\text{prob}(X > Y)$:

$$\text{prob}(X > Y) = \Phi\left(-\frac{1}{\sqrt{\alpha}}\right)$$

Using this we can ask, after five years, what is the probability that an alternative rebalancing strategy (X) will outperform the logarithmic strategy (Y)?

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>prob($X &gt; Y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.399</td>
</tr>
<tr>
<td>.5</td>
<td>.439</td>
</tr>
<tr>
<td>1.0</td>
<td>.479</td>
</tr>
<tr>
<td>1.5</td>
<td>.481</td>
</tr>
<tr>
<td>2.0</td>
<td>.441</td>
</tr>
<tr>
<td>2.5</td>
<td>.402</td>
</tr>
</tbody>
</table>

This analysis suggests that the longer we wait the better the logarithmic strategy will do. Indeed, we have now arrived quite easily at a key well-known result of financial economics:

*as $t \to \infty$, the probability that the logarithmic utility strategy will outperform any other continuously rebalanced strategy goes to 1.*

To see this, if $X$ is the result of maximizing the expected logarithmic return and $Y$ is the result of any other different strategy, then $\alpha > 0$. As $t \to \infty$, then $\alpha \sqrt{t} \to \infty$; so that $\text{prob}(X > Y) = \Phi(\alpha \sqrt{t}) - 1$.

Ask now, how long it will take for the logarithmic utility strategy to have at least a .95 probability of outperforming alternative strategies?

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$t$ (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>208</td>
</tr>
<tr>
<td>.5</td>
<td>569</td>
</tr>
<tr>
<td>1.0</td>
<td>4700</td>
</tr>
<tr>
<td>1.5</td>
<td>6136</td>
</tr>
<tr>
<td>2.0</td>
<td>621</td>
</tr>
<tr>
<td>2.5</td>
<td>220</td>
</tr>
</tbody>
</table>

Although the logarithmic utility strategy will almost surely in the long run outperform any other different strategy set against it, the long run may be long indeed. To be even 95% sure of beating an all cash strategy, an investor must be prepared to wait 208 years (longer than the time from the presidency of George Washington to the present) and to be at least 95% sure of beating an all stock strategy will take 4,700 years (about the time from the unification of Lower and Upper Egypt to the present).

We should not become too enamored of the logarithmic utility strategy. In particular, we might be tempted to think that relative to a fixed return benchmark, the logarithmic strategy has a higher probability of outperforming that benchmark than any other strategy. This would be a serious mistake. So let us ask: after a prespecified horizon, how does the logarithmic utility strategy perform relative to other strategies in beating a low or high fixed return benchmark? To answer this question, standardize our initial investment to $1$ and let $Y$ (now a constant) be the level of the fixed return. Our problem is to calculate:
$$\text{prob}(X > Y) = N\{[(\alpha \mu + (1-\alpha) \ln r + \frac{1}{2} \alpha (1-\alpha) \sigma^2 t - (\ln Y)/[\alpha \sigma^2 t]]\}$$

Here we will need to specify both $\mu$ and $\ln r$ separately to make the calculation. Again following Liebowitz and Krasker, set $\mu = .106$ and $\ln r = .081$. Say our horizon is $t = 5$ years. To get a feel for how we might expect to do, if we used an all cash strategy the value of our portfolio at the horizon would be $1.09^5 = 1.54$. Let us look at fixed targets $Y$ which are about $1/3$ and $3$ times this amount:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>prob$(X &gt; .5)$</th>
<th>prob$(X &gt; 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
<td>.000</td>
</tr>
<tr>
<td>.5</td>
<td>1.000</td>
<td>.000</td>
</tr>
<tr>
<td>1.0</td>
<td>.999</td>
<td>.004</td>
</tr>
<tr>
<td>1.251</td>
<td>.992</td>
<td>.018</td>
</tr>
<tr>
<td>1.5</td>
<td>.979</td>
<td>.037</td>
</tr>
<tr>
<td>2.0</td>
<td>.930</td>
<td>.083</td>
</tr>
<tr>
<td>2.5</td>
<td>.864</td>
<td>.117</td>
</tr>
</tbody>
</table>

Logarithmic utility strategy is highlighted

This illustrates that the criterion of choosing a strategy that is likely to beat any other strategy set against it is not the same as choosing the strategy with the highest probability of beating a particular prespecified benchmark.

For example, suppose our benchmark strategy were a fixed return equal to $.5$ after five years. The all stock strategy ($\alpha = 1$) has a higher probability of outperforming this benchmark than the logarithmic strategy. This illustrates a general proposition about strategies. Say we compare three strategies $X$, $Y$ and $Z$ (think of $X$ as the logarithmic strategy, $Y$ as the all stock strategy, and $Z$ as the fixed return benchmark):

If prob$(X > Y) > \frac{1}{2}$, it does not follow that prob$(X > Z) > \text{prob}(Y > Z)$.

Thus, there is a kind of intransitivity in an investment criterion which relies solely in the probability of outperforming alternative strategies.

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