The Prepayment Uncertainty of Collateralized Mortgage Obligations

by

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The Prepayment Uncertainty of Collateralized Mortgage Obligations

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Abstract

Prepayment risks are today regarded as the most important risks and the least hedgeable risks of mortgages and mortgage-backed securities. It is argued here that the riskiness of a mortgage-related security depends positively on the variance of the duration of that security (and other factors). The CMO tranche structure that arranges investors in "batting order" to receive amortization payments is the most common form of CMO today. It is shown here that the tranches formed through such a CMO may have less duration uncertainty and less investment risk than underlying mortgages collateralizing the CMO. Under some circumstances ALL CMO tranches must have less prepayment risk. Hence the popularity of the CMO could be explained by its effectiveness at reducing prepayment risk.
I. INTRODUCTION

Over the past decade collateralized mortgage obligations (CMO's) have been one of the fastest growing financial innovations in the American mortgage markets. Today there are hundreds of billions of dollars of CMO's outstanding. CMO's, also known as real estate mortgage investment conduits (REMICs), have become one of the most important forms for mortgage-backed securities (MBS). Their increased use has important implications for the housing market, for thrift risk exposure, and for institutional investor opportunities.¹

Research on CMO's, which were first introduced in the early 1980s in America and more recently in a few other countries, has concentrated largely on the pricing issue. As is the case for other mortgage-backed securities pricing is not simple, due to the role of prepayment risks.²

A different question that has received less attention is why CMO's exist in the first place. What financial function, if any, do they play? Is not that function performed by other instruments? This is the focus of this paper.

¹For reviews of the features and history of the CMO market, see Hu (1988), and Roll (1987a, 1987b).
²Recent papers on pricing MBS's include Schwartz and Torous (1989a, 1989b), and Jacob and Toevs (1988).
Most CMO's are in effect stripped mortgages or mortgage-backed securities. Part of the answer to the question is of course that CMO's, like other stripped securities, allow investors to "specialize" in the maturities or components of the underlying security that they desire. In this paper it will be argued that an additional reason and perhaps the main one for the development and rapid growth of CMO's is that the CMO may serve to reduce prepayment uncertainty, which may be the most important risk for investors in mortgages and MBS's. We will show this to be the case by proving that under some conditions each tranche formed in a "batting order" CMO must have less prepayment risk and less uncertainty about its duration than the underlying mortgages from which they were formed.

In the next section we will review the CMO as an instrument and describe the role of prepayment risk in MBS markets. Following that we will develop a formal model of the CMO and prepayment risk, and then relate prepayment risk to total financial risk.

II. PREPAYMENT RISK AND MBS's

It is generally agreed that the greatest risk in fixed-rate mortgage (FRM) and in most mortgage-backed security investment is the risk of early repayment of mortgage loans, i.e., the "calling" of the mortgage by the borrower. This prepayment often (but not always) accompanies refinancing of the housing purchased by the borrower. The incidence or probability of prepayment tends to rise whenever market interest rates decrease, although the exact
relation is quite difficult to model and predict.

Prepayment is a difficult phenomenon to analyze (and to hedge) because the "call provision" in mortgages is not exercised in the same way as in ordinary callable bonds (see Quigley and Van Order (1990)). First, it is often exercised for reasons other than to realize profit when the call option is "in the money". Borrowers often prepay when they sell their house for any reason, even when the option is out of the money. Second, borrowers do not seem to follow the option exercise strategies first suggested by Brennan and Schwartz (1977, 1980), and in particular do not exercise immediately when the option moves into the money. This may be due to high transaction costs for refinancing, to more complex strategic considerations that dictate the timing of the exercise of the call, or to other reasons. Even the forecasting of prepayment behavior has proved difficult; prepayment shows sharp nonstationarity over time.

The inability to hedge the prepayment risks from FRM's (and to a lesser extent from other mortgages) has been a serious problem for thrifts and other mortgage lenders. For virtually every other risk associated with mortgage lending and investing, some kind of solution or neutralizing methodology exists. For example, default risks on mortgages may be covered through mortgage insurance of one form or another and of course through portfolio diversification. Title insurance, where used, may resolve risks associated with title. "Ordinary" interest risk, that is the risk of capital gains
and losses on mortgages arising from interest fluctuations abstracting from prepayment risks, may be hedged in ways similar to the hedging of interest risks on any fixed-income portfolio, e.g., through appropriate duration hedging, futures coverage, etc.

Indeed the reason thrifts and other investors have been so concerned with prepayment risks is probably because they are capable of hedging these other risks, and are left with prepayment risks as their primary residual unhedgeable exposure. Prepayment risk can only be "diversified away" if the weight of all fixed-rate mortgages and MBS's in the portfolio approaches zero.

One innovation that partly resolves the prepayment problem has been the adjustable-rate mortgage (ARM). ARM's reduce the prepayment problem because their interest rates are periodically adjusted to market levels during the life of the mortgage contract. In theory the borrower need not refinance to benefit from a fall in interest rates; his interest payments are automatically adjusted according to the pricing formula used.

ARM's of course do not by themselves represent a method through which the prepayment risks of fixed-rate mortgages (FRM's) held in a portfolio may be hedged. Moreover, it is now recognized that ARM's have their own prepayment risks (see Cunningham and Capone 1990), due to the fact that some ARM borrowers refinance with FRM's when market rates fall (locking in the lower rates), and
hence ARM's frequently include prepayment penalties. ARM's increasingly carry conversion features to allow such alterations without bearing recomtracting costs.

While prepayment risks are difficult to hedge, they may nevertheless be reduced through financial innovation. Specifically it will be shown that CMO's may reduce those risks.

What exactly is a CMO? It is essentially a set of claims issued against a pool of underlying mortgages (or MBS's) sold to different groups of investors. Unlike ordinary pass-throughs or mortgage participation certificates, the claims are not identical. Different investor classes, known as tranches, are created with different features and risks. The aggregate of the tranches is similar to, and is "collateralized" by, the underlying mortgage pool.\(^3\)

Originally introduced by investment banks in 1982, CMO's have grown into a large and highly heterogeneous market. CMO initiators today include Wall Street firms, commercial banks, thrifts and the federal housing agencies.

The most common type of CMO, and the type we will be concerned

\(^3\)The pricing of CMO tranches was analyzed recently in McConnell and Singh (1990).
with here, might be referred to as a "batting order" CMO based on underlying FRM's. Here investors are divided into tranches according to their turn "at bat" or for receiving payments of principal amortization. For example a pool of 30-year FRM's could be converted into three tranches, where the first is to receive all of the payments for the first ten years of amortization. The second tranche could receive the amortization payments from the next ten years, and the third tranche thereafter. Typically (except for a zero-coupon tranche) each class would receive some interest payments as well until the time that it begins its turn at the amortization "bat". The actual timing of amortization is uncertain due to prepayment risk. When prepayment occurs, all principal received early is paid to the investors whose turn it is "at bat". Hence the duration of each tranche is uncertain.

While probably the most common form of CMO, this is not the only form. CMO's involving underlying floating-rate mortgages or ARM's have become common, often constructed together with swaps.\textsuperscript{4} CMO's where the interest and principal components of mortgage payments are each assigned to different tranches are also now common.\textsuperscript{5} Some "batting order" CMO's include a special tranche

\textsuperscript{4}Floating-rate CMO's are analyzed in Gordon (1987) and Mann and Fedak (1989).

\textsuperscript{5}Interest only/principal only CMO's are analyzed by Carlson and Sears (1987) and Asay and Sears (1989).
isolated from all prepayments known as a planned-amortization class (PAC).\textsuperscript{6}

It is difficult to measure, analyze and hedge the financial risks of mortgages, MBS's and CMO's using traditional duration techniques. Because of the role of prepayment, the duration of these securities is not knowable in advance. The duration is stochastic, depending on future prepayment behavior. Prepayment in turn is related to interest rates, but the relationship is noisy and seems to be unstable over time.

Trying to hedge an asset portfolio with uncertain duration is somewhat similar to the classical problem of trying to hedge (using futures) a commodity position with both unknown quantity and price of the commodity. Hedging can never be complete and any exogenous reduction in the variances of the underlying variables must benefit the trader. A holder of a mortgage portfolio could try to hedge his financial risk on the basis of the mean duration of these assets, but would still retain considerable residual risk.

Interest risks may be fully hedged only when duration is determinate. Formally it can be shown that the elasticity of the value of any asset (or group of assets) with respect to \((1+r)\) is precisely the duration, where \(r\) is the market interest rate. For

\textsuperscript{6}CMO's continuing PAC tranche are analyzed by Hancock (1988).
determinate duration the variance of the value of the asset (or set of assets) is approximately $D^2 V(r)$, where $V(r)$ is the variance of $r$ and $D$ is the duration.\textsuperscript{7} The portfolio may be structured to calibrate the effects of changes in $r$ such that equity remains insulated. This is done through judicious duration matching.

If the duration is itself uncertain, then the uncertainty of the value of any asset or set of assets (A) is a function of both interest rate uncertainty and duration uncertainty. To see this, suppose that $\ln A = C - D \ln (1+r)$. The expected value of $\ln A$ would be $E(\ln A) = C - E(D)E[\ln(1+r)] - \text{cov}[D, \ln(1+r)]$. What is the uncertainty of $A$? If $D$ were independent of $A$ then the variance of $\ln A$ would be

$$V(\ln A) = E(D^2)\ln(1+r) + E[\ln(1+r)]^2 V(D) \approx (ED)^2 V(r) + (E r)^2 V(D) + V(D)V(r).$$

The uncertainty of $A$ would depend on the uncertainties of both interest rates and duration. In the more general case where duration and interest rates are correlated the above expression for $V(\ln A)$ would have additional covariance terms.\textsuperscript{8} If $D$ and $r$ were positively correlated, as would be the case if interest decreases caused prepayment, these covariance terms would have net negative

\textsuperscript{7}It is exactly $D^2 V(\ln(1+r))$ if $D$ does not change over values of $r$, i.e., if "convexity" is zero.

\textsuperscript{8}The additional terms would be $\text{cov}(D^2, r^2) - \text{cov}^2(D, r) + 2E(D)E(r)\text{cov}(D, r)$. 

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value, reducing \( V(\ln A) \) below the level for independent \( D \) and \( r \).

Total investment risk would depend on the variance of duration and the correlation of duration with interest rates.

Financial risk in mortgage and MBS investing can be thought to be a complex function of two risk components: "ordinary" interest risk and prepayment risk. The former is the risk that capital gains or losses will occur due to interest rate fluctuations abstracting from prepayment, holding the maturity of all payments constant. This risk would be hedgeable using standard duration hedging and other techniques. The second component is prepayment risk or duration uncertainty caused by changes in the timing of payments.

Any interest rate change affects the value of an investment (A) according to

\[
d\ln A = \left[ -D + \frac{\partial \ln A}{\partial \ln D} \frac{\partial \ln D}{\partial \ln (1+r)} \right] d\ln (1+r) \approx \left[ -D + \frac{\partial \ln A}{\partial \ln D} \frac{\partial \ln D}{\partial r} \right] dr.
\]

The term itself may be stochastic for any \( dr \). The two terms inside the brackets could be thought to represent the two components of investment risk: "ordinary" and prepayment risk.

If "ordinary" interest risk were completely hedged, an investor would be left with prepayment risk. As seen above
prepayment risk can be thought of as a function of duration uncertainty and interest-duration correlation. The strategy here will be to first address the duration uncertainty factor independently of the interest rate (ignoring the interest-duration correlation) and then return to address the role of all the risk components together.

As noted, the variance in the duration of mortgages and MBS's stems from prepayment. Duration and duration uncertainty for mortgages have been analyzed in the literature empirically and through simulation.\(^9\)

What is the relationship between the prepayment risk (measured by duration uncertainty) of CMO tranches and that of the underlying mortgages or MBS's? In the next section it will be shown that under certain assumptions prepayment risk must be lower for all CMO tranches than for the underlying mortgages. This may appear at first to be a paradoxical assertion. After all, CMO tranches are merely strips from underlying mortgages. The sum of the tranches is equivalent to the underlying mortgages. How can the duration uncertainty for all tranches be less than that of the underlying mortgage? Should not the "whole" be equal to the sum of the parts?

\(^9\)The duration of mortgage-backed securities was analyzed using simulations in Davidson (1987), Jacob, Lord and Tilley (1987), and Asay, Guillaume and Mattu (1987).
The answer is negative. By analogy, coupon and principal strips from Treasury notes and bonds have zero duration uncertainty, yet the Treasury security's duration itself will change when interest rates change and so is uncertain. For Treasuries the timing of individual coupon and principal payments does not change (except for callable bonds), but for mortgages the timing may change as well due to prepayment. But as is true for Treasuries, mortgage strips (i.e., CMO's) may have less duration risk than the underlying mortgages from which they are formed. Before proving this formally a simple example will be presented to illustrate that this is possible, using an extremely simplified mortgage and CMO structure.

The Example

All mortgages in the market are identical. The representative mortgage has principal of $2000 and is initially issued with maturity of 30 periods. Some prepayment however always occurs. Repayment of the mortgage is made continuously from time 0 through termination or full amortization. Mortgage repayments will be recorded in terms of their discounted or present values, rather than current nominal values, in order to make duration computation simple. That is, mortgage payment in the amount of $1 at time $t$ will be recorded here at its discounted value or $\dfrac{1}{(1+r)^t}$. $r$ is the contractual interest rate and never changes. When all payments are measured by their discounted (to time zero) or present values, duration is the simple average of the maturities or timing of those
Any mortgage repayment schedule is feasible as long as the amortization components \( A_t \) (for time \( t, 0 \leq t \leq 30 \)) satisfy

\[
\int_0^{30} A_t \, dt = 2000.
\]

It will be assumed here and throughout that the timing of mortgage interest payments to lenders/investors is such that each $1 of amortization is coupled with \( e^{rt} - 1 \) dollars of interest at the time of payment. Amortization of \( A_t \) dollars at time \( t \) means that the borrower actually writes a check for \( A_t e^{rt} \) dollars, whose present (discounted) value is precisely \( A_t \). The same assumption regarding the timing of interest payments will be applied to CMO tranches below. The result of the assumption is that we can ignore interest while calculating duration, which is then just a simple average of the \( A_t \)'s. The duration of the mortgage would be simply

\[
\frac{1}{2000} \int_0^{30} tA_t \, dt.
\]

There are two states of nature. In State I, whose probability is 1/2, amortization is constant for the first 20 periods of the mortgage, and full amortization is complete at 20 periods. If \( M \)

\[\frac{dM}{dt}\]

is the outstanding mortgage principal, \( A_t = -\frac{1}{dt} \) is the rate of continuous amortization and is a constant 100. (See Figure 1.)
Figure 1. Amortization for CMO Illustration
Initial principal = $2000.

State I (50% probability)

State II (50% probability)
In State II amortization follows a linear declining continuous schedule. The instantaneous amortization is once again \( A_t = \frac{dM}{dt} \), but now it is initially 200 and declines linearly until 20 periods have passed. (See Figure 1.) \[ -\frac{dA_t}{dt} = \frac{d^2M}{dt^2} = 10. \]

Once again full amortization occurs by time 20.

It can be easily shown that the duration of the mortgage is 10.00 in State I and 5.86 in State II, with an expected duration of 7.93 and a standard deviation of duration of 2.07.

Next let us form a three-tranche CMO using a pool of these underlying mortgages. The CMO creates an A-tranche for the first $200 of principal, a B-tranche for the next $1500, and a C-tranche for the last $300. Assume each has the same interest yield as the underlying mortgage and, as before, \((e^{zt}-1)\) dollars of interest are paid to a tranche-holder together with each $1 of principal paid at time \(t\). The duration of each CMO tranche is calculated as

\[
\frac{1}{X} \int_{t_1}^{t_2} t A_t \, dt \text{ and } X = \int_{t_1}^{t_2} A_t \, dt,
\]

where \(t_1\) and \(t_2\) are the timing for the commencement and termination (respectively) of amortization for the specific tranche.

The durations and their uncertainty for the three tranches are as follows:
Exhibit 1. Example of CMO Reducing Prepayment Risk

1) Mortgage Principal = $2000 for 30 periods

2) State of Nature I

   Probability of Occurrence = 50%

   Amortization Schedule: $100 per period, paid
   \[ \frac{dM}{dt} = 100 \]

   continuously, for 20 periods or

3) State of Nature II

   Probability of Occurrence = 50%

   Amortization schedule is diminishing function of \( t \).

   \[ \frac{dM}{dt} = 200 - 10t \]

   In both cases full amortization by period 20. All mortgage
   payments expressed in terms of discounted (to time zero)
   values.

4) Underlying Mortgage has duration of 10.00 in State I, 5.86 in
   State II, Mean Duration 7.93 and standard deviation of
   duration 2.07.

5) A CMO is created with three tranches.

<table>
<thead>
<tr>
<th>Tranches</th>
<th>Amortization Commences</th>
<th>Duration of Tranche</th>
<th>Mean Duration</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>State I</td>
<td>State II</td>
<td>State I</td>
<td>State II</td>
</tr>
<tr>
<td>A - First $200</td>
<td>0</td>
<td>0</td>
<td>1.00</td>
<td>0.50</td>
</tr>
<tr>
<td>B - Next $1500</td>
<td>2</td>
<td>1.02</td>
<td>9.50</td>
<td>5.51</td>
</tr>
<tr>
<td>C - Next $300</td>
<td>17</td>
<td>12.25</td>
<td>18.50</td>
<td>14.52</td>
</tr>
</tbody>
</table>
As is seen each and every tranche in the CMO has a standard deviation of duration smaller than that of the underlying mortgages from which they were derived. The whole is not the same as the sum of the parts, at least in terms of prepayment risk. But how can this be?

In effect by specializing in parts of the cash flow of an underlying mortgage, CMO tranche investors are defining new securities whose cash flows are more predictable than the mortgage itself. The intuition is somewhat the same as a situation where one wished to predict the height of a pedestrian chosen at random. If a sample of ten pedestrians is chosen, it is possible that the uncertainty regarding the height of the shortest person in the sample and that of the tallest are both less than the uncertainty regarding heights in the general population. Similarly the uncertainty regarding the duration of the first and last tranches (and other tranches) in a CMO may be less than the uncertainty regarding the duration of mortgages themselves.

In the next section it will be proved that the prepayment risks (as measured by duration variance) of each of the CMO tranches for a "batting order" CMO based on underlying FRM's must be less than for the underlying mortgages for certain conditions.

III. MODEL

Let us begin with a standard fixed-rate mortgage that is
initially not callable. We will image that the payments under the mortgage are made continuously at a fixed rate throughout the life of the mortgage and with a fixed mortgage interest rate. The mortgage must satisfy

\[
M = \int_0^T R e^{-rt} dt = \frac{R}{r} \left[ 1 - e^{-rt} \right],
\]

where \( M \) is the mortgage principal, \( T \) is the (promised) termination time of the contract, \( R \) is the fixed repayment per standard unit of time, and \( r \) is the fixed (continuous) contractual mortgage rate.10

The duration of a non-callable FRM would be

\[
D^* = \frac{1}{M} \int_0^T R e^{-rt} dt,
\]

and this can be shown to equal

\[
D^* = \frac{1}{r} \left[ 1 - \frac{rT}{e^{rt}-1} \right].
\]

Let us now allow prepayment to take place in any period \( t \) at some stochastic non-negative proportion \( P_t \) of \( R \). It is assumed that \( P_i \) is independent of \( P_j \) for all \( i \neq j \) and \( \frac{dP_t}{dt} \) is defined and continuous. Then the FRM must satisfy

10As we are concentrating on the duration uncertainty component of total investment risk, we will ignore for now the impact of current interest rate fluctuations on market value.
\( M = \int_0^\tau R(1+P_t)e^{-rt}dt \)

for all states of nature, that is, for all values of \( P_t \) and for all \( t \). Here \( \tau \) is the actual (stochastic) termination date given prepayment, and \( \tau \leq T \). For any given \( T \), \( M \) and \( R \), a greater \( P_t \) (for any \( t<\tau \)) must lower \( \tau \). Formally

\[
\left. \frac{d\tau}{dP_t} \right|_M = -\frac{e^{r(\tau-t)}}{1+P_\tau} \leq 0,
\]

for all \( t<\tau \) and for all the states of nature.

The duration of this callable FRM is a random variable. For any set of \( P_t \) values, the actual duration would be

\[
D_0 = \frac{\int_0^\tau tR(1+P_t)e^{-rt}dt}{M}.
\]

This of course must be less than or equal to \( D^* \) in (2).

Let us now convert the mortgage into a two-tranche CMO of the following form. There are no fees nor operating expenses for the CMO. The investor in the first tranche receives all payments stemming from the underlying mortgage until he has been paid the equivalent (in terms of present value) of \( BM \). The second tranche receives all payments thereafter. The second tranche receives no interest payments during the first period, during which amortization payments are made only to the first tranche. The yield on all tranches is \( r \). Formally,
(6) \[ BM = \int_0^k R(1+P_t)e^{-rt} dt, \]

and

(6') \[ (1-\beta)M = \int_k^T R(1+P_t)e^{-rt} dt. \]

Here \( k \) is the cutoff date when tranche-one principal payments cease, and tranche-two payments begin, and \( \frac{dk}{d\beta} \geq 0. \)

The sum of the present values of payments from each of the two tranches equals the principal of the original mortgage. The duration of the first tranche is

(7) \[ D_1 = \frac{\int_0^k tR(1+P_t)e^{-rt} dt}{BM}, \]

and that of the second is

\[ D_2 = \frac{\int_k^T tR(1+P_t)e^{-rt} dt}{(1-\beta)M}. \]

Both of these durations are stochastic. Note that the duration of the underlying mortgage in each state of nature is still \( D_0 \) and satisfies

(8) \[ D_0 = \beta D_1 + (1-\beta) D_2. \]

It follows that

(8') \[ E(D_0) = \beta E(D_1) + (1-\beta) E(D_2), \]

where \( E \) is the expected value operator. The variance of \( D_0 \) is

(9) \[ V(D_0) = \beta^2 V(D_1) + (1-\beta)^2 V(D_2) + 2\beta(1-\beta) \text{ cov}(D_1, D_2), \]
where \( V(\cdot) \) indicates variance and \( \text{cov}(\cdot) \) is covariance. The cutoff time \( k \), when the first tranche terminates and payments to the second tranche commence, is stochastic, with \( \frac{\partial k}{\partial P_t} \leq 0 \) for \( 0 \leq t \leq k \). In addition it can be shown that

\[
\frac{dD_1}{dP_t} = \frac{\partial D_1}{\partial k} \frac{\partial k}{\partial P_t} \leq 0 \text{ for } 0 \leq t \leq k, \text{ and }
\]

\[
\frac{dD_2}{dP_t} = \frac{\partial D_2}{\partial \tau} \frac{\partial \tau}{\partial P_t} - \frac{\partial D_2}{\partial k} \frac{\partial k}{\partial P_t} \leq 0 \text{ for } 0 \leq t \leq T.
\]

This brings us to our central propositions.

**Proposition I:** The variance of the duration of the first tranche of the CMO must be always less than or equal to that of the underlying mortgage. Formally \( V(D_1) \leq V(D_0) \).

**Proof:** First let us note that when \( B=0 \) the first tranche vanishes, as do both its duration and the variance of its duration. When \( B=1 \) the first tranche is identical with the underlying mortgage, and so \( D_1 = D_0 \), and \( V(D_1) = V(D_0) \). These end points are shown in Figure 2.

In order to prove the proposition it is sufficient to prove that the \( V(D_1) \) curve connecting these two end points is everywhere non-decreasing. Formally, we wish to show

\[
\frac{\partial V(D_1)}{\partial B} \geq 0 \text{ for all } 0 \leq B \leq 1.
\]
Now since \( V(D_1) = E(D_1^2) - E^2(D_1) \), then

\[
(11) \quad \frac{\partial V(D_1)}{\partial B} = \frac{\partial V(D_1)}{\partial k} \frac{\partial k}{\partial B} = 2E[D_1 \frac{\partial D_1}{\partial k} \frac{\partial k}{\partial B}] - 2(ED_1)E[\frac{\partial D_1}{\partial k} \frac{\partial k}{\partial B}]
\]

\[
= 2 \text{cov} \left[D_1, \frac{\partial D_1}{\partial k} \frac{\partial k}{\partial B} \right] = \frac{2}{B} \text{cov}(D_1,k).
\]

If this covariance were always non-negative the proof would be complete. Now note that the only source of randomness in the model is prepayment behavior \( P_t \). For \( t > k \), \( P_t \) has no effect on either \( D_1 \) or \( k \). For \( 0 \leq t \leq k \), any increase (decrease) in any \( P_t \) must lower (raise) \( D_1 \) since

\[
(12) \quad \frac{dD_1}{dP_t} = \frac{\partial D_1}{\partial P_t} + \frac{\partial D_1}{\partial k} \frac{\partial k}{\partial P_t} = \frac{Re^{-rt}}{BM} (t-k) \leq 0.
\]

This must be non-positive since \( k \geq t \).

The effect of \( P_t \) on \( k \) is, from (6), equal to

\[
(13) \quad \left. \frac{dk}{dP_t} \right|_{BM} = \frac{-e^{-rt}}{(1+P_k)e^{-rk}} \leq 0
\]

\( \frac{dk}{dP_t} \)

Hence must also be negative. \( D_1 \) and \( k \) must always move in the same direction when \( P_t \) changes. Therefore the covariance of \( D_1 \) with \( k \) must be non-negative.\(^{11}\) Q.E.D.

\(^{11}\)Actually for all \( t \) not equal to \( 0 \) or \( k \), the covariance is strictly positive, and so for all \( \beta \) not equal to \( 0 \) or \( 1 \), \( V(D_1) \) is strictly less than \( V(D_0) \).
Figure 2.

Figure 3.
Proposition II: The variance of the duration of the second tranche is less than or equal to the variance of the duration of the underlying mortgages. Formally, \( V(D_2) \leq V(D_0) \).

**Proof:** First note that when \( \beta = 0 \) the second tranche is identical to the underlying mortgage, and so \( V(D_2) = V(D_0) \). When \( \beta = 1 \), the second tranche vanishes as does its duration and the variance \( V(D_2) \). These end points are shown in Figure 3.

In order to prove the proposition it is sufficient to prove that the \( V(D_2) \) curve connecting these end-points is everywhere non-increasing. Formally, we wish to show

\[
\frac{\partial V(D_2)}{\partial \beta} \leq 0 \text{ for all } 0 \leq \beta \leq 1.
\]

Noting that \( \beta \) does not affect \( \tau \) (nor \( D_0 \)), and that

\[
V(D_2) = E(D_2^2) - E^2(D_2),
\]

we derive

\[
(14) \quad \frac{\partial V(D_2)}{\partial \beta} = \frac{\partial V(D_2)}{\partial k} \frac{\partial k}{\partial \beta} = 2E[D_2 \frac{\partial (D_2)}{\partial k} \frac{\partial k}{\partial \beta}] - 2(ED_2)E\left[\frac{\partial (D_2)}{\partial k} \frac{\partial k}{\partial \beta}\right]
\]

\[
= 2 \text{cov}\left[D_2, \frac{\partial (D_2)}{\partial k} \frac{\partial k}{\partial \beta}\right] = -\frac{2}{1-\beta} \text{cov}(D_2, k).
\]

If this covariance were always non-negative the proof would be complete. Again note that the only source of randomness in the model is \( P_t \). In evaluating this covariance we need only consider noise deriving from \( P_t \) with \( 0 \leq t \leq \tau \). This is because for all \( k \leq t \leq \tau \),
$P_t$ would not effect $k$ and so the covariance conditional on $k \leq t \leq \tau$ would be zero.

Now for $0 \leq t \leq k$, the impact of $P_t$ on $D_2$ is

$$\begin{align*}
\frac{dD_2}{dP_t} &= \frac{\partial D_2}{\partial t} \frac{\partial t}{\partial P_t} - \frac{\partial D_2}{\partial k} \frac{\partial k}{\partial P_t} = -\frac{(\tau-k)e^{-\nu t}}{(1-B)M} \leq 0,
\end{align*}$$

since $t \leq k$.

The impact of $P_t$ on $k$ (for the same range) was shown above to be negative. $D_2$ and $k$ must move in the same direction when $P_t$ changes. Therefore, the covariance of $D_2$ and $k$ must be non-negative.\footnote{As in the previous proposition, for $B$ not equal to 0 or 1, the covariance is in fact strictly positive, and $V(D_2)$ is strictly less than $V(D_0)$.} Q.E.D.

**Corollary:** For any "batting order" CMO with more than two tranches, the variance of the duration of every tranche must be less than that of the underlying mortgage.

**Proof:** If there are more than two tranches, they can always be grouped so that they reduce to two aggregate "tranches", one original tranche preceded by a second consisting of all others. Both must have duration variance less than the underlying mortgage by the above propositions. Leaving off the former effectively defines a new mortgage composed of the "all other" tranche. The "all other" tranche may then be further split into two sub-tranches, each of which must have even smaller duration variance, and so on.
IV. The Reinvestment Risk of Mortgages and CMO Tranches

Consider a mortgage with principal $M$, stochastic duration $D$, and contractual interest rate $r$. Suppose that mortgage investors/lenders invest on the basis of a well-defined investment horizon $H$, which must always be greater than the termination of any mortgage purchased.\textsuperscript{13} The investor faces reinvestment risk. Any dollar received from the mortgage at any time $t < H$ must be reinvested at prevailing interest rates from $t$ until $H$.

The value of the mortgage as of $H$ when all payments reinvested are received may be shown to equal

$$X = M e^{rD} + (H-D)i,$$

where $i$ is the properly weighted reinvestment rate based on prevailing market rates (maturing at $H$) when mortgage payments are received. Finally suppose that the investor has a simple investment utility function that depends only on the expected value and variance of the log of $X$.\textsuperscript{14}

The expected value of the log of $X$ is equal to

$$\ln(M) + rE(D) + HE(i) - E(i)E(D) - \text{cov}(i,D).$$

The variance of the log of $X$ is equal to

$$r^2V(D) + H^2V(i) + V(Di).$$

\textsuperscript{13}Actually it is not necessary for this to be so. Any horizon will do, but if it comes before mortgage termination it makes it difficult to interpret $V(i)$ below as reinvestment risk.

\textsuperscript{14}For simplicity no other assets or liabilities are held.
As was discussed above in section II the $V(D_i)$ term is itself a positive function of $V(D)$, $V(i)$, and a negative function of $\text{cov}(D,i)$. So investment risk with $V(D)$ and $V(i)$ and falls with $\text{cov}(D,i)$.

In the previous section it was shown that CMO tranches would have lower duration uncertainty than underlying mortgages under the assumptions of the model. If the probability distribution of the reinvestment rate (average yield to maturity $H$ from time of mortgage payment) is stationary, which is probably not too far from reality, the difference in the investment risk between CMO tranches and underlying mortgages depends only on their values for $V(D)$ and $\text{cov}(D,i)$. We have shown that $V(D)$ is lower for CMO tranches than for underlying mortgages. What about $\text{cov}(D,i)$?

The sign of $\text{cov}(D,i)$ should be non-negative both for CMO tranches and for whole mortgages, since lower interest rates increase prepayment, lowering $D$. As shown in equation (8) above the duration of the underlying mortgage is itself a weighted average of the duration of the tranches. Now if the value of $\text{cov}(D,i)$ for the underlying mortgage is smaller than or equal to that of the CMO tranches, the total financial risk as defined in (16) must be higher for the whole mortgage than the tranches.\(^{15}\) The $\text{cov}(D,i)$ in turn will be smaller for the whole mortgages as

\[^{15}\text{Continuing to assume that } V(i) \text{ is roughly the same for all securities.}\]
long as duration and reinvestment interest are equally or less correlated for entire mortgages than for the CMO tranches. The correlation between duration and reinvestment will depend on the specific prepayment functions. However one would expect it to be lower for whole mortgages than for CMO tranches because the whole mortgages are averages of tranches and the relationship between D and i should be "noisier", especially if the prepayment function itself fluctuates over time (or the sensitivity of prepayment to reinvestment rates fluctuates). If so, it would follow that the investment risks of mortgages, including both "ordinary" and prepayment risks, must exceed those of CMO tranches.

V. Conclusion

In this paper the workings of a "batting order" CMO were described, where different investment tranches receive their principal payments according to their turn "at bat". This structure allows investors to "specialize" in the maturity ranges that they desire.

But more importantly this CMO structure creates new investment instruments, each of which has lower prepayment uncertainty than the underlying mortgages. This was proved formally using duration variance as the measure of prepayment uncertainty.

\[ \text{cov}(D,i) = \text{cor}(D,i) \sqrt{\text{V}(D)\text{V}(i)}, \text{ where } \text{cor} \text{ is correlation and } \text{V}(D) \text{ must be lower for the tranches.} \]
Since prepayment risk is difficult to hedge and is often regarded as the most important factor in mortgage and MBS investment, its reduction through CMO's represents a major market improvement and welfare-increasing innovation for investors.
References


Gordon, Frank D., "Floating-Rate Collateralized Mortgage Obligations", Housing Finance Review 6(2), Summer 1987, 155-165.


