Optimal Consumption-Portfolio Policies: A Convergence from Discrete to Continuous Time Models

by

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OPTIMAL CONSUMPTION-PORTFOLIO POLICIES: A CONVERGENCE FROM DISCRETE TO CONTINUOUS TIME MODELS

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Abstract

We establish a convergence of optimal intertemporal consumption and portfolio policies from a sequence of discrete time multivariate and multinomial models to a continuous time multidimensional diffusion model. *Journal of Economic Literature* Classification Numbers: 022, 521, 522.
1 Introduction

Individuals' optimal intertemporal consumption and portfolio policies have traditionally been analyzed using both discrete and continuous time models, see, for example, Mossin [14], Samuelson [15], or Hakansson [7] for a discrete time approach and Merton [13] for a continuous time approach. As a result, the connection between discrete and continuous time models has become an important issue in finance. While economists commonly maintain that discrete and continuous time models of intertemporal consumption and portfolio policies approximate each other, no one has actually established their convergence. This would require finding, for a given continuous time economy, a sequence of discrete time economies in which securities prices as well as optimal consumption and portfolio policies converge to their corresponding continuous time limits.

Merton [12] argues that if security returns are lognormally distributed but observed only discretely in time, then an investor with an isoeelastic utility function will never choose to borrow or short sell. However, the same investor may well choose to borrow or short sell in the continuous time analysis. Thus, even though security prices in the discrete time analysis converge pathwise to lognormal processes, the optimal consumption and portfolio policies derived from the discrete time analysis might not converge to their corresponding continuous time limits. Merton's example at least shows that finding a correct sequence of discrete time price processes for convergence of individuals' optimal policies might not be simple.

In this paper we establish a weak convergence of individuals' optimal intertemporal consumption and portfolio policies from discrete time multivariate multinomial models to a general continuous time multi-dimensional diffusion model. The continuous time diffusion model consists of \( N \) risky stocks and one riskless bond, where the stocks and the bond form a dynamically complete securities market in the sense of Harrison and Kreps [8]. We approximate the \( N \) stock prices and the bond price by a sequence of \( N+1 \)-variate, \( N+1 \)-nomial Markov processes so that the stocks and the bond in the discrete time models also form a dynamically complete securities market. The securities price processes in discrete time models converge weakly to those of the continuous time models. Under the standard assumption of a time-additive utility function, we show that an individual's optimal consumption policies, optimal portfolio policies, and maximum expected utilities derived from the discrete time models converge weakly to their corresponding continuous time limits.

There are at least two reasons for establishing a convergence theory as such. The first one is that it may be easier to derive analytic or numerical results in discrete time than in continuous time (or visa versa). As in the binomial option pricing model (Cox, Ross and Rubinstein [4]), the discrete time analysis may provide an elegant numerical procedure for the computation of
the optimal consumption and portfolio policies which may otherwise be calculated numerically by solving a partial differential equation (PDE) of often more than two state variables. The second reason has to do with the behavioral predictions of the two models. Since both models are at best approximations to reality, if we perturb one model slightly in some reasonable way and the behavioral predictions generated by the model change drastically, we may suspect that the model can't tell us much about the real world behavior. Our convergence theory demonstrates that for a fairly broad class of continuous time models, there exists a class of discrete time models whose behavioral predictions approximate those of the continuous time counterparts.

The rest of the paper is organized as follows. In Section 2 we formulate a continuous time economy. Section 3 gives an example which illustrates some numerical advantages of the discrete time model over the continuous time model. In Section 4 we approximate the continuous time economy by a sequence of discrete time economies, using the approximation procedure developed in He [9], which establishes a convergence of contingent claims prices under this approximation. We present our main convergence results in Section 5, and Section 6 contains some concluding remarks and suggestions for future research.

2 The Continuous Time Economy

Consider a continuous time economy in which there are $N$ risky stocks and one locally riskless bond. The $N$-dimensional vector of stock prices, $S$, and the bond price, $B$, are described by the stochastic differential equations:

$$dS_t = b(S_t) \, dt + \sigma(S_t) \, dw_t, \quad (1)$$

$$dB_t = B_t \, r(S_t) \, dt, \quad B_0 = 1, \quad (2)$$

where $w$ is an $N$-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The time horizon for this economy is $[0, 1]$. We assume that $b : \mathbb{R}^N \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \to \mathbb{R}^{N \times N}$, and $r : \mathbb{R}^N \to \mathbb{R}$ are continuous and satisfy the following regularity conditions: $\exists L, K > 0$ such that for all $x, y \in \mathbb{R}^N$,

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|,$$

$$|x^2 r(x)| \leq K(1 + |x|^2).$$

We further assume that $\sigma(S)$ is non-singular (almost surely) and that $r$ is non-negative. Next, define $\kappa(x) \equiv -\sigma(x)^{-1}(b(x) - r(x)x)$. We assume throughout that $\kappa$ is continuous in $x$ and uniformly bounded. This implies that there exists a unique equivalent martingale measure or
a risk neutral probability measure for the price system defined by (1) and (2), i.e., the stock prices discounted by the bond price become martingales under this measure. In fact, if \( \xi_t \equiv \exp \left( \int_0^t \kappa(s).dw_s - \frac{1}{2} \int_0^t |\kappa(s)|^2 ds \right) \), then the martingale measure, denoted by \( Q \), has the form
\[
Q(A) \equiv \int_A \xi_t(\omega) \, dP(\omega).
\]
The existence of a unique equivalent martingale measure implies that markets are dynamically complete, c.f., Harrison and Kreps [8]. That is, contingent claims such as options written on stocks can be spanned by dynamically trading in the stocks and the bond.

One can easily verify that the stochastic process \( \{\xi_t\} \) satisfies the stochastic differential equation
\[
d\xi_t = \kappa(S_t) \xi_t \, dW_t, \quad \xi_0 = 1.
\tag{3}
\]
In the literature \( \{\xi_t(\omega)\} \) is usually interpreted as the implicit Arrow-Debreu state prices for a security that pays off one dollar at time \( t \), state \( \omega \) and nothing otherwise (Cox and Huang [3]). More specifically, the price of a contingent claim that pays off \( X_t \) dollars (in flow) for \( t \in [0, 1] \) and \( Y \) dollars (in lump sum) at time 1 can be calculated as
\[
E \left[ \int_0^1 \xi_t(\omega) \frac{X_t(\omega)}{B_t(\omega)} \, dt + \xi_1(\omega) \frac{Y(\omega)}{B_1(\omega)} \right].
\]

Given the system of implicit Arrow-Debreu state prices, an individual's intertemporal consumption and portfolio problem can be formulated in terms of the following static expected utility maximization problem:\(^1\)
\[
\sup_{c, W \geq 0} E \left[ \int_0^1 u(c_t, t) \, dt + V(W) \right] \quad \text{s.t. } E \left[ \int_0^1 \xi_t \frac{c_t}{B_t} \, dt + \xi_1 \frac{W}{B_1} \right] \leq W_0, \tag{4}
\]
where \( u \) is the utility function for the intertemporal consumption rate, and \( V \) is the utility function for the final wealth, which may represent a bequest to the next generation. Both \( u \) and \( V \) are assumed to be continuously differentiable, strictly increasing, and strictly concave. The individual's consumption decisions \( (c, W) \) are required to depend upon only the currently available information, i.e., the history of the Brownian motions up to that date.\(^2\) Equation (4) is called the budget constraint, which says that the present value of a feasible consumption-final wealth pair must be less than or equal to the initial wealth \( W_0 \).

---

\(^1\)Traditionally, the dynamic consumption and portfolio problem is solved using stochastic dynamic programming (Merton [13]). However, for our purposes, the martingale approach developed by Cox and Huang [3] turns out to be much easier to work with.

\(^2\)Let \( \mathcal{F}_t \) be the smallest sigma-field containing all of the \( F \)-measure zero set with respect to which \( \{w_s, 0 \leq s \leq t\} \) is measurable. We require that \( c_t \) be measurable with respect to \( \mathcal{F}_t \) and \( W \) be measurable with respect to \( \mathcal{F}_1 \).
The individual's maximization problem can be solved using standard Lagrangian method. Specifically, if we define
\[ f(y, t) = \inf \{ x \geq 0 : u'_x(x, t) \leq y \}, \]
\[ g(y) = \inf \{ x \geq 0 : V'_x(x) \leq y \}, \]
then there exits a \( \lambda_0 > 0 \) such that the optimal consumption and final wealth are
\[ c_t = f(\lambda_0 \xi_t / B_t, t), \quad W = g(\lambda_0 \xi_1 / B_1), \]
where \( \lambda_0 \) is determined through the budget constraint
\[ E \left[ \int_0^1 \xi_t f(\lambda_0 \xi_t / B_t, t) \frac{B_t}{B_t} dt + \xi_1 g(\lambda_0 \xi_1 / B_1) \right] = W_0. \quad (5) \]
Since \( u \) and \( V \) are strictly concave, \( \lambda_0 \) is uniquely determined. To characterize the portfolio policy that produces the optimal consumption policy, we introduce a new state variable \( Z \), where \( Z_t = B_t / \lambda_0 \xi_t \), and the optimal wealth function \( F \). At any time, the optimal wealth function gives the present value of the future consumption and the final wealth. Since \((S, Z)\) forms a Markov process, the optimal wealth function is a function of \( Z_t, S_t, \) and \( t \), defined by
\[ F(S, Z, t) = B_t E_Q \left[ \int_t^1 f(Z_s^{-1}, s) ds + g(Z_1^{-1}) \bigg| S_t = S, Z_t = Z \right] \]
\[ = E \left[ \int_t^1 Z_s^{-1} f(Z_s^{-1}, s) ds + Z_1^{-1} g(Z_1^{-1}) \bigg| S_t = S, Z_t = Z \right], \]
where \( E_Q \) denotes the expectation under \( Q \). The optimal wealth function satisfies the linear partial differential equation
\[ \frac{1}{2} \text{trace}[\sigma^T F_{SS} \sigma] + \frac{1}{2} |Z|^{2} F_{ZZ} + 2Z \sigma^T F_{SZ} + rS^T F_S + rZF_Z + F_t - rF + f(Z^{-1}, t) = 0, \quad (6) \]
\[ F(S, Z, 1) = g(Z^{-1}). \quad (7) \]
The dynamic portfolio strategy that produces the optimal consumption policy is determined by
\[ \theta_t = F_S + (\sigma_t^T \sigma_t)^{-1} (b_t - rS)ZF_Z, \]
\[ \alpha_t = \frac{1}{B_t} \left( F - \theta_t^T S_t \right). \quad (8) \]
We refer the reader to Cox and Huang [3] for details. In general, the above PDE has to be solved numerically, which could be quite involved since there are at least two state variables.
3 An Example

We now present a simple example to illustrate why discrete time models may have numerical advantages over their continuous time counterparts. Consider the Black-Scholes economy with two stocks and one bond. The movements of the stock prices, $S_1$ and $S_2$, and the bond price, $B$, can be described by the stochastic differential equations:

$$dS_{t,1} = \mu_1 S_{t,1} dt + \sigma_1 S_{t,1} dw_{t,1},$$
$$dS_{t,2} = \mu_2 S_{t,2} dt + \sigma_2 \rho S_{t,2} dw_{t,1} + \sigma_2 \sqrt{1 - \rho^2} S_{t,2} dw_{t,2},$$
$$dB_t = rB_t dt,$$

where $w_1$ and $w_2$ are two independent Brownian motions, $\sigma_i^2$ is the volatility of the instantaneous return on stock $i$, $\rho$ is the correlation coefficient between the instantaneous returns of the two stocks, and $r$ is the instantaneous riskless rate. Consider an individual who wants to invest his/her wealth $W_0 > 0$ in the stocks and the bond in order to maximize the expected utility of the final wealth. We assume that the time horizon for this investor is $[0,1]$ and that the utility function is $u(x) = -e^{-x} + \sqrt{x}$.

Under the continuous time setting formulated in the previous section, the investor would have to solve the partial differential equation (6) together with the boundary condition (7). Since there are three space variables and one time variable, standard numerical procedure (such as finite difference method) would be difficult to implement although it is not impossible.\(^3\) However, this problem can be solved in a simple discrete time setting.

We approximate the increments of the Brownian motions by two trinomial random variables $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$, defined by

$$P[\tilde{\epsilon}_1 = \frac{\sqrt{3}}{\sqrt{2}}, \tilde{\epsilon}_2 = \frac{1}{\sqrt{2}}] = \frac{1}{3}, \quad P[\tilde{\epsilon}_1 = 0, \tilde{\epsilon}_2 = -\frac{2}{\sqrt{2}}] = \frac{1}{3}, \quad P[\tilde{\epsilon}_1 = -\frac{\sqrt{3}}{\sqrt{2}}, \tilde{\epsilon}_2 = \frac{1}{\sqrt{2}}] = \frac{1}{3}.$$

These two random variables have the desired properties

$$E(\tilde{\epsilon}_1) = E(\tilde{\epsilon}_2) = 0,$$

$$\text{Var}(\tilde{\epsilon}_1) = \text{Var}(\tilde{\epsilon}_2) = 1, \quad \text{Cov}(\tilde{\epsilon}_1, \tilde{\epsilon}_2) = 0.$$

We obtain the following trinomial approximation for $B$, $S_1$ and $S_2$ with three equally-probable states:

$$B_{k+1}^n = \begin{cases} B_k^B (1 + \frac{\tilde{\epsilon}_1}{n}) & \text{with probability } \frac{1}{3}, \\ B_k^B (1 + \frac{\tilde{\epsilon}_2}{n}) & \text{with probability } \frac{1}{3}, \\ B_k^B (1 + \frac{\tilde{\epsilon}_3}{n}) & \text{with probability } \frac{1}{3}. \end{cases}$$

\(^3\)The partial differential equation for this problem is even more difficult to solve under the dynamic programming approach, since the PDE would be nonlinear.
\[ S_{k+1,1}^n = \begin{cases} S_{k,1}^n + \frac{\mu_1 S_{k,1}^n}{n} + \sigma_1 S_{k,1}^n \frac{\sqrt{3}}{\sqrt{2n}} \\ S_{k,1}^n + \frac{\mu_1 S_{k,1}^n}{n} \\ S_{k,1}^n + \frac{\mu_1 S_{k,1}^n}{n} - \sigma_1 S_{k,1}^n \frac{\sqrt{3}}{\sqrt{2n}} \end{cases} \]

\[ S_{k+1,2}^n = \begin{cases} S_{k,2}^n + \frac{\mu_2 S_{k,2}^n}{n} + \frac{\sqrt{3} \sigma_2 S_{k,2}^n}{\sqrt{2n}} + \frac{\sigma_3 \sqrt{1 - \rho^2} S_{k,2}^n}{\sqrt{2n}} - \frac{1}{n} \\ S_{k,2}^n + \frac{\mu_2 S_{k,2}^n}{n} - \frac{\sqrt{3} \sigma_2 S_{k,2}^n}{\sqrt{2n}} - \frac{1}{n} \\ S_{k,2}^n + \frac{\mu_2 S_{k,2}^n}{n} - \frac{\sqrt{3} \sigma_2 S_{k,2}^n}{\sqrt{2n}} - \frac{1}{n} \end{cases} \]

We will show in the next section that the above trinomial processes for the stock prices converge weakly to the original lognormal processes \( S_1 \) and \( S_2 \).

For ease of exposition, let us take \( n = 2 \) and draw the price movements for all three assets in a single tree, see Figure 1. To solve the final wealth problem, we first derive the Arrow-Debreu state prices for each time-and-state contingent claim. It is easy to verify that the Arrow-Debreu state price vector for each of the sub-tree are

\[ \pi_u = \frac{1}{3} \left( 1 + \frac{\sqrt{3} \kappa_1 + \kappa_2}{2} \right) \left( 1 + \frac{r}{2} \right)^{-1}, \]

\[ \pi_m = \frac{1}{3} \left( 1 - \kappa_2 \right) \left( 1 + \frac{r}{2} \right)^{-1}, \]

\[ \pi_d = \frac{1}{3} \left( 1 + \frac{-\sqrt{3} \kappa_1 + \kappa_2}{2} \right) \left( 1 + \frac{r}{2} \right)^{-1}, \]

where \( u \) (or \( d \)) corresponds the state that \( S_1 \) moves upwards (horizontally or downwards), and

\[ \kappa_1 = -\frac{\mu_1 - r}{\sigma_1}, \]

\[ \kappa_2 = \frac{\rho(\mu_1 - r)}{\sigma_1 \sqrt{1 - \rho^2}} - \frac{\mu_2 - r}{\sigma_2 \sqrt{1 - \rho^2}}. \]

Hence, \( \pi_w_1 = \pi_u \pi_u, \pi_w_2 = \pi_u \pi_m, \pi_w_3 = \pi_u \pi_d, \ldots, \) and \( \pi_w_0 = \pi_d \pi_d \). The investor's final wealth problem can now be formulated as follows:

\[ \max_{\pi \geq 0} \sum_{i=1}^{9} \left\{ -e^{W_i \pi_i} + \sqrt{W_i} \right\} \]

\[ \text{s.t.} \sum_{i=1}^{9} \pi_w W_i \leq W_0. \]

This is a simple nonlinear maximization problem, which can be solved (numerically) using the Lagrangian method.

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*We note that by collapsing those states that have the same levels of stock prices, the number of states in the trinomial model grows at a rate \( (n+1)(n+2)/2 \). If the drift and diffusion terms were not linear in \( S \), then the number of the states in the trinomial model would grow at a rate \( 3^n \) (and \( (N+1)^n \) for \( N+1 \)-nomial processes).
4 The Discrete Time Economy

In this section we construct a sequence of discrete time economies in which the stock and bond prices follow an \(N+1\)-variate, \(N+1\)-nomial Markov process that converges weakly to the \(N+1\)-dimensional diffusion process defined by (1) and (2), in the sense of Billingsley [2]. We then formulate the individual's optimal consumption and portfolio problem for the discrete time economies.

We adopt the \(N+1\)-nomial approximation procedure developed in He [9], which establishes a weak convergence of contingent claims prices under this approximation. The basic idea of He's approach is to approximate the increments of \(N\) independent Brownian motions by \(N\) random variables. Naturally, this yields a finite difference approximation of (1) and (2) determined by the stochastic difference equations

\[
S^0_{k+1} = S^0_k + \frac{b(S^0_k)}{n} + \sigma(S^0_k) \frac{\hat{\epsilon}^k}{\sqrt{n}},
\]

\[
B^0_{k+1} = B^0_k \left( 1 + \frac{r(S^0_k)}{n} \right),
\]

where \(\hat{\epsilon}^k = (\hat{\epsilon}^1, \ldots, \hat{\epsilon}^N)^T\) is a random vector satisfying \(E[\hat{\epsilon}^k] = 0\) and \(\text{Cov}(\hat{\epsilon}^k, \hat{\epsilon}^s) = I_{N \times N}\), \(S^0_k\) and \(B^0_k\) denote the \(N\)-dimensional vector of stock prices and the bond price at time \(\frac{k}{n}\), and \(S^0_0 = S_0\) and \(B^0_0 = 1\).

Obviously, there are many ways of constructing the random vector \(\hat{\epsilon}^k\). One possible approach is to require \((\epsilon^j, j = 1, \ldots, N)\) to be independent of each other. Unfortunately, this approach will not lead to a dynamically complete securities market. If we instead require \((\epsilon^j, j = 1, \ldots, N)\) to be uncorrelated with each other, then we can construct \(N\) random variables, each of which takes \(N+1\) values. This will allow us to obtain an \(N+1\)-variate, \(N+1\)-nomial process for the stock and bond prices so that the stocks and the bond in the discrete time economy form a dynamically complete market as in its continuous counterpart.

Following He [9], we construct \(\tilde{\epsilon}^k\), \(k = 1, 2, \ldots, n\), as follows. Let \(A\) be an \((N+1) \times (N+1)\) real, orthogonal matrix such that the last column of \(A\) is \((1/\sqrt{N+1}, \ldots, 1/\sqrt{N+1})^T\). Define \(e_{s,j} \equiv a_{sj}\sqrt{N+1}\), where \(a_{sj}\) is the \(s\)-th element in \(j\)-th column of \(A\). Let \(\Omega = \{\omega_1, \ldots, \omega_{N+1}\}\) be the sample space on which \(\tilde{\epsilon}^k\) is defined. We assign an equal probability to each state \(\omega_s\), i.e., \(P[\omega_s] = \frac{1}{N+1}\), for \(s = 1, \ldots, N+1\). Define \(\tilde{\epsilon}^k_s(\omega_s) = e_{s,j}\), \(s = 1, \ldots, N+1\). It is easy to verify that \(\tilde{\epsilon}^1_k, \ldots, \tilde{\epsilon}^N_k\) are uncorrelated with each other, and have mean zero and variance 1.

\[^6\]Let \(X^n\) and \(X\) be the stochastic processes defined on the probability spaces \((\Omega, \mathcal{F}, P_n)\) and \((\Omega, \mathcal{F}, P)\) with values in \(D^M[0,1]\), the space of functions from \([0,1]\) to \(\mathbb{R}^M\) that are right-continuous with left limits. The sequence of processes \(X^n\) is said to converge to \(X\) weakly, if for any bounded continuous mapping from \(D^M[0,1]\) to \(\mathbb{R}\), we have \(E_n[h(X^n)] \to E[h(X)]\), see Billingsley ([2], Chapter 1) for details.

\[^\dagger\]Such a matrix always exists!
\( \Omega = \Omega \times \cdots \times \Omega \) (n times) be the natural product space and \( P_n \) be the natural product measure on \( \Omega_n \); then \( \{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n\} \) can be treated as a sequence of independent and identically distributed N-dimensional random vectors defined on \( (\Omega_n, P_n) \). The N-dimensional vector of stock prices, \( \mathbf{S}^n \), and the bond price, \( B^n \), are well defined according to (9) and (10), and are \( N+1 \)-nominal. To define the stock and bond prices on the entire time horizon \([0, 1]\), we set \( \tilde{B}^n_t = B^n_{[nt]} \) and \( \tilde{S}^n_t = S^n_{[nt]} \), where \( [\ ] \) denotes the largest integer that is less than or equal to \( nt \). The sample paths of \( \tilde{S}^n \) and \( \tilde{B}^n \) are piecewise constant and have jumps only at \( t = \frac{k}{n} \).\(^7\)

Since there are \( N+1 \) securities traded and \( N+1 \) possible uncertain states following each trading date, and since \( \sigma \) and \( A \) are invertible, it can be verified that markets are dynamically complete.\(^8\)

The unique Arrow-Debreu state price \( \pi(\omega_i; S^n_k) \) at time \( \frac{k}{n} \), conditional on the stock price at time \( \frac{k}{n} \) being \( S^n_k \), for a security that pays off one dollar only at time \( \frac{k+1}{n} \), state \( \omega_i \), must satisfy the following conditions:\(^9\)

\[
\begin{align*}
\sum_{i=1}^{N+1} \pi(\omega_i; S^n_k) S^n_{k+1}(\omega_i) &= S^n_k, \\
\sum_{i=1}^{N+1} \pi(\omega_i; S^n_k) B^n_{k+1}(\omega_i) &= B^n_k.
\end{align*}
\]

Solving (11) and (12) by substituting (9) and (10) into these equations, we obtain

\[
\pi(\omega_i; S^n_k) = \frac{1}{N+1} \left( 1 + \frac{\kappa(S^n_k)^\top}{\sqrt{n}} \varepsilon^k(\omega_i) \right) (1 + \frac{r(S^n_k)}{n})^{-1}.
\]

To obtain the Arrow-Debreu state price at time 0 for a security that pays off one dollar at time \( \frac{k}{n} \), we multiply together all of the one period Arrow-Debreu state prices from period 1 to period \( k \), where period \( i \) is from time \( \frac{i-1}{n} \) to time \( \frac{i}{n} \). Letting \( \pi^n_k \) denote this state price, then

\[
\pi^n_k = \pi(\omega_i; S^n_{k-1}) \pi(\omega_i; S^n_{k-2}) \cdots \pi(\omega_i; S^n_0), \quad k = 1, \ldots, n
\]

with \( \pi^n_0 = 1 \). Moreover, \( \pi^n_k \) satisfies the stochastic difference equation:

\[
\pi^n_{k+1} = \frac{\pi^n_k}{N+1} \left( 1 + \frac{\kappa(S^n_k)^\top}{\sqrt{n}} \varepsilon^k(1 + \frac{r(S^n_k)}{n})^{-1} \right).
\]

Since \( \kappa \) is bounded, all of the \( \pi \)'s are non-negative for sufficiently large \( n \). Next, we introduce a new process \( \xi^n_k \), defined by

\[
\xi^n_k = \pi^n_k B^n_k (N+1)^k.
\]

\(^7\)Let \( \mathcal{F}_n = \{F_k, k \leq n\} \), where \( F_k = \sigma(\varepsilon^1, \ldots, \varepsilon^k) \), then \( (\tilde{S}^n, \tilde{B}^n) \) is a process defined on the probability space \( (\Omega_n, \mathcal{F}_n, P_n) \) with values in \( \mathcal{D}^{N+1}[0, 1] \).

\(^8\)If \( \sigma \) and \( A \) are invertible, the \( (N+1) \times (N+1) \) matrix formed by the prices of the \( N \) stocks and the bond at the \( N+1 \) states for any subtree is also invertible.

\(^9\)Rigorously speaking, we should write \( S^n_{k+1}(\omega_i; S^n_k) \) for \( S^n_k \), since \( S^n \) is a Markov chain and its value at time \( \frac{k+1}{n} \) depends upon its value at time \( \frac{k}{n} \).
Using the fact that \( \sum_{i=1}^{N+1} \pi(\omega; S_{h}^{n}) = \left(1 + \frac{r(S_{h}^{n})}{n}\right)^{-1} \), one can verify that \( E_{n}[\xi_{n}^{n}] = 1 \), where \( E_{n} \) denotes the expectation under the probability measure \( P_{n} \). Now defining

\[
Q_{n}(A) = \int_{A} \xi_{n}^{n}(\omega) P_{n}(d\omega),
\]

then (11) and (12) imply that \( Q_{n} \) is an equivalent martingale measure, i.e., the discounted stock price processes become martingales under this measure. We next define \( \tilde{\xi}_{t}^{n} = \xi_{[nt]}^{n} \) for \( t \in [0,1] \) and call \( \tilde{\xi}^{n} \) the implicit Arrow-Debreu state price process, similar to the convention we adopted in the continuous time case. Analogous to (3), \( \xi^{n} \) can be represented by the stochastic difference equation:

\[
\xi_{t+1}^{n} = \xi_{t}^{n} \left(1 + \kappa(S_{t}^{n})^{\top} \frac{\xi^{n}}{\sqrt{n}}\right).
\] (13)

The following proposition has been proved in He [9].

Proposition 1 \((\tilde{S}^{n}, \tilde{B}^{n}, \tilde{\xi}^{n}) \) converges weakly to \((S, B, \xi)\).

This proposition comes at no surprise, since the stochastic difference equations for \( S^{n}, B^{n}, \) and \( \xi^{n} \) are exactly the finite difference approximations of the stochastic differential equations for \( S, B, \) and \( \xi \).

Given the system of implicit Arrow-Debreu prices, we can formulate an individual’s intertemporal consumption and portfolio problem in discrete time. Since the securities prices are constant over \([k/n, (k+1)/n]\), the optimal consumption and portfolio policies must be constant over this interval. Hence, instead of integrating the consumption over \([0,1]\), we sum the consumption over \( n \) different periods. Analogous to the continuous time model, we define an individual’s consumption and portfolio problem as follows:

\[
\max_{c_{k}^{n} \geq 0} E_{n} \left[ \frac{1}{n} \sum_{k=0}^{n-1} u(c_{k}^{n}, t_{k}^{n}) + V(W^{n}) \right]
\]

s.t.

\[
E_{n} \left[ \frac{1}{n} \sum_{k=0}^{n-1} \xi_{k}^{n} \frac{c_{k}^{n}}{B_{k}^{n}} + \xi_{n}^{n} \frac{W^{n}}{B_{n}^{n}} \right] \leq W_{0},
\]

where \( t_{k}^{n} = \frac{k}{n} \) and the above equation is the budget constraint. Applying the standard Lagrangian method, we can find a \( \lambda_{n} > 0 \) such that the optimal consumption and portfolio policies are

\[
c_{k}^{n} = f(\lambda_{n} \xi_{k}^{n}/B_{k}^{n}, t_{k}^{n}), \quad W^{n} = g(\lambda_{n} \xi_{n}^{n}/B_{n}^{n}),
\]

for \( k = 0, 1, \ldots, n-1 \), where \( \lambda_{n} \) is determined by the budget constraint

\[
E_{n} \left[ \frac{1}{n} \sum_{k=0}^{n-1} \xi_{k}^{n} f(\lambda_{n} \xi_{k}^{n}/B_{k}^{n}, t_{k}^{n}) \frac{c_{k}^{n}}{B_{k}^{n}} + \xi_{n}^{n} g(\lambda_{n} \xi_{n}^{n}/B_{n}^{n}) \right] = W_{0}.
\] (15)
As in the continuous time model, the optimal consumption policy for the discrete time model can be expressed explicitly as a function of the implicit Arrow-Debreu state prices. Computationally, the above maximization problem is not difficult to solve, since we only need to search for a \( \lambda_n > 0 \) that satisfies (15).

In order to characterize the portfolio policy that produces the optimal consumption policy, we introduce a new state variable \( Z^n \), defined by \( Z^n_k \equiv \lambda_n B^n_k / \xi^n_k \), and the optimal wealth function \( F^n \). The optimal wealth function at any time gives the present value of the future consumption and the final wealth. The optimal wealth function \( F^n \) is a function of \( (S^n_k, Z^n_k, t^n_k) \), defined by

\[
F^n(S^n_k, Z^n_k, t^n_k) = B^n_k E^n \left[ \frac{1}{n} \sum_{m=k}^{n} \frac{f([Z^n_m]^{-1}, t^n_m)}{B^n_m} + g([Z^n_n]^{-1}) \left| S^n_k, Z^n_k \right. \right] \]

\[
= Z^n_k E^n \left[ \frac{1}{n} \sum_{m=k}^{n} [Z^n_m]^{-1} f([Z^n_m]^{-1}, t^n_m) + [Z^n_n]^{-1} g([Z^n_n]^{-1}) \left| S^n_k, Z^n_k \right. \right].
\]

Hence, \( F^n \) satisfies the recurrent equation

\[
F^n(S^n_k, Z^n_k, t^n_k) = \frac{1}{n} f([Z^n_k]^{-1}, t^n_k) + \sum_{i=1}^{N+1} \pi(\omega_s; S^n_k) F^n(S^n_{k+1}(\omega_s), Z^n_{k+1}(\omega_s), t^n_{k+1}).
\]

(16)

\[
F^n(S^n_n, Z^n_n, t^n_n) = g([Z^n_n]^{-1}).
\]

(17)

Let \( \theta^n \) be the portfolio policy that produces the optimal consumption policies, then \( \theta^n \) at time \( \frac{k}{n} \) is determined by the system of linear equations

\[
\alpha^n_k B^n_{k+1}(\omega_s) + \theta^n_{k+1,1} S^n_{k+1,1}(\omega_s) + \cdots + \theta^n_{k,N} S^n_{k+1,N}(\omega_s) = F^n(S^n_{k+1}(\omega_s), Z^n_{k+1}(\omega_s), \frac{k+1}{n}),
\]

for \( s = 1, \cdots, N+1 \). Since there are \( N+1 \) equations and \( N+1 \) unknowns, and since \( \sigma \) and \( A \) are invertible, the solution for \( (\alpha^n_k, \theta^n_k) \) is uniquely determined. Moreover, \( \alpha^n_k \) and \( \theta^n_k \) are functions of \( S^n_k, Z^n_k \), and \( t^n_k \). To define the discrete time processes \( c^n, Z^n, \alpha^n \), and \( \theta^n \) on \( [0,1] \), we set \( c^n_t = c^n_{[nt]} \), etc., for \( t \in [0,1] \).

5 Weak Convergence Theorems

In this section we present our main convergence theorems for optimal consumption policies, optimal portfolio policies, and maximum expected utilities. As we have mentioned earlier, these convergence results should enhance our understanding of the connection between discrete and continuous time models. Moreover, since the discrete time problem is easier to solve computationally, the discrete time analysis can be employed as a numerical alternative to the partial differential equation derived from the continuous time analysis.
We start with two lemmas, which are used in the proofs of the main theorems. The proofs of these two lemmas can be found in Theorems 1.5.1 and 1.5.4, Billingsley ([2], p.30 and p.32).

Lemma 1 (Continuous Mapping Theorem) Assume that $X^n$ converges weakly to $X$.

i) Suppose that $h(x)$ is a continuous mapping from $\mathbb{R}^N$ to $\mathbb{R}^M$. Let $Y^n_t = h(X^n_t)$ and $Y_t = h(X_t)$, then $Y^n_t$ converges weakly to $Y$.

ii) Suppose that $H$ is a continuous mapping from $D^M[0,1]$ to $\mathbb{R}$ (under the Skorohod topology), then $H(X^n)$ converges to $H(X)$ in distribution.

Lemma 2 Let $X_n$ and $X \in \mathbb{R}^1$ be random variables such that $X_n$ converges to $X$ in distribution as $n \to \infty$. If $\{|X_n|\}$ is uniformly integrable, then $E_n[X_n] \to E[X]$ as $n \to \infty$.

We first present a convergence theorem for optimal consumption policies. All the proofs in the rest of the paper are in the Appendix.

Theorem 1 (Consumption Policies) Suppose that

$$
\int_0^1 \xi^n_t \frac{f(\gamma \xi^n_t / \bar{B}^n_t, [nl] \xi^n_t)}{\bar{B}^n_t} \, dt,
$$

are uniformly integrable for every constant $\gamma > 0$, then the optimal discrete time consumption policies $\bar{c}^n$ and $\bar{w}^n$ converge weakly to the optimal continuous time consumption policies $c$ and $w$ as $n \to \infty$.

The basic conclusion of Theorem 1 is that if the securities price processes and the implicit Arrow-Debreu state price process converge weakly to their continuous time limits, then, subject to some regularity conditions on the utility functions and on the parameters of the securities price processes, the optimal consumption processes in the discrete time models always converge weakly to the corresponding continuous time limit. The proof for this theorem is quite straightforward. Since $\xi^n$ converges weakly to $\xi$, Lemma 1 implies that $\bar{c}^n$ and $\bar{w}^n$ converge weakly to $c$ and $w$ if and only if $\lambda^n$ converges to $\lambda_0$. We prove that $\lambda^n$ converges to $\lambda_0$ in the Appendix.

Obviously, for purposes of applications, checking the uniform integrability conditions for every price system and every utility function of interest would be cumbersome. The following proposition provides explicit conditions on the parameters of the price processes and the individual’s utility functions that ensure the convergence of optimal consumption policies. We first need a definition.

Definition 1 A function $h(z,t) : \mathbb{R}^N \times [0,1] \rightarrow \mathbb{R}$ is said to satisfy a polynomial growth condition if there exist $A > 0$ and $\beta > 1$ such that

$$
|h(z,t)| \leq A(1 + |z|^{\beta}) \quad \forall (z,t) \in \mathbb{R}^N \times [0,1].
$$
The function $h(x,t)$ is said to satisfy a negative polynomial growth condition if there exist $A' > 0$ and $\beta' > 1$ such that 

$$|h(x,t)| \leq A'(1 + |x|^{-\beta'}) \quad \forall (x,t) \in \mathbb{R}^N \times [0,1].$$

**Proposition 2** Suppose that $r$ and $\kappa$ are uniformly bounded and that $f(x,t)$ and $g(x)$ satisfy a negative polynomial growth condition. Then $\tilde{e}^n$ and $W^n$ converge weakly to $c$ and $W$.

**Remark 1** The negative polynomial conditions for $f$ and $g$ are satisfied if $u(x) = e^{-\delta t} \frac{x^b}{b}$ and $V(x) = \frac{x^{b'}}{b'}$, where $b, b' < 1$.

Next, we present a convergence theorem for optimal portfolio policies. The idea of this theorem is similar to that of Theorem 2 in He [9].

**Theorem 2** (Portfolio Policies) Suppose that $\tilde{Z}^n$ converges weakly to $Z$ (or equivalently $\lambda_n \rightarrow \lambda_0$). Suppose further that function $F$ is continuously differentiable up to the third order and all of its derivatives up to the third order satisfy a polynomial growth condition. Then $\tilde{e}^n$ and $e^n$ converge weakly to $\alpha$ and $\theta$ as $n \rightarrow \infty$. In particular, $\tilde{e}_0^n$ and $e_0^n$ converges (numerically) to $\alpha_0$ and $\theta_0$.

**Remark 2** The assumption that $F$ is differentiable and satisfies the polynomial growth condition can be guaranteed if the following three conditions are satisfied:

1. $\sigma$, $r$, and $\kappa$ are continuously differentiable up to the sixth order, with all these derivatives satisfying a polynomial growth condition.

2. $r(x)y$ and $\kappa(x)y$ satisfy a uniform Lipschitz condition for $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^1$.

3. $f(x^{-1}, t)$ and $g(x^{-1})$ are continuously differentiable with respect to $x$ up to the sixth order, and all of these derivatives satisfy a polynomial growth condition.

See He ([10], p.70) for details.

The basic approach of the proof is to write the discrete time optimal wealth function as follows:

$$F^n(S^n, Z^n, [nt]_n) = F(\tilde{S}^n_t, \tilde{Z}^n_t, [nt]_n) - \left[ F(S^n, Z^n, [nt]_n) - F^n(\tilde{S}^n_t, \tilde{Z}^n_t, [nt]_n) \right].$$

Since $F$ is continuous, and $(S^n, Z^n)$ converges weakly to $(S, Z)$, the stochastic process defined by the first term on the right-hand-side of the above equation converges weakly to the optimal wealth process. We show in the Appendix that the stochastic process defined by the second term on the right-hand side of the above equation converges weakly to zero. This result allows us to show that
the portfolio policies that produce the optimal wealth processes in discrete time models converge to the corresponding continuous time limit.

Finally, we record a convergence of maximum expected utilities, the proof of which follows immediately from the Continuous Mapping Theorem (Lemma 1).

**Theorem 3 (Expected Utilities)** Suppose that the sufficient conditions for Theorem 1 are satisfied. Suppose further that \( \{ \int_0^1 u(c_t^n, \frac{int}{n}) dt \} \) and \( |V(W^n)| \) are uniformly integrable. Then

\[
E_n \left[ \frac{1}{n} \sum_{k=1}^{n-1} u(c_{k}, t_{k}^n) + V(W^n) \right] = E_n \left[ \int_0^1 u(c_t^n, \frac{nt}{n}) dt + V(W^n) \right] \to E \left[ \int_0^1 u(c_t, t) dt + V(W) \right],
\]

as \( n \to \infty \).

**Remark 3** A sufficient condition for \( \int_0^1 |u(c_t^n, \frac{nt}{n}) dt \) and \( |V(W^n)| \) to be uniformly integrable is that there exist constants \( A, B, C \), and \( 0 < b < 1, \beta > 0 \) such that for all \( x > 0 \)

\[
|u(x, t)| + |V(x)| \leq A + B \frac{t^{1-b}}{1-b},
\]

\[
f(x, t) + g(x) \leq C(1 + x^{-\beta}).
\]

The claim can be proved using the approach in Proposition 1.

This concludes our demonstration of the convergence of optimal consumption policies, optimal portfolio policies, and maximum expected utilities.

### 6 Concluding Remarks

We have shown that for a fairly broad class of diffusion prices \((S, B)\), and for a particular sequence of discrete time multinomial approximations \((S_n, B_n)\), the individual's optimal consumption and portfolio policies in discrete time models converge to their corresponding continuous time limits. It is important to note that there are other discrete time approximations that produce similar convergence results. For example, any discrete time approximation that leads to a convergence of Arrow-Debreu state prices would admit a convergence of optimal consumption processes (subject to some regularity conditions). A more general examination for a complete class of discrete time models that would lead to behavioral predictions similar to their continuous counterpart is an important subject to pursue in the future. We refer the readers to Duffie and Portter [5] for discussions on convergence of financial gain processes. Duffie and Sun [6] contains some interesting results on the connection between the discrete and continuous time models when there are transaction costs associated with trading.
The mode of convergence used in this paper is weak convergence. Establishing a convergence of consumption and portfolio policies under a weak topology instead of a strong one might be preferable, as Kreps [11] suggests. While Kreps establishes a convergence of an individual's optimal consumption policies under $L^2$-norm for a two-period economy, he finds that using the strong topology requires excessive restrictions on the structure of discrete time price processes and on the continuity of the preferences. Kreps then constructs a trinomial approximation for the Black-Scholes model and conjectures that under somewhat weaker assumptions on the utility function, a weak convergence of optimal intertemporal consumption and portfolio policies would obtain. Willinger [16] establishes a strong convergence for optimal consumption policies, but he fails to show a convergence for the corresponding optimal portfolio policies that produce the optimal consumption policies. Therefore, it should be clear that our convergence results depend crucially on the mode of convergence we used.

Our convergence results also require the important assumption of dynamic completeness in the continuous time model. Because of dynamically complete markets, we could formulate an individual's intertemporal consumption and portfolio problem in terms of a static expected utility maximization problem. This static formulation, as we have demonstrated, effectively establishes the convergence of optimal consumption and portfolio policies. It would be interesting to investigate how our method might be extended to allow dynamically incomplete markets. We conjecture that similar convergence results would be obtained in this setting, where one would expect markets in discrete time models to be dynamically incomplete as well. The exact implementation is subject to further research.
References


Proof of Theorem 1: Without loss of generality, we assume that \( V = r = 0 \). According to the Continuous Mapping Theorem (Lemma 1), it is sufficient to demonstrate that \( \lambda_n \to \lambda_0 \).

Suppose that \( \lambda_n \to \lambda_0 \) is not true, then there exists a subsequence of \( \lambda_n \), denoted again by \( \lambda_n \), that converges to some number \( \bar{\lambda} \neq \lambda_0 \) (\( \bar{\lambda} \) could be infinity, but is always non-negative). Let us first assume that \( \bar{\lambda} \) is finite and positive. According to Lemma 1, the stochastic process \( \tilde{\xi}^n \) converges weakly to \( \xi^n \), where \( \xi_i = f(\bar{\lambda} \xi_i, t) \). We have that \( \int_0^1 \xi^n f(\bar{\lambda} \xi_i, t) dt \) converges to \( \int_0^1 \xi f(\bar{\lambda} \xi_i, t) dt \) in distribution, by (ii) of Lemma 1.

If \( \bar{\lambda} > \lambda_0 \), then there exists a sufficiently small number \( \epsilon > 0 \) such that \( \lambda_n > \bar{\lambda} - \epsilon > \lambda_0 \) for \( n > n_0 \). Since \( f(x, t) \) is decreasing in \( x \), \( f(\lambda_n \tilde{\xi}_i^n, \frac{[nt]}{n}) \leq f((\bar{\lambda} - \epsilon) \tilde{\xi}_i^n, \frac{[nt]}{n}) \) for \( n > n_0 \). Because of the uniform integrability of \( \int_0^1 \tilde{\xi}_i^n f(\gamma \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \) for \( \gamma = \bar{\lambda} - \epsilon \), we have

\[
E_n \left[ \int_0^1 \tilde{\xi}_i^n f(\lambda_n \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \right] \leq E_n \left[ \int_0^1 \tilde{\xi}_i^n f((\bar{\lambda} - \epsilon) \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \right] \to E \left[ \int_0^1 \xi f((\bar{\lambda} - \epsilon) \xi_i, t) dt \right],
\]

where the convergence on the right-hand side of the above equation follows from Lemma 2. The left-hand side of (18) is equal to \( W_0 \) by the budget constraint (14). Hence, we obtain that

\[
E \left[ \int_0^1 \xi f((\bar{\lambda} - \epsilon) \xi_i, t) dt \right] \geq W_0.
\]

Since \( \lambda - \epsilon > \lambda_0 \) and \( f(x, t) \) is strictly decreasing in \( x \), the above inequality contradicts (5), i.e,

\[
E \left[ \int_0^1 \xi f(\lambda_0 \xi_i, t) dt \right] = W_0.
\]

If \( \bar{\lambda} < \lambda_0 \), then there exists a sufficiently small number \( \epsilon > 0 \) such that \( \lambda_n < \bar{\lambda} + \epsilon < \lambda_0 \) for \( n > n_0 \). Hence, \( f(\lambda_n \tilde{\xi}_i^n, \frac{[nt]}{n}) \geq f((\bar{\lambda} + \epsilon) \tilde{\xi}_i^n, \frac{[nt]}{n}) \) for \( n > n_0 \). We have

\[
E_n \left[ \int_0^1 \tilde{\xi}_i^n f(\lambda_n \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \right] \geq E_n \left[ \int_0^1 \tilde{\xi}_i^n f((\bar{\lambda} + \epsilon) \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \right] \to E \left[ \int_0^1 \xi f((\bar{\lambda} + \epsilon) \xi_i, t) dt \right].
\]

Since the left-hand side of (19) is equal to \( W_0 \) by (14), we obtain that

\[
E \left[ \int_0^1 \xi f((\bar{\lambda} + \epsilon) \xi_i, t) dt \right] \leq W_0.
\]

The above inequality again contradicts (5), because \( \lambda + \epsilon > \lambda_0 \) and \( f(x, t) \) is strictly decreasing in \( x \).

We now consider the cases when \( \bar{\lambda} \) is either zero or infinity. If \( \bar{\lambda} = 0 \), then there exists a sufficiently small number \( \epsilon > 0 \) such that \( \lambda_n < \epsilon < \lambda_0 \) for \( n > n_0 \). We have \( f(\lambda_n \tilde{\xi}_i^n, \frac{[nt]}{n}) \geq f(\epsilon \tilde{\xi}_i^n, \frac{[nt]}{n}) \) for \( n > n_0 \), which gives

\[
W_0 = E_n \left[ \int_0^1 \tilde{\xi}_i^n f(\lambda_n \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \right] \geq E_n \left[ \int_0^1 \tilde{\xi}_i^n f(\epsilon \tilde{\xi}_i^n, \frac{[nt]}{n}) dt \right] \to E \left[ \int_0^1 \xi f(\epsilon \xi_i, t) dt \right].
\]
This implies that
\[ E \left[ \int_0^1 \xi_t f(\varepsilon \xi_t, t) dt \right] \leq W_0. \]

The above inequality also contradicts (5). We omit the proof for the case when \( \lambda = \infty \). ■

**Proof of Proposition 1:** We need to check that the uniform integrability conditions required by Theorem 2 are satisfied. Note that a sufficient condition for a sequence \( X^n \) to be uniformly integrable is that there exists an \( \eta > 0 \) such that \( \sup_n E_n |X^n|^{1+\eta} < \infty \).

Without loss of generality, we may assume that \( \beta' = 1 + m \) for some non-negative integer \( m \). Since \( r \) is bounded and non-negative, we can find a constant \( L > 0 \) such that \( 1 \leq B_p^r \leq L \). We have
\[
\xi^n_n \leq \xi^n_n \leq K \xi^n_n \left[ 1 + \left( \frac{\gamma}{L} \right)^{-\beta} (\xi^n_n)^{-\beta} \right].
\]
Hence,
\[
\int_0^1 \xi^n_n f(r\xi^n_n / B_p^r, [nt]/n) dt \leq K' K \left[ \frac{1}{n} \sum_{k=0}^{n-1} (\xi^n_k + (\xi^n_k)^{1-\beta}) \right],
\]
where \( K' = \max(1, (\gamma/L)^{-\beta}) \). We show below that
\[
\sup_n E_n \left[ \frac{1}{n} \sum_{k=0}^{n-1} (\xi^n_k + (\xi^n_k)^{-m}) \right]^2 < \infty.
\]
In fact,
\[
E_n \left[ \frac{1}{n} \sum_{k=0}^{n-1} (\xi^n_k + (\xi^n_k)^{-m}) \right]^2 \leq E_n \left[ \frac{1}{n} \sum_{k=0}^{n-1} (\xi^n_k + (\xi^n_k)^{-m})^2 \right] \leq 2 E_n \left[ \frac{1}{n} \sum_{k=0}^{n-1} ((\xi^n_k)^2 + (\xi^n_k)^{-2m}) \right],
\]
where the first inequality follows from the fact that \( x^2 \) is convex. Direct computation shows that
\[
E_n |\xi^n|^2 \leq E_n |\xi^n_{n-1}|^2 \left( 1 + \frac{D}{n} \right) \leq \left( 1 + \frac{D}{n} \right)^n,
\]
where \( D \) is the upper bound of \( \kappa^2 \). Since
\[
[\xi^n_{k+1}]^{-1} = [\xi^n_k]^{-1} \left( 1 + \frac{\kappa^T \xi^n_k}{\sqrt{n}} \right)^{-1}
\]
\[
= [\xi^n_k]^{-1} \left( 1 - \frac{\kappa^T \xi^n_k}{\sqrt{n}} + \frac{(\kappa^T \xi^n_k)^2}{n} - \frac{(\kappa^T \xi^n_k)^3}{1 + \frac{\kappa^T \xi^n_k}{\sqrt{n}}} \right),
\]
we have that
\[ E[\xi_{k+1}^n]^{-2m} = E[\xi_k^n]^{-2m} \left(1 + O\left(\frac{1}{n}\right)\right), \]
where \(O\left(\frac{1}{n}\right)\) denotes the sum of those terms that are of order \(n^{-1}\) or higher. Hence, there exist constants \(A > 0\) and \(n_0 > 0\), independent of \(k\), such that when \(n > n_0\),
\[ E[\xi_{k+1}^n]^{-2m} \leq E[\xi_k^n]^{-2m} \left(1 + \frac{A}{n}\right). \]
Hence \(E[\xi_k^n]^{-2m} \leq \left(1 + \frac{A}{n}\right)^n\), for all \(k\). It is now clear that
\[ \sup_n E_n \left[ \frac{1}{n} \sum_{k=0}^{n-1} \left( \xi_k^n + (\xi_k^n)^{1-f} \right)^2 \right] < \infty. \]
Similar inequalities can be established for \(g\). We conclude that \(\bar{\varepsilon}^n\) and \(W^n\) converge weakly to \(\varepsilon\) and \(W\). 

Proof of Theorem 2: The proof for this theorem is similar to the proof of Theorem 2 in He [9]. Therefore we shall be brief, and refer the readers to that paper for more details. We demonstrate the proof only for \(N = 1\) (the proof for \(N > 1\) is analogous).

Let \(\bar{F}_t^n = F^n(\bar{S}_t^n, \bar{Z}_t^n, [nt])\) and \(\bar{F}_t = F(S_t, Z_t, t)\), where \(F\) and \(F^n\) are solutions to (6) and (15), respectively. We first prove that \(\bar{F}_t^n\) converges weakly to the optimal wealth process \(\bar{F}\). Rewrite \(F^n\) as follows,
\[ \bar{F}_t^n = F(\bar{S}_t^n, \bar{Z}_t^n, [nt]) - F(\bar{S}_t^n, \bar{Z}_t^n, [nt]) - F^n(\bar{S}_t^n, \bar{Z}_t^n, [nt]). \]
The first term on the right-hand side of the above equation converges weakly to the optimal wealth process by the Continuous Mapping Theorem. Thus, we need only to prove that the second term on the right-hand side of the above equation converges weakly to zero. To do so, we substitute the function \(F\) into the finite difference equation that defines \(F^n\) to get an estimate for the truncation errors:
\[ \bar{\varepsilon}_n^k = F(S_k^n, Z_k^n, \frac{k}{n}) - F^n(S_k^n, Z_k^n, \frac{k}{n}). \]
For simplicity, we use + and − to denote the states \(\{\bar{\varepsilon}^k = 1\}\) and \(\{\bar{\varepsilon}^k = -1\}\) respectively. Define
\[
\begin{align*}
S_{k+1}^n &= \frac{b(S_k^n)}{n} + \frac{\sigma(S_k^n)}{\sqrt{n}}, \\
S_{k-1}^n &= \frac{b(S_k^n)}{n} - \frac{\sigma(S_k^n)}{\sqrt{n}}, \\
Z_{k+1}^n &= Z_k^n \left(1 + \frac{r(S_k^n)}{n}\right)(1 + \kappa(S_k^n)\sqrt{\frac{1}{n}})^{-1}, \\
Z_{k-1}^n &= Z_k^n \left(1 + \frac{r(S_k^n)}{n}\right)(1 - \kappa(S_k^n)\sqrt{\frac{1}{n}})^{-1},
\end{align*}
\]
and define two functions:

\[ f^+_n(r) = F(S^n_k + r(S^n_{k+1} - S^n_k), Z^n_k + r(Z^n_{k+1} - Z^n_k), t^n_k + r(t^n_{k+1} - t^n_k)), \]
\[ f^-_n(r) = F(S^n_k + r(S^n_{k+1} - S^n_k), Z^n_k + r(Z^n_{k+1} - Z^n_k), t^n_k + r(t^n_{k+1} - t^n_k)). \]

Since \( F \) is continuously differentiable up to the third order, \( f^+_n \) and \( f^-_n \) admit the following Taylor expansions (we omit the superscripts for \( f \)):

\[ f_+(1) = f_+(0) + f'_+(0) + \frac{1}{2} f''_+(0) + R^n_k, \]
\[ f_-(1) = f_-(0) + f'_-(0) + \frac{1}{2} f''_-(0) + Q^n_k, \]

where

\[ R^n_k = \frac{1}{6} \int_0^1 (1 - s)^3 f''_+(s) ds, \]
\[ Q^n_k = \frac{1}{6} \int_0^1 (1 - s)^3 f''_-(s) ds. \]

We have

\[ f'_+(0) = F_S(S^n_{k+1} - S^n_k) + F_Z(Z^n_{k+1} - Z^n_k) + \frac{1}{n} f_t, \]
\[ f'_-(0) = F_S(S^n_{k+1} - S^n_k) + F_Z(Z^n_{k+1} - Z^n_k) + \frac{1}{n} f_t, \]
\[ f''_+(0) = F_{SS}(S^n_{k+1} - S^n_k)^2 + F_{ZZ}(Z^n_{k+1} - Z^n_k)^2 + F_{tt} \frac{1}{n^2} \]
\[ + 2 F_{SZ}(S^n_{k+1} - S^n_k)(Z^n_{k+1} - Z^n_k) + 2 F_{St} \frac{1}{n} (S^n_{k+1} - S^n_k) + 2 F_{Zt} \frac{1}{n} (Z^n_{k+1} - Z^n_k), \]
\[ f''_-(0) = F_{SS}(S^n_{k+1} - S^n_k)^2 + F_{ZZ}(Z^n_{k+1} - Z^n_k)^2 + F_{tt} \frac{1}{n^2} \]
\[ + 2 F_{SZ}(S^n_{k+1} - S^n_k)(Z^n_{k+1} - Z^n_k) + 2 F_{St} \frac{1}{n} (S^n_{k+1} - S^n_k) + 2 F_{Zt} \frac{1}{n} (Z^n_{k+1} - Z^n_k), \]

where the function \( F \) and all of its partial derivatives are evaluated at \((S^n_k, Z^n_k, t^n_k)\). Now, in order to obtain an estimate for \( e^n_k \), we substitute the expressions for \( f^+_n(1) \) and \( f^-_n(1) \) into the recurrent equation (16). Using the fact that \( F \) satisfies (6), we obtain, as in the proof of Theorem 2, He [9], that

\[ \pi(\cdot; S^n_k) f_+(1) + \pi(\cdot; S^n_k) f_-(1) + \frac{1}{n} f(Z^n_k) = F(S^n_k, Z^n_k, t^n_k) + O \left( \frac{1}{n^{2/3}} \right) \]

where \( O \left( \frac{1}{n^p} \right) \) denotes the sum of those terms that are of order \( \frac{1}{n^p} \) or higher. Combining this equation with (17) results in the following relation for the truncation errors:

\[ e^n_k = 0 \]
\[ e^n_k = \pi(\cdot; S^n_k) e^n_{k+1}^+ + \pi(\cdot; S^n_k) e^n_{k+1}^- + O \left( \frac{1}{n^{2/3}} \right). \]
Again, using an argument similar to that in the proof of Theorem 2, He [9], we can conclude that

\[ |e^n_k| = O \left( \frac{1}{n^{1/2}} \right) \Rightarrow 0, \]

i.e., \( \tilde{e}^n \) converges weakly to zero in the order of \( \frac{1}{\sqrt{n}} \), and

\[ \sqrt{n}(e_{k+1}^n - e_{k+1}^{-n}) \Rightarrow 0. \]

We now show that \( \tilde{\theta}^n \) converges weakly to \( \theta \). Substituting the expressions of \( f_{+}^n \) and \( f_{-}^n \) directly into the expression for \( \theta_k^n \), we obtain

\[
\theta_k^n = \frac{F^n(S_{k+1}^+, Z_{k+1}^+, t_{k}^n) - F^n(S_{k+1}^-, Z_{k+1}^-, t_{k}^n)}{S_{k+1}^+ - S_{k+1}^-} \\
= \frac{f_+^n(1) - f_-^n(1)}{S_{k+1}^+ - S_{k+1}^-} - \frac{e_{k+1}^+ - e_{k+1}^-}{S_{k+1}^+ - S_{k+1}^-} \\
= F_S(S_k^n, Z_k^n, t_k^n) + \frac{b(S_k^n) - r(S_k^n)S_k^n}{\sigma^2(S_k^n)} F_Z(S_k^n, Z_k^n, t_k^n) + e_k^n - \frac{2\sqrt{n}}{\sigma}(e_{k+1}^+ - e_{k+1}^-),
\]

where \( e_k^n \) is the sum of several terms, all of which are of order \( n^{-1} \) or higher. We conclude that \( \tilde{\theta}^n \) converges weakly to \( \theta \) as defined in (8).

Finally, since \( \tilde{F}^n, \tilde{\theta}^n, \tilde{B}^n, \) and \( \tilde{S}^n \) converge to \( \tilde{F}, \theta, B, \) and \( S \), we conclude from \( \tilde{F}_t^n = \alpha_t^n \tilde{B}_t^n + \tilde{s}_t^n \tilde{S}_t^n \) that \( \tilde{\alpha}^n \) converges weakly to \( \alpha \). The claim that \( \alpha_0^n \) and \( \theta_0^n \) converge follows directly from the definition of the Skorokhod topology (Billingsley p.121) and from the fact that they are deterministic. \[\square\]
\[ \begin{align*}
\omega_1 &= (u_1 u_{S0,1}, u_2 u_{S0,2}) \\
(\omega_2) &= (u_1 m_{S0,1}, u_2 m_{S0,2}) \\
\omega_3 &= (u_1 d_{S0,1}, u_2 d_{S0,2}) \\
(\omega_4) &= (m_1 u_{S0,1}, m_2 u_{S0,2}) \\
(m_1 S_{0,1}, m_2 S_{0,2}) &= (m_1 m_{S0,1}, m_2 m_{S0,2}) \\
\omega_5 &= (m_1 d_{S0,1}, m_2 d_{S0,2}) \\
(\omega_6) &= (d_1 u_{S0,1}, d_2 u_{S0,2}) \\
(d_1 S_{0,1}, d_2 S_{0,2}) &= (d_1 m_{S0,1}, d_2 m_{S0,2}) \\
\omega_7 &= (d_1 d_{S0,1}, d_2 d_{S0,2}) \\
\omega_8 &= (d_1 d_{S0,1}, d_2 d_{S0,2}) \\
\omega_9 &= (d_1 d_{S0,1}, d_2 d_{S0,2}) \\

u_1 &= 1 + \frac{\mu_1}{2} + \frac{\sqrt{3}}{2} \sigma_1, \quad u_2 = 1 + \frac{\mu_2}{2} + \frac{\sqrt{3}}{2} \sigma_2 \rho + \frac{1}{2} \sigma_2 \sqrt{1 - \rho^2} \\
m_1 &= 1 + \frac{\mu_1}{2}, \quad m_2 = 1 + \frac{\mu_2}{2} - \sigma_2 \sqrt{1 - \rho^2} \\
d_1 &= 1 + \frac{\mu_1}{2} - \frac{\sqrt{3}}{2} \sigma_1, \quad d_2 = 1 + \frac{\mu_2}{2} - \frac{\sqrt{3}}{2} \sigma_2 \rho + \frac{1}{2} \sigma_2 \sqrt{1 - \rho^2}
\end{align*} \]

Figure 1
Event tree representation of the stock price processes