Variations in Economic Uncertainty and Risk Premiums on Capital Assets

by

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and

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Abstract

We investigate the movements in equity market risk premiums caused by variation over time in uncertainty about corporate dividends. The framework for our analysis is a general equilibrium model which we fit to NYSE index returns using a method of simulated moments (MSM). Using the model and the MSM estimates, we find: in general, equity risk premiums are not proportional to return volatility; GARCH-M specifications in which the equilibrium risk premium is linear in return variance can give a reasonable approximation, at least at the level of the market; asset return volatility is about twice as high as dividend volatility, even though equity prices in our model are rationally determined; dividend-price ratios “predict” subsequent equity returns with about the same precision as found in practice; long-run market returns are negatively autocorrelated; and, consistent with empirical evidence, risk premiums on levered equity are a monotonically increasing function of junk bond yield spreads.
I. Introduction

Variations in equity prices can be ascribed to three sources: the features of the market in which equities are traded; factors affecting the uncertainty about future dividends, or investor perceptions of those dividends (loosely referred to as "investor confidence or uneasiness"); and differences among investors with respect to their valuation of the different segments of the dividend distribution. In this paper, we focus on the second of these and analyze a model of the behavior of stock prices when dividend uncertainty changes over time.

The representative investor in our model faces a simple buy and hold portfolio decision. In this way, we isolate the impact of dividend uncertainty on stock prices by ruling out heterogeneity among investors and the effects of the trading mechanism. Specifications for risky asset returns, interest rates, and thus equilibrium risk premiums, are endogenous to the changes in dividend uncertainty, and are derived from an equilibrium model in the Lucas (1978), Brock (1982), Cox-Ingersoll-Ross (1985a) mould.

Our analysis is motivated in part by the cumulation of evidence that stock return volatilities and risk premiums vary over time, e.g. Mandelbrot (1963), Rosenberg (1972), Merton (1980), Pindyck (1984), and Bollerslev (1987), among others. Further work by Breeden (1979), Malkiel (1979), Chen, Roll, and Ross (1984), Keim and Stambaugh (1984), and Bernstein (1985a,b) has suggested that these variations tend to be related to movements in dividend yields and junk bond yield spreads, though perhaps not always in the same way.
There is also a reason to suspect that risk-premium and volatility changes are not random over time. We computed the Brock, Dechert, and Scheinkman (BDS) (1986) portmanteau "W statistic," which is asymptotically unit normal under the hypothesis of return independence, to be 7.232 and 4.385 for the equal-weighted and value-weighted NYSE market over the period January 1926 to December 1985. These statistics are a quite general indication that there is a time series structure in monthly market returns which remains after linear dependence is removed.¹

At the same time, brute force empirical methods seem unlikely to shed much light on the nature of these return volatility and risk premium changes. For example, French, Schwert and Stambaugh (1987) fit an ARIMA model to time series of market volatility estimates and then express market risk premium movements as a linear function of estimates of the predictable volatility changes implied by that model. However, even ignoring the problems of inferring the time series properties of volatilities from a time series of their estimates,² the latter step makes sense only if the volatility changes are independently distributed through time, in which case the ARIMA model is trivial, or if investors have time-additive logarithmic utility, in which case the ratio of risk premium to volatility is already known to be unity. In fact, there is in general a potential

¹The statistics were computed after removing linear dependence, and thus the test is conditional on the model (an AR(12)) used for that removal. Simple interocular analysis of the history of U.S. market returns since 1926 suggests that volatilities in the 1930s and, to a lesser extent, the early 1970s, were much higher than in the intervening period. Officer (1971) presented more formal tests for the differences in volatility. Over the period 1926.1 to 1941.1, the W statistic is 4.38 and 7.23 for the value-weighted and equal-weighted index; from 1942.1 to 1983.12, it is 1.07 and 2.45.

²See, for example, Ansley (1980) and Pagan and Ullah (1987) for a discussion.
inconsistency between ARCH-m models and equilibrium asset pricing models (e.g. Brock (1986)).

Our model, which is developed in Section II, indicates the potential ways in which changes in economic uncertainty can leave their "footprint" in the time series properties of asset returns at the market level. The equilibrium risk premium is in general nonlinear in the uncertainty "factor," and in general surprise changes in the uncertainty cause both a revision in future risk premiums and a contemporaneous change in stock prices and hence realized risk premiums. In the absence of leverage effects, an increase (decrease) in the volatility of asset payoffs will, if the level of payoffs is unaffected, cause a simultaneous decrease (increase) in asset prices\(^3\) if the market is less (more) risk averse than a logarithmic investor. Such a result was subsequently obtained in independent comparative-static analysis by Barsky (1986) in a two-period model, and Abel (1987) in a discrete time, infinite horizon model which is similar to our's. However, for all levels of risk aversion, if increases in uncertainty adversely impact the level of dividends, or if leverage effects are important, stock prices may decrease in response to increased uncertainty.

In Section III, we examine how leverage affects the response of risk premiums on stocks to changes in economic uncertainty in our model. When the value of levered equity is contingent upon both the all-equity value of assets and the endogenous level of interest rates, we find that equity values can decrease in response to increases in uncertainty about the underlying asset returns. The direction of this response occurs across the

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\(^3\)As envisaged in the constant-elasticity-of-variance (CEV) model (e.g. Black (1976) and Cox (1975)).
range of empirically observed leverage ratios. We also argue in this
section that earlier empirical evidence regarding the positive relation
between junk bond returns and equity risk premiums is consistent with the
comovements between dividend volatility and all-equity asset firm value
such as those contemplated in our model. In fact, we show that the
relation between equity risk premium and junk bond yield spreads in our
equilibrium model—in which interest rates and asset values as well as
yields are endogenous—is positive and monotonically increasing.

In Section IV, we estimate our model by minimizing the
discrepancies between the return moments predicted by simulations of the
model and corresponding moments computed from NYSE value-weighted index
returns from January 1926 to December 1985. This method-of-simulated-
moments (MSM) which we apply was developed by McFadden (1989), Pakes
(1986), and Pakes and Pollard (1986). Using the parameter estimates so
obtained, we investigate the behavior of risk premiums when asset
volatility changes. The price-to-dividend ratio turns out to be nonlinear
in asset return volatility. The equity risk premium is a nonlinear
function of return volatility as measured by the standard deviation of
returns, but is reasonably close to linear with respect to the return
variance. Measurement problems aside, we suggest that this last result
provides some justification for one of the GARCH-M model specifications
most frequently put forward for market risk premiums. In our structural
model, the risk premium is estimated to be about 1.8 times return variance,
and averages about 0.8 times return volatility measured by standard
deviation of returns.

We also show in Section IV that if returns are generated by our
model with the parameters set equal to their simulated moment estimates,
dividend-price ratios have about a 4% $R^2$ in "predicting" subsequent returns. The fit of the "predictive" equation is about the same as that observed empirically by Shiller (1981) and Campbell and Shiller (1988). Moreover, the negative autocorrelation in long-run returns in our model, induced by the regressivity in dividend uncertainty, is observationally equivalent to that produced by "fads"-type models (e.g. Summers (1986)), albeit investors behave rationally in our model. Finally, we show that the stock returns produced by our model with parameters set equal to the simulated moment estimates are about twice as variable as dividends (or price variability in a lognormal world), thus presenting one more means of explaining Mehra and Prescott's (1985) finding that a constant output variability is insufficient to justify historical differences between U.S. equity returns and interest rates simultaneously with the level of interest rates.¹

¹Mehra and Prescott (1985) themselves indicated that their conclusion was potentially sensitive to assumed output (consumption) variability: "the average equity premium was roughly proportionate to [that variability] squared" (p.156).
II. Endogenous Stock Price Dynamics

The economy

In this section, we derive an empirically testable model of asset prices with stochastic risk premiums. It is a general equilibrium model in the spirit of Lucas (1978), Brock (1982), and Cox, Ingersoll and Ross (1985a), in that the levels of asset prices depend upon the structure of the uncertainty. This endogeneity of asset prices differentiates the model from the earlier partial equilibrium models (e.g. Merton (1971)) in which asset prices are given and the implications of a change in risk for equilibrium risk premiums are derived.

We will consider a one-good pure exchange economy with identical, competitive agents. Real investments are assumed to be fixed and production proceeds to accrue continuously. Agents can continuously trade shares of the "harvest" and borrow or lend risklessly. Asset prices respond to accruals of information concerning the level of production at date $t$. Agents are able to continuously observe the growth of production until the current date $t$. The economy is defined by the probability distribution of the amount of the good available for consumption at all future dates $t$, by agents' preferences and by the set of admissible trading strategies.

By analogy with discounted dividend models of asset prices we will refer to the stochastic amount of real output available at date $T$ as the "dividend", $D_T$. Assume that the distribution of the dividend at date $T$ conditional on all the information available to agents as of the current date (date $t$) is fully characterized by a set of state variables $Y_t$. 
Agents seek to maximize the expected utility of lifetime consumption:

\[ E_t \left( \int_t^\infty U(C(s),s) \, ds \right) \]

where \( U \) is a Von Neuman-Morgenstern utility function and \( C(s) \) the consumption flow at date \( s \). The utility function \( U \) is assumed to be increasing, strictly concave and twice differentiable. \( E_t \) denotes the expectation operator, conditional on all information available at date \( t \).

Following Lucas (1978), an equilibrium in this economy is defined by a set of consumption choices, trading strategies, prices and price expectations. Agents observe asset prices and select their investment strategy on the basis of their view of the stochastic process for asset prices. Market clearing requires that each agent consumes an equal share of available aggregate output and holds the same equal share of each of the traded securities. The requirement that agents' expectations be rational, i.e. that the resulting price structure coincides with their beliefs, closes the model. The approach adopted by Lucas (1978) to solve for equilibrium prices was to specify the set of trading strategies and agents' price expectations and to impose the market clearing and rational expectations conditions.

Since any agent holds an equal share of the claims, it must also be the case that in equilibrium each agent consumes an equal share of the aggregate dividend at all dates. Since the distribution of future dividends is exogeneously specified, the derivation of the rates of intertemporal substitution is immediate and the Euler equation yields the equilibrium price of any asset. Consequently, we will obtain the following
expression for the equilibrium price of a claim \( X \) to a stream \( X_t \), \( t \leq T \leq \infty \), of the consumption good per unit time\(^5\):

\[
\rho_t(X) = E_t\left( \int_t^\infty \frac{X_T U'(D_T)}{U'(D_t)} \, dT \right). \tag{II.1}
\]

Equation II.1 gives the price of a claim on real production when \( X_t \) is equal to \( D_t \).

We now proceed to construct the equilibrium, and show that the price of a claim on real production is indeed given by equation II.1. We assume that changes in the state of the production process are described by the following subordinated process:

\[
dD_t = D_t \, m \, dt + D_t \, /\sigma_t \, dZ_t, \tag{II.2}
\]

where changes in the instantaneous variance \( \sigma_t \) are given by:

\[
d\sigma_t = \beta(\sigma_t - \bar{\sigma}) \, dt + /\sigma_t \, \alpha \, dZ_\alpha \tag{II.3}.
\]

The instantaneous correlation coefficient between the Brownian motions \( dZ_t \) and \( dZ_\alpha \) is a constant, \( \pi \).

In (II.2), the instantaneous uncertainty about percentage changes in production, \( \sigma_t \), itself changes over time in the manner prescribed in (II.3)\(^6\). These changes in uncertainty could be considered changes in the

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\(^5\)Provided that the integral converges.

\(^6\)Of course, the diffusion term for cash flows \( D_t \) in (II.2) could in general be made a function of a vector of state variables as in Cox, Ingersoll, and Ross (1985a).
time scale which maps clock time into an operational time in units of which the dividend process has a constant variance (e.g. Clark (1973), Stock (1983)). It seems economically sensible to make the parameter $\sigma_i$ mean-reverting while allowing the level of production to be nonstationary.

The specification (II.2, II.3) could be fleshed out in the way suggested by Brock (1982) to include firm-level decisions about real production levels, and, in that context, the market-wide uncertainty in asset returns might intuitively be considered to arise from exogenous shocks to demand or to the economy’s capital-labor ratio (cf. Merton (1975) where they are attributable to a stochastic labor supply). Alternatively, $D_i$ might be regarded not as the dividend process per se, but rather as investor perceptions concerning that process, as a function of available information. In this interpretation, $\pi$ would become a measure of dependence between changes in investor "uneasiness" or "confidence" about the state of the economy and current perceptions, while II.3 would be a stipulation that "uneasiness" is resolved in an autoregressive fashion. At present, our hope is that, except for leverage effects like those considered in Section III, (II.2) and (II.3) will prove rich enough, and yet parsimonious enough, to enable us to isolate the effects of shifts in uncertainty along the lines entertained by (say) Black (1976).

**Equilibrium prices and interest rates**

At each instant, agents choose the share of wealth $\omega$ invested in real production purchased and their level of consumption, $C$. The balance of their wealth is invested risklessly at a rate $r$ which is endogenously determined in equilibrium. This dynamic programing optimization problem can
be restated in terms of the indirect utility function $J$,

$$J(W, Y, t) = \max_{C, \omega} E_t \left( \int_t^\infty U(C(s), s) \, ds \right) \quad (II.4)$$

As in Merton (1971), $J$ is the expected utility of wealth, given the current state of the economy characterized by the set of state variables $Y$. $J$ is the expected utility of lifetime consumption maximized over the set of feasible consumption streams and investment decision (given by $C$ and $\omega$) at every future date. The probability distribution of future wealth, $W$, and hence future consumption, depends on the the state variables $Y$ which determine the expected return on investment and the variance-covariance matrix of returns and their evolution through time. The vector of state variables $Y$ consists of $D$ and $\sigma$ in the present case. The current market value of real production, $V_t$, is endogenously determined in equilibrium. Since the values for wealth, for the dividend, and for the instantaneous variance $\sigma$, determine the state of the economy, the current value of production $V_t$ is in general a function of the three state variables. We will assume that $V_t$ is given by $V(\sigma, D)$, where $V$ is a twice differentiable function. The nature of the dividend stochastic process allows us a first simplification. The relative change in the dividend is independent of the dividend level. Consequently if the dividend doubles today, all future dividends are multiplied by a factor of two. We can thus reasonably expect the value process $V_t$ to be of the form $D_t \cdot V(\sigma_t)$ and the indirect utility function to depend only on $W_t$ and $\sigma_t$.

We now proceed to derive the equilibrium price, $V_t$, of real production. To obtain $V_t$, we solve the optimization problem where the choice variables are the level of consumption and the amount invested in the risky asset. Since riskless lending is in zero aggregate supply, the riskless rate is determined by the equilibrium in the market for the risky
asset. In fact, the equilibrium in that market does not depend on whether financial claims are traded or not. The price of any financial claim will result from the equilibrium in the real economy. We show below how to solve for the equilibrium price of the risky asset directly, but we start by assuming that riskless lending is available so as to obtain the instantaneous riskless rate r. The process for wealth, W, is then:

\[ dW = W \left[ \omega \left( \frac{dV}{V} + \frac{D}{V} dt - r dt \right) + r dt \right] - C dt \]

The capital gain on the risky asset, dV/V, is a function of the rate of dividend growth specified by (II.2) and of the rate of change in the price to dividend ratio dv/v which will be determined in equilibrium, v being a function of σ, dv/v can be determined as a function of the stochastic process for v with respect to σ by applying Ito's lemma:

\[ \frac{dV}{V} = (m + \frac{v}{v} (\beta (\sigma, \sigma) + \alpha \sigma) + \frac{1}{2} \frac{v}{v} v \sigma^2 \sigma) dt + \sigma dz_1 + \frac{v}{v} v \sigma \, \sigma \, dz_2 \] (II.5)

To simplify our notation, we denote the expected return on the risky asset not inclusive of the dividend payout (the expected capital gain E(dV/V)) as μ and the instantaneous variance of the return, as Σ. The explicit functional form of these parameters in the above equation lets us examine the effects of changes in the expected profitability of capital --μ-- in our model-- and changes in the instability of the economy --σ in our model-- on required equity returns and the market value of capital. Malkiel (1979) considered both these factors as candidate explanations for increases in U.S. equity premiums in the 1970's.

The theorem of dynamic programming yields the equation

\[ U(C^*) + L(J) = U(C^*) + J + J \left[ W \left( \omega (\mu - r + 1) + r \right) - C^* \right] + J \beta (\sigma - \sigma \cdot r) \]

---

7See Cox, Ingersoll and Ross (1985a).
\[ t \quad W \quad v \quad \sigma = \infty \]
\[ + \frac{1}{2} \omega^2 J_{W^2} w^2 \Sigma + \omega J_{W \sigma} W \Sigma_{V \sigma} + \frac{1}{2} J_{\sigma \sigma} \sigma \alpha^2 = 0 \]

at the optimum. \( \Sigma_{V \sigma} \) denotes the covariance between the instantaneous return on the risky asset and the instantaneous change in the variance of the dividend growth rate, \( d\sigma \).

The first order conditions are
\[ U_C(C^*(t), t) = J_W \quad \text{(envelope condition, II.7)} \]
\[ J_{W}(\mu - r + \frac{1}{v}) + J_{W\sigma} W \Sigma_{V\sigma} + \omega J_{WW} W^2 \Sigma^2 = 0 \quad \text{(II.8)} \]

In an economy with a representative agent, equilibrium borrowing is zero and the agent holds the risky asset. Furthermore, the agent consumes the exogenously specified dividend and his wealth is equal to the value of the risky asset. With these restrictions, equations II.6 and II.7 characterize the functional form of \( J \) and \( v \) and equation II.8 defines the interest rate which prevails in equilibrium. To render our model tractable and useful for empirical inference, we will make the customary assumption that the utility function is isoelastic and that the rate of time preference is constant,
\[ U(C,t) = e^{-\phi t} \frac{C^{1-\gamma}}{1-\gamma} \quad 0 \leq \gamma \leq \infty \]
where $\phi$ is the rate of time preference and $\tau$ is the coefficient of relative risk aversion. In this case, it is easily shown\(^8\) that the indirect utility function $J$ takes the simple form\(^9\)

$$J(W, \sigma, t) = e^{-\phi t} \frac{W^{1-\tau}}{1-\tau} j(\sigma).$$

The system of equations then becomes\(^10\)

$$v(\sigma) = \left[ j(\sigma) \right]^T$$

$$\frac{1}{2} \sigma^2 v_{\sigma \sigma} + [(1-\tau) \alpha \sigma + \beta(\sigma - \sigma_0)] v_{\sigma} + \left[ \frac{1}{2} (\tau - 1) \sigma + (1-\tau) m - \phi \right] v + 1 = 0 \quad (II.10)$$

$$r = \phi + m \tau - \frac{1}{2} (1+\tau) \sigma \sigma$$

(II.11)\(^{11}\)

Equation II.9 shows that the price to dividend ratio, $v$, --the inverse of the payout rate-- is directly related to agents' preferences. This equation in conjunction with II.6 yields the pricing equation II.10. The solution of the differential equation is the price to dividend ratio which prevails in equilibrium. Its functional form determines the instantaneous expected return on the risky asset and its return variance. It depends on the stochastic process for the instantaneous variance of the dividend rate $\sigma$ --through the parameters $\alpha$, $\beta$, $\sigma_0$, and $\pi$-- and on the representative

\(^{8}\)Hakansson (1970), Merton (1971), and Samuelson (1969) demonstrate this in a different context.

\(^{9}\)The function $j$ does not depend on time because the horizon is infinite and the stochastic processes are independent of time.

\(^{10}\)Equation II.9 is obtained by substituting the utility functions $U$ and $J$ in the envelope condition (II.7). Equations II.10 and II.11 then follow from II.6 and II.8 combined with II.9.
agent's preferences --through the relative risk aversion coefficient $\tau$ and the rate of time preference $\phi"$.

Alternatively, we can obtain the differential equation (II.10) for $v$ more directly using the relationship between $v$ and $j$ (in II.9) and the definition of the indirect utility function $J:\n v_t = E_t\{ \int_0^\infty e^{-\phi s} \frac{D_{t+s}}{D_t^1-\tau} ds \}.$ (II.12)

The price to dividend ratio is thus simply equal to the expected utility of future consumption divided by the utility of current consumption. Note that equation II.1 specializes to II.12 when agents' preferences are isoelastic. $v$, does not depend on the current dividend level, justifying the functional form of the equilibrium price $V_t = D_t v(\sigma_t)$ assumed above.

Equation II.12 is the integral form of II.10 and thus provides the boundary conditions. Equation II.12 leads to an interesting result concerning the form of the function $v(\sigma)$. If the representative investor is less risk averse than a logarithmic investor ($\tau<1$), $v$, is the expectation of a concave function of future dividends. In this case, an increase in the variance $\sigma$ would trigger an immediate decrease in the asset price, provided that the dividend level is unchanged. For higher degrees of risk aversion ($\tau>1$), however, the asset price increases if the variance of the dividend process increases. This might at first seem counter to the

"A similar differential equation arises for any dividend variance process; we retained the process defined in II.2 only to simplify our empirical analysis. This result also applies for any utility function and dividend process. The choice of a utility function in the Hyperbolic Absolute Risk Aversion (HARA) class is not very restrictive but simplifies the equations. And, if the dividend growth rate depended on the dividend level, the differential equation would become a partial differential equation with respect to $\sigma$ and $D$.\)
intuition that "if there is more risk to be borne, assuming that the expected payoffs from business investments don't change, then stock prices must fall, so that investors will continue to hold the existing stocks" (Black (1976, p.179)). For $\tau$ greater than one, however, an increase in the uncertainty of future consumption induces risk averse investors to demand a higher level of expected future consumption and hence to bid up the price of the risky asset".

Previous empirical research suggests that the representative investor is more risk averse than the logarithmic investor. Gennette and Marsh (1985) test the CEV model and conclude that the variance of the return on the market portfolio was actually inversely related to the level of the market over the period 1925 to 1978. This would occur in our framework if an increase in the volatility of dividends was on average associated with a decrease in the level of expected dividends (i.e. the correlation, $\pi$, between the "surprises" in $D$ and $\sigma$ is strictly negative).

In equation II.11, the instantaneous riskless interest rate is expressed as a function of the parameters of the dividend and price processes. The risk premium, which is the difference between $E(dV/V)$ in (II.5) and $r$ in (II.11) plus the dividend payout rate, is also equal to the covariance between the rate of change in aggregate consumption --the rate of dividend change in our model-- and the return on the risky asset multiplied by the coefficient of relative risk aversion,

$$\mu + \frac{1}{V} - r = \tau \text{Cov} \left( \frac{dD}{D}, \frac{dV}{V} \right) = \tau (\sigma + \frac{V}{V} \sigma \alpha \pi).$$

(II.13)

\footnote{As noted earlier, Barsky (1986) and Abel (1987) have obtained a similar result.}
Hence, the equilibrium risk premium is a complicated function of the variance of the return on the risky asset. It is well known that if preferences are logarithmic (which correspond to the limit of $\tau$ at one) the indirect utility function does not depend on the state variables. Consequently, $\nu$ is a constant in this case (equation II.9), the variance of the return on the asset reduces to the variance of the dividend growth rate $\sigma$, and the risk premium is equal to the variance of the return on the risky asset. Another interesting special case is the familiar lognormal process for asset prices. If dividends are lognormally distributed, that is if $m_\tau$ and $\sigma_\tau$, the instantaneous mean and variance of the growth in dividends, are constant ($\alpha=\beta=0$), the risky asset price is lognormally distributed as well.

Integrating II.12 directly yields\(^\text{13}\)

\[
V_t = D_t v_t = \frac{D_t}{\phi - (1-\tau) m + \frac{1}{2} \tau (1-\tau) \sigma}.
\]  

(II.14)

The instantaneous variance of the return on the asset is equal to the variance of the dividend growth rate. The expected return (cum dividend) becomes

\[
\mu + \frac{1}{v} = \phi + \tau m + \frac{1}{2} \tau (1-\tau) \sigma
\]

The interest rate (II.11) and the risk premium (II.13) are, respectively,

\[
r = \phi + \tau m - \frac{1}{2} \tau (\tau+1) \sigma
\]

and

\[
\mu + \frac{1}{v} - r = \tau \sigma.
\]

\(^{13}\)The integral converges provided that the the denominator of (II.14) is strictly positive.
An increase in the variance causes an increase in the risky asset price if the coefficient of risk aversion \( \tau \) is greater than one. Since the expected dividend payout is unaffected by changes in risk, the expected return then decreases. The interest rate, however, decreases for any value of \( \tau \), leading to an increase in the risk premium\(^6\). The risk premium is a linear function of the instantaneous variance, the proportionality factor being the coefficient of relative risk aversion\(^5\).

Comparative statics exercises for the lognormal case are of limited interest, however, because the functional form derived for \( V \), in II.14 holds only if the variance is constant. Hence, a change in risk induces a proportional change in the risk premium only if investors believe that the variance is constant and an unforecasted change in regime occurs. Such a violation of rational expectations is unrealistic in light of the non-parametric results presented in section II and earlier evidence.

Returning to the stochastic risk environment, the risk premium is the sum of two terms: the "myopic" proportional response to a change in instantaneous dividend risk and the non-linear adjustment to a permanent change in the investment opportunity set (equation II.13). The latter occurs because a risk increase adversely affects the expected utility of future consumption, so risk averse investors seek to hedge changes in risk through their investment in the risky asset. The supply of the risky asset being fixed, however, this demand for hedging against future shifts in the

\(^6\)Barsky (1986) and Campbell (1986) derive similar results in the context of a lognormal model for dividends. Hendershott (1981, p. 910) noted the importance of endogenizing interest rates when examining the impact of risk changes on share values: "... when the debt yield is endogenous, most of the increase in risk premiums will be reflected in a lower debt yield, rather than a higher equity yield and thus lower share values." (Here, for \( \tau > 1 \), equity yields actually decrease, and share values increase, in response to an increase in risk.)

\(^5\)This relationship was empirically investigated by Merton (1980).
investment opportunity set is reflected in the risk premium. Equation (II.8) leads to the following expression for the risk premium as a function of the instantaneous return variance and the hedging demand:

\[ \mu + \frac{1}{\nu} - r = \tau \sigma \left[ 1 + (\alpha_{\nu})^2 + 2\alpha \frac{\nu_{\alpha}}{\nu} \right] - \tau \sigma \left[ (\alpha_{\nu})^2 + \alpha \frac{\nu_{\alpha}}{\nu} \right] \]  

(II.15)

The first term is equal to the coefficient of risk aversion \( \tau \) multiplied by the instantaneous return variance, \( \Sigma \). The return variance is equal to the variance of the dividend growth rate plus two other terms which stem from the change in price directly triggered by the change in \( \sigma \). The second term results from the hedging demand.

It can be seen from II.15 that the linear relationship between the risk premium and the variance of the asset return will hold approximately if dividend uncertainty is fairly constant (small \( \alpha \)) or if changes in risk have little effect on price levels (small \( \nu_{\alpha} \)).

In the absence of compelling a priori evidence that the price level is not affected by changes in risk (i.e. that \( \nu \) does not depend on \( \sigma \))\(^6\), there is an inherent contradiction in estimating risk premium movements as a linear function of movements in return variance: either the linear functional form is approximately correct and the permanent component of risk changes is negligible, or the risk changes are important, but the functional form is incorrect.

It can also be seen from equation II.15 that \( m \), the expected rate of "profitability" of the risky asset, does not affect the equilibrium risk

\(^6\)Equation II.14 shows that even in the lognormal case the derivative of the price level with respect to a change in dividend risk is sizeable.
premium, though it does affect interest rates. Factors such as corporate taxation and regulation which affect only expected profitability will not affect movements in risk premiums. Moreover, if this is true when the supply of the risky asset is fixed, as it is in our analysis, it will be a fortiori true when substitution away from the risky asset is allowed unless (say) differential taxes on asset returns are introduced. Although \( m \) does not affect the risk premium, it does affect the risky asset value, and thus it will affect the equity risk premium when leverage is included.

We have not been able to find a closed form solution for the price function. We have, however, derived the closed form for the price of a pure discount bond (see Appendix A). A simple transformation of this closed form leads to the following expression for \( v_t \):

\[
v_t = \int_t^\infty e^{-\phi_s} e^{[(1-\tau) m + \sigma \beta B_i] s} \frac{2\beta \sigma}{\sigma^2 \bar{e}^2 (1-Z) \sigma} \, ds \tag{II.16}
\]

where \( B_1, B_2 \) and \( Z \) are the following functions:

\[
B_1 = \frac{(\beta+(\tau-1)\alpha \pi) - [((\beta+(\tau-1)\alpha \pi)^2 - \alpha^2 \tau (\tau-1))]^{1/2}}{\alpha^2}
\]

\[
B_2 = \frac{(\beta+(\tau-1)\alpha \pi) + [((\beta+(\tau-1)\alpha \pi)^2 - \alpha^2 \tau (\tau-1))]^{1/2}}{\alpha^2}
\]

Thus, at least at the level of abstraction of our model, its predictions on this point are consistent with Malkiel's assertion that it "... is not so much the direct cost of regulation that has inhibited investment (the price of capital in our model with fixed capital stock) but rather the unpredictability of future regulatory changes".
\[ Z = \frac{B_z - B_i}{B_z - B_j + \frac{1}{2} \alpha^2 (B_z - B_i)s} \]

We numerically integrated the function \( v(\sigma) \) in equation II.16. The result is presented in Figures 2(a) and 2(b) where the function \( v(\sigma) \) is evaluated for alternative values of \( \beta \) and \( \pi \), respectively, and remaining parameters are set equal to their moment estimates given below in Section IV. It can be seen from both figures that \( v \) increases reasonably steeply in \( \sigma \), especially at the estimated values \( \beta = 0.45 \) and \( \pi = -0.14 \). Holding other parameters constant, the rate of increase in \( v \) with respect to \( \sigma \) is higher when \( \beta \) and \( \pi \) are smaller. That is, the more persistent the shocks to dividend volatility, or the higher the inverse correlation between dividend levels and volatility shocks, the bigger the impact of a shock to dividend volatility on the unlevered price-dividend ratio.
III. The Impact of Leverage

In this section, we show that the relationship between economic uncertainty and an asset’s price level depends on whether the asset is a levered claim on the dividend process. The introduction of leverage reconciles one of the empirical implications of our model with the view expressed by Black (1976) that increases in risk tend to lower stock prices and with the empirical evidence in Gennette and Marsh (1985) that a CEV model with an elasticity parameter of approximately 1.6 best fits monthly returns data.

The intuition behind the effect of leverage on equilibrium asset prices is very simple. Inspection of equation II.12 shows that the price of a perpetuity paying a continuous coupon of 1 per unit of time is given by:

\[ P_t = E_t \left( \int_0^\infty e^{-\phi s} \frac{D_t s}{D_t^\tau} ds \right). \]  

(III.1)

Hence the consol bond price is equal to the price to dividend ratio in a world where investors have a relative risk-aversion coefficient equal to \( \tau + 1 \). Consequently, the value of a claim with payoffs \( D_t - 1 \) per unit of time (a crude approximation to levered equity) will decrease with an increase in

---

^18 We thank Vasant Naik for his thoughts on the set of issues discussed in this section.

^19 Black offered leverage as an explanation of the negative correlation he observed between stock price levels and volatility. The mechanism he describes is different from our’s, however. His conjecture, which is the rationale for the Constant Elasticity of Variance (CEV) model, is that a decrease in price causes an increase in the volatility of equity because the option to compensate bondholders becomes less likely to be exercised. In our model, the uncertainty shock affects the levels of riskless bond prices and of equity prices.
risk provided that the stock is sufficiently "out of the money" (i.e. D, not too large relative to 1).

The pricing of claims whose payoffs are contingent on the value of the market portfolio, or of fixed income assets, can be done by arbitrage using the market portfolio and a riskless bond such as the consol bond. This approach has the advantage of incorporating the correlation between the level of the market portfolio and the term structure in the valuation. While we haven't been able to derive a closed-form solution for the price of a call on the market for the general dividend process in Section II,\(^{20}\) we can illustrate the effects of volatility changes on the value of levered equity on the market portfolio for the lognormal case. In this case, the Black and Scholes (1973) analysis can be applied to derive the value of equity, a call option on the risky asset.

In our case, the value of the underlying asset and the riskless rate both depend on the return volatility. The direct effect of an increase in return volatility is to increase the value of the option, but its indirect effects are to increase the value of the underlying asset (the present value of earnings or dividends) and also to reduce the interest rate.

In Fig. 2, we plotted the value of equity as a function of the return volatility of the underlying asset for debt-to-asset value ratios of 20%.

\(^{20}\)The approach described above does, of course, lead to a differential equation analogous to the Black and Scholes heat equation which can be solved numerically. Where the distribution of the dividend level depends on a set of state variables, the general procedure to price a financial asset by arbitrage consists of pricing a number of non-perfectly correlated fixed income assets equal to the number of underlying state variables and then using those and the market portfolio to form the hedge portfolio.
30%, and 40% and a debt maturity of seven years.\textsuperscript{2} To derive the value of equity, we calculated the value of the asset and the interest rate using the estimated parameters of our model and applied the Black-Scholes formula. In Fig. 2, the value of equity was normalized by dividing it by the fixed value of the asset when dividend volatility is 20%. It can be seen that, for all levels of leverage, the value of equity decreases when return volatility increases. When volatility is below 21%, the fall in equity value in response to an increase in volatility is more rapid for higher levels of leverage. When return volatility is above approximately 22%, the sensitivity of equity value to changes in return volatility is about the same irrespective of the degree of leverage.

Intuitively, the value of levered firms decreases with an increase in asset risk because the interest rate effect dominates the usual option effect. The value of equity drops below the exercise value of the option for large levels of risk because the real interest rate becomes negative (for a volatility level of 21.5%).

In this analysis of leverage, neither the price of equity nor the price of debt depend upon the expected return or the ex ante risk premium on the productive asset. In particular, yields on the debt do not depend upon the expected risk premium for the economy's productive assets, \((\mu + 1/v - r)\) in (II.15). Given the contingent claim nature of the debt (and levered equity), this is what we'd expect. However, evidence provided by Breeden (1984) and Keim and Stambaugh (1984) suggests that the yields on highly levered debt, "junk bonds," can in fact be used to predict

\textsuperscript{2}The average maturity of corporate debt in the U.S. over the time period of our sample is of the order of seven years, and the various estimates of the average debt to asset ratio range between 20% and 40%, see Taggart (1985) for example.
subsequent return premiums on equities. How, then, does our analysis
square with this evidence?

By definition, junk bonds can be interpreted as out-of-the money
options on the value of the economy’s productive assets. Variations in
leverage per se can be expected to have little effect on the spreads
between the promised yields on these out-of-the-money options and the
riskless rate (e.g. Merton (1974, Fig. 1)). This leaves us with the
conjecture that the empirical association between spreads on junk bonds--
gua options on corporate cash flows--and risk premiums on corporate
equities reflects: (i) the dependence of corporate bond yields on
corporate cash flow volatilities, together with (ii) comovements between
these cash flow volatilities and the equilibrium equity risk premium.29

In fact, a quick glance at the data supports this reasoning that
corporate cash flow volatility changes are related to movements in both
risk premiums and junk bond yields. For example, visual inspection of
frames one and two in Fig. 3 suggests that prior-period risk premium
squared -- a rough measure of return variance, plotted in frame two, is as
highly correlated with the market risk premium as are junk bond yields.

This correlation between the series plotted in Fig. 3 can be
expressed in terms of an OLS regression. If the realized premium for the
equally weighted NYSE stock portfolio is regressed on the squared
prior-period risk premium and the junk bond yield spread over the interval
February 1926 to December 1978, the result is:

29Even if investors have constant relative risk aversion and their
investment opportunity sets are constant (or they have log utility), then at
the level of "the" market, changes in ex ante premiums will be proportional to
changes in the variance of market returns (e.g. Merton (1980)).
Risk Premium\textsubscript{t} = -0.003 + 0.624 [Risk Premium\textsubscript{t-1}]^2 \\
(-0.006) (0.118) \\
(III.2) \\
+ 1.822 Junk Bond Spread\textsubscript{t-1} + e\textsubscript{t} \\
(1.058) \\
(standard errors in parentheses) \\
R^2 = 6\% \\
DW = 1.76

The estimates in (III.2) tend to confirm the impression gained from Fig. 3 that squared prior-period risk premiums explain risk premiums at least as well as do junk bond yields.

Of course, the point of the analysis in the previous section is that prior-period equity premium squared can at best be interpreted as a crude approximation to return variance; indeed, such simple measures of association between equity volatility and equity risk premiums are fraught with potential errors-in-variables and errors-in-specification problems.\textsuperscript{23} Turning to our model, we find that it does, in fact, predict that the equity risk premium is a monotonic, but nonlinear, function of the junk bond yield spread. This can be seen from Fig. 4 where, using the simulated moments estimates of our model, we computed the relation between the equity risk premium and junk bond yield spread--where a junk bond is defined as a bond with 90% leverage--as the volatility of dividends changes. The relation between equity risk premium and the yield spread is positive,

\textsuperscript{23} French, Schwert, and Stambaugh (1987, p. 11, n.5) point out that if a correction is made for heteroscedasticity in estimating (III.2), the standard error of the squared prior-period risk premium variable roughly doubles to 0.25 (or 0.23). Of course, as we make clear in previous drafts, the regression hardly qualifies as a model; indeed, we've always labelled it a "crude approximation." The issue of whether the t-statistic in such a crude look at the data is really 5.2 or 3.0 seems to us to be unimportant relative to the problems of specification and errors in measuring the stock return variance; indeed, it's not even clear how to interpret the heteroscedasticity correction in the presence of those problems.
which is consistent with the earlier empirical evidence as well as that in (III.2).

In fact, the negative point estimate for the constant in (III.2) already provides a glimpse of the problems in using the squared prior-period risk premium as a proxy for return variance in a linear regression framework: although the point estimates of the coefficients of both independent variables in III.2 are positive, the negative intercept makes it possible for the predicted ex ante risk premium to be negative, which doesn't make much economic sense. We turn now to a detailed explanation of the estimation of the model developed in Section II.
IV. Simulated Moment Estimates of Model Parameters and Empirical Implications

In this section, we present some preliminary results for the fit of the equilibrium model (II.10) to time series of monthly NYSE value-weighted stock returns over the period January 1926-December 1985. A method of simulated moments (MSM) (e.g. McFadden (1989), Pakes (1986), and Pakes and Pollard (1986)) is used to estimate the parameters of the model by matching the moments of the simulated returns which it produces with observed returns.\textsuperscript{24}

There are seven parameters, $\theta' = [m, \beta, \sigma, \alpha, \tau, \phi, \pi]'$ in our model. We have chosen the following ten unconditional moments, $[m_{0}(\theta), \ldots, m_{9}(\theta), \ldots, m_{10}(\theta)]'$ to fit to the data:

\textsuperscript{24}We use MSM because we do not want to fit only the specialized version of our model for which an analytic solution for the distribution of price changes is available. In using a procedure which matches data moments to the moments of the returns produced by our model, we are ignoring some distributional information, viz. the Ito process assumption which was imposed on the underlying shocks to dividends and the uncertainty about dividends. Our parameter estimates may thus be inefficient relative to maximum likelihood estimates, but unfortunately to obtain the discrete-time likelihood function, we would have to numerically solve the forward equation for the probability density of returns—a partial differential equation in two variables $D$, and $\sigma$, whose coefficients involve $v(\sigma)$ which itself must be solved numerically.
where $R_i$ is a monthly rate of return on the market index, and $r_i = R_i - E(R_i)$.

These unconditional moments were chosen because they capture current evidence of regressivity (e.g. Lo and MacKinlay (1988), Fama and French (1988), and Poterba (1986)) and heteroscedasticity (Bollerslev (1987)) in stock returns. An advantage of our approach is that term structure moments can also be used along with stock returns in fitting it, but we do not use these moments in this paper.

The selection of these moments is admittedly ad hoc. We would like to use those moments which provide the most power in discriminating between the returns produced by our model and alternatives such as the geometric
random walk. Only a few of the moments can be interpreted within a systematic framework. For example, we know that second moment tests for serial dependence, such as the autoregression and variance ratio tests, can be expressed in terms of the autocorrelation structure of returns (e.g. Lo and MacKinlay (1988, p. 48); Richardson (1988)) and hence can be nested together. However, we have not been able to derive any such unifying representation for all the moments in (IV.1).

Estimation of \( \theta \) involves minimizing the distance between the model moments \( m_k(\theta) \), \( k=1,...,10 \), and the respective moments computed from the data. The method becomes one of simulated moments here because, in the absence of analytical expressions for \( m_k(\theta) \), we approximate them by simulation. As can be verified by reference to the adaptation of McFadden (1989), which is briefly described in the Appendix B, MSM estimation of \( \theta \) here amounts to setting a linear combination of the discrepancies between actual and model moments to zero\(^{25}\). To derive the respective moments of monthly returns implied by the continuous time model, we simulated it by making 100 draws of the innovations \( dZ(1) \) and \( dZ(2) \) per month for 120,000 months.\(^{26}\)

---

\(^{25}\)Gibbons and Ramaswamy (1988) have independently used an unconditional moment matching procedure to fit the Cox-Ingersoll-Ross (1985b) model for interest rates.

\(^{26}\)We verified that, at least when our returns model is restricted to the lognormal special case, .01 of a month is close enough to an "infinitesimal" interval to produce returns whose first two moments accord with those of the (theoretical) continuous lognormal distribution. A random search algorithm with fixed step size was used to minimize the distance between simulated and actual moments with respect to the parameters \( \phi, \gamma, \ldots \). We also experimented with alternative step sizes as an informal substitute for an adaptive step size search (e.g. Schumer and Steiglitz (1968)) to assure ourselves that we were indeed at a minimum.
The MSM estimate of \( \tau \), the coefficient of relative risk aversion, is -1.19. This value of \( \tau \) would, in an unlevered economy, imply that the representative investor is more risk averse than the logarithmic investor, and thus that, if dividends are held constant, stock prices rise on average in response to an increase in uncertainty. However, consistent with previous studies, the estimate of \( \tau \) is not sharp--its standard error is 1.065. Also, the estimate of \( \pi \) is -0.137 with a standard error of 0.072. That is, a higher level of dividend (cash flow) uncertainty is on average associated with lower dividends. As noted in Section II, such a negative correlation is consistent with results we obtained in fitting the CEV model to this returns series.

The parameters \( \beta, \sigma_{\alpha}, \) and \( \alpha \) determine the instantaneous volatility of dividends. Their point estimates and standard errors are, respectively, 0.451 (0.215), 0.038 (0.017), and 0.158 (0.051). Parameters are stated in annualized units, so the 0.038 estimate for \( \sigma_{\alpha} \), for example, is approximately 20% per annum. The point estimate of \( \beta \) implies a half life for regressivity of the dividend uncertainty parameter of approximately 1.54 years, while the half life would be 33 years (0.78 years) if \( \beta \) were two standard errors below (above) its point estimate.

The variance of cum-dividend returns \( \Sigma \), given in equation II.5, equals

\[
\Sigma = \sigma \left[ 1 + \left( \frac{\nu}{\sigma_v} \right)^2 + 2\alpha \frac{\nu}{\pi_v} \right]
\]
Evaluated at the MSM parameter estimates just given, this variance is roughly twice \( \sigma \), the variance of dividends which would be the same as the variance of returns if they were lognormal. Thus, simple as our stripped-down model may be, it constitutes yet another candidate explanation for at least a portion of the "excess volatility" of stock returns discussed by LeRoy and Porter (1981) and Shiller (1981).\(^{27}\)

As stated in the introduction, our analysis of the model developed in Section II is in part motivated by our interest in the potential characteristics of movements in stock market risk premiums and return volatilities. Equation (II.13) gives the risk premium as a nonlinear function of the state variable \( \sigma \) in our model. Using the MSM estimates of the model's parameters, we calculated the equity risk premium as a function of the return volatility, which is approximately the function that others have tried to fit if the error in measuring volatility is ignored.

\(^{27}\)Unfortunately the MSM procedure does not per se produce an estimated time path of dividend uncertainty (\( \sigma \)) over the sample period. Though beyond the scope of this paper, such an estimate could be obtained, still without requiring the observability of dividends, as we now briefly discuss. In our model where \( V = D, v(\sigma) \), it follows that:

\[
\frac{dV}{V} = \frac{dv}{v} + \frac{dD}{D} + \frac{1}{2} \frac{E(dv \cdot dD)}{vD}
\]

where the function \( v(\sigma) \) is known. Thus, when \( D \) and \( v(\sigma) \) are unobservable, we can think intuitively of \( dD/D \) as "noise" and \( dv/v \) as the "signal" (actually a function of the signal \( d\sigma/\sigma \)) that we would like to extract from \( dV/V \). Estimation of the trajectory for \( (\sigma) \) becomes, then, one of nonlinear filtering where the variance of the noise process \( dD/D \) depends on the signal \( d\sigma/\sigma \). Of course, just as in the MSM estimation above, this filtering approach invokes the certainty equivalence principle. It can be be justified rigorously in our continuous time diffusion context (intuitively anything other than linear-quadratic terms are higher order in \( t \)). That is, the continuous time limit provides a reference point for the discrete time approximation which is just as compelling as (say) the linear quadratic approximation made directly in Kydland and Prescott (1982), or an implicit approximation made in the initial specification of the discrete-time model.
The result is plotted in Fig. 5 (a) and (b) where risk is measured by the standard deviation and the variance of returns, respectively. The risk premium is convex with respect to the return volatility, but is reasonably linear with respect to return variance. In fact, the function \( \nu(\sigma) \) can be fit quite well by the exponential \( K \cdot e^{2.66\sigma} \) over the range of \( \sigma \) which is plotted. Leverage aside, this result provides some justification, at the level of the market index, for the usual GARCH-M model assumption that the risk premium is linear in return variance.

The ratio of risk premium to return variance over the entire range of variance considered in Fig. 5(b) is 1.86. Over the lower half of the range of variance, the ratio is 1.87; over the upper half, 1.85. The ratio would be equal to the slope coefficient in a linear regression of noiseless observations of ex ante market risk premium on noiseless observations of volatility.\(^{28}\) It can be compared to the values of 1.51-2.03 which Merton (1980) obtained in his study of movements in market risk premiums. The average ratio of risk premium to return standard deviation is 0.73, and 0.53 (0.85) in the lower (upper) half of the range of return volatility plotted in Fig. 5.

Our model provides one means of explaining the association between stock returns and lagged dividend yields reported by Shiller (1981),

\(^{28}\)Of course, the \( R^2 \) on empirical regressions of realized risk premiums on proxies are generally quite low, but that is to be expected both because ex ante risk premiums are difficult to measure and because there will be errors-in-variables attenuation due to errors in measuring the changing volatility over time.
Campbell and Shiller (1988), and others. We simulated our model with the parameters set equal to the MSM estimates in order to produce a time series of monthly "model" returns. These model returns were then regressed on the model's lagged dividend-price ratio. The result is:

\[
\frac{V_{t+1}+D_{t+1}}{V_t} - 1 = 0.922 + 17.471 \frac{D_t}{V_{t-1}} + \epsilon_t, \quad (\text{IV.2})
\]

\[
R^2 = 0.039 \quad \text{(standard errors in parentheses)}
\]

where \(D_{t+1}\) is the cumulated month \(t+1\) dividend and \(V_t\) is the beginning-of-month \(t\) price. Shiller (1981, p. 433) reported a slope coefficient of 3.533 (t-statistic of 2.67), and \(R^2\) of about 0.06 when (IV.2) is fitted to annual data. This compares with the t-statistic of 4.89 and \(R^2\) of 0.039 for this regression. Thus, in spite of the fact that our equilibrium model is hardly a blueprint of "the real world," it offers a relatively simple explanation of why dividend-yield ratios might have the observed degree of power in explaining subsequent index returns.

Of course, since the month-\(t\) dividend-price ratio (equal to \(1/V\) in our model) and expected month \(t+1\) return (which depends on the interest rate, market risk premium, and \(V\)) are both functions of the state variable \(\sigma\), higher \(D/P\) ratios do not "cause" higher subsequent returns--one could just as validly think of the \(D/P\) ratio "responding" to changes in the expected future paths of \(r\), \(\mu\), etc. A model such as ours implies (nonlinear) restrictions across the jointly endogenous interest rate, dividend-yield, and risk premium variables which are in principle testable.
We refrain from introducing observed aggregate dividend data to carry out such tests here because there is a wide gap between measured dividend data and the concept of dividends in our model, and attempting to bridge the gap would take us far afield of the objective of this paper.

Finally, we briefly examine the impact of the estimated mean reversion in dividend volatility in our model on the serial dependence in stock returns. This mean reversion induces a negative serial dependence in the returns. Thus, our model, which is based on rational investor behavior, is potentially observationally equivalent to "fads" models (e.g. Summers (1986)), insofar as serial dependencies in returns are concerned. Our model produces a variance ratio for two, three, six, and twelve month returns of 0.97, 0.96, 0.91, and 0.69, respectively. The variance ratios are less than one and decline monotonically as the interval length increases. The implied negative autocorrelation is a feature of the long-run NYSE index returns which were used to derive the MSM estimates of the parameters (e.g. Fama and French (1988)). However, it is stronger than that in the historical returns data—for example, the autocorrelation of the one-month model returns using the MSM estimates is approximately -3%, whereas the univariate reduced-form estimate of this one-month autocorrelation is about zero.\textsuperscript{29} Also, the monotonicity would not be

\textsuperscript{29}Some readers may wonder why the variance ratios computed from our simulated returns are not closer to the variance ratios computed by others for actual index return data. It must be borne in mind that no matter how long the series of simulated returns, they are still a function of the MSM (point) estimates of the parameters in our model. Moreover, even though several "time series" moments were included among those used to fit our model, these long-run serial dependence moments themselves tend to have a substantial variance and thus are downweighted by MSM. The net result is that the standard errors of parameter-based measures like the variance ratio are likely to be quite high.
consistent with the "U-shape" in negative autocorrelation estimates as a function of return interval reported by Fama and French (1988), although Richardson's (1988) point is that the U-pattern shouldn't be interpreted too aggressively.
V. Conclusion

Our objective in this paper has been to analyze the nature of the response of risk premiums on capital assets to changes over time in the uncertainty about the cash flows which the assets will generate. We show that, in a general equilibrium framework, the market risk premium is a complicated function of the cash flow uncertainty, implying that the simple regression and time series fits of the relation between equity risk premiums and asset price volatility are likely to be misspecified. Further, even though our equilibrium model is a simple exchange model, it suggests that stock price volatility exceeds dividend volatility, that long-run returns can be negatively correlated, and that dividend-price ratios could appear to predict subsequent stock returns. These are all stylized facts that have been documented in recent years. Finally, in a levered economy, our approach provides a rationale for the empirical association between equity risk premiums and junk bond yields.

Since our model is developed in an exchange economy, changes in investment demands in response to changes in uncertainty about dividends affect the prices of assets, not their quantities. Probably the most fruitful extension of our analysis would be to incorporate an endogenous supply of assets by incorporating a production technology, e.g. along the lines of Kydland and Prescott (1982). Nevertheless, it seems that such extensions would underscore, rather than invalidate, our general points.
APPENDIX A: DERIVATION OF THE $v$ FUNCTION

The integral in equation II.12 is absolutely convergent, hence we can rewrite II.12 as:

$$ v_t = \int_0^\infty E_t\left( e^{-\phi s} \frac{D_t^{1-\tau}}{D_t^{1-\tau}} \right) ds. \quad (A.1) $$

We noted in section III that the price to dividend ratio is equal to the price of a console bond with unit payments per unit of time in an economy where the risk aversion coefficient is $\tau - 1$, instead of $\tau$. Here we derive $v$ by integrating the prices at date $t$ of pure discount bond associated with a risk aversion coefficient of $\tau - 1$ for maturities between zero and infinity. To derive the price of a pure discount bond, and subsequently $A.1$, we will first evaluate the function:

$$ g(t,D,\sigma) = E_t(D_t) $$

$g(t,D,\sigma)$ is a martingale, hence it verifies the following equation:

$$ g_t + g_{Dm} + \frac{1}{2} g_{D\sigma \sigma} D^2 + g_{D\sigma} D \sigma \Pi + g_{\sigma} \beta(\sigma, -\sigma) + \frac{1}{2} g_{\sigma \sigma} \sigma^2 \sigma = 0. $$

The solution $g$ is homogeneous of degree $k$ in $D$: $g(t,D,\sigma) = D^k h(t,\sigma)$. The function $h(t,\sigma)$ verifies the equation:

$$ \frac{1}{2} h_{\sigma \sigma} \sigma^2 \sigma + h_{\sigma} [k \sigma \Pi + \beta(\sigma, -\sigma)] + h \left[ \tau m + \frac{1}{2} k(1-k) \right] + h = 0. $$

We will derive a solution of the form: $h(t,\sigma) = \exp(A(t)+B(t)\sigma)$

By substitution in the differential equation, we obtain:

$$ A'(t) = -\beta \sigma \sigma B(t) - \tau m \quad \text{and,} $$

$$ \frac{1}{2} B^2(t) \sigma^2 + B(t) [k \sigma \Pi - \beta] + \frac{1}{2} k(1-k) + B'(t) = 0. $$
We then solve the second equation subject to the boundary condition $B(T)=0$, please refer to Murphy (1960), p229. Once $B(t)$ is derived, $A(t)$ is obtained by integration subject to the boundary condition $A(T)=0$. The price at date $t$ of a pure discount bond maturing at date $T$ is then equal to $h(t,\sigma)\exp[-\phi(T-t)]$ where $h(\ldots)$ is the solution for $k=-\tau$. And the price to dividend ratio $v_i$ is given by the integral of $h(t,\sigma)\exp[-\phi(T-t)]$ where $h(\ldots)$ is the solution for $k=1-\tau$, given in equation II.16.
APPENDIX B: DETAILS OF THE MSM ESTIMATION

Let \( f(\theta) \) be the vector of simulated moments from the model which correspond to those in (IV.1), and \( S(\theta) \) be the vector of the sample unconditional moments, as follows:

\[
S(\theta) = \frac{1}{t} \sum_{t=1}^{T} \begin{bmatrix}
R_t \\
r_t^2 \\
r_t^4 \\
r_t^2 \cdot r_{t-1}^2 \\
r_t^2 \cdot r_{t-2}^2 \\
r_t^2 \cdot r_{t-12}^2 \\
r_t \cdot r_{t-1} \\
r_t \cdot r_{t-2} \\
r_t \cdot r_{t-12} \\
|r_t| \cdot |r_{t-1}| \cdot |r_{t-2}| \cdot |r_{t-12}|
\end{bmatrix}
\]

(B.1)

where \( r_i \) is defined in B.1 the deviation about the sample mean of \( R_i \).

Then, defining the discrepancies between the ten actual and corresponding model moments as \( e'(\theta) = [S(\theta) - f(\theta)] \), the weighted average of discrepancies to be set equal to zero to determine \( \hat{\theta} \) is:

\[
F' e(\theta) = 0
\]

(B.2)

7x10 10x1 10x1
If \( F' \) is chosen to minimize the asymptotic variance-covariance matrix of the estimates of \( \theta \), Hansen (1982) shows that \( F' \) should be set equal to 
\[ D' \Gamma' \], where:

\[
D' = \begin{bmatrix}
\frac{\delta f_1}{\delta \theta_1} & \frac{\delta f_2}{\delta \theta_1} & \ldots & \frac{\delta f_0}{\delta \theta_1} \\
\frac{\delta f_1}{\delta \theta_2} & \frac{\delta f_2}{\delta \theta_2} & \ldots & \frac{\delta f_0}{\delta \theta_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\delta f_1}{\delta \theta_7} & \frac{\delta f_2}{\delta \theta_7} & \ldots & \frac{\delta f_0}{\delta \theta_7}
\end{bmatrix}
\]

and

\[
[\Gamma]_{ij} = \frac{1}{T-1} \sum_{t=0}^{T-1} [S - \mu^t]^i (S - \mu^t)^j
\]  \( \text{ (B.3)} \)

So long as \( D \) and \( \Gamma \) are consistent estimates of their population counterparts, the asymptotic variance-covariance matrix \( H \) is:

\[
H' = F' \Gamma' F
\]  \( \text{ (B.4)} \)

With 10 moments and 7 parameters, the statistic \( [\Gamma \varepsilon(\theta)' \Gamma' \varepsilon(\theta)] \) which provides a test for the overidentifying restrictions is asymptotically distributed chi-square with 3 degrees of freedom.

To make the best use of the returns data available, we use overlapping returns to compute the moments in (B.1) (Hansen and Hodrick (1980)). To assure positive definiteness of the \( \Gamma \) matrix, we use the Newey-West (1987) correction:

\[
\Gamma = \Gamma_0 + \sum_{k=1}^{m} w(k,m) [\Gamma_k + \Gamma_k']
\]  \( \text{ (B.5)} \)
\[ w(k,m) = 1 - \frac{1}{1+m} \]

\[ [\Gamma]_k = \frac{1}{T-1} \sum_{t=k+1}^{T-1} [S_{it} - f_{it}] [S_{j(t-k)} - f_{j(t-k)}] \]  

To compute \( \hat{\theta} \), \( W \) is set equal to 1 to produce an initial consistent estimate of \( \hat{\theta} \) which is then used to construct a new \( W \), which in turn produces a second estimate of \( \hat{\theta} \), etc. Unfortunately, the justification for these covariance matrix procedures is only asymptotic. Richardson and Smith (1988) present some simulation evidence showing that estimates of \( \Gamma \), \( \Delta \), and the Newey-West correction, perform reasonably poorly in samples of the size here---720 observations.
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Cox, J. C., 1975, "Notes on option pricing I: Constant elasticity of variance diffusions," Unpublished Note, Stanford University, Graduate School of Business, Stanford, CA 94305.


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Equilibrium price-to-dividend ratio as a function of market uncertainty about dividends, $\sigma$, and $\beta$, the speed at which dividend uncertainty regresses to a steady-state.

\[ V(\Sigma(t); \beta) \]

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The equilibrium price-to-dividend ratio is given by equation (II.16) in the text.
Figure 1(b)

Equilibrium price-to-dividend ratio as a function of market uncertainty about dividends, \( \sigma_i \), and \( \pi \), the correlation between the level of dividends and dividend uncertainty\(^3\).

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\(^3\)The equilibrium price-to-dividend ratio is given by equation (II.16) in the text.
Figure 2

The ratio of levered equity value to asset value as a function of the volatility of the asset value, for three levels of the promised debt-to-asset value.

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Note: The asset, debt, and equity values are all endogenous to the uncertainty about asset values. The value of debt is derived using a Black-Scholes approximation.
Figure 3

Time series of the returns on the NYSE equally-weighted index, yield-premiums on junk bonds, and prior-month risk premium squared (a crude measure of volatility), over the period 1926.1-1980.1
Figure 4

Equity risk premium on an asset with 90% debt as a function of the spread between the yield on the bond ("junk" bond) and the riskless rate, when both the risk premium and yield spread change in response to changes in the volatility of the dividends on the asset.
Figure 5(a)

Market risk premium is a function of the standard deviation of return when both risk premium and return standard deviation are endogenous to variations in the level of dividends and uncertainty about those dividends.
Market risk premium is a function of the variance of return when both risk premium and return variance are endogenous to variations in the level of dividends and uncertainty about those dividends.