FINANCE WORKING PAPER NO. 214

Supershares

by

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August 1991

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Prepared for The New Palgrave Dictionary of Money and Finance
The idea of a complete market, first introduced by Arrow (1953) and Debreu (1959), is generally viewed as both elegant and unattainable, whether contemplated in the financial arena or elsewhere. The unattainability dimension stems from the requirement that a complete market must have at least as many securities as contingencies or "states". It is not hard to come up with a realistic set of states in which the number of states is in the billions.

As a practical matter, no one knows for sure what a complete market is, or when it will be reached, especially in the presence of transaction costs. But it is reasonably clear that we still have a long way to go. We will continue to move toward a complete market by adding even more financial instruments. Unfortunately, in adding new securities we are also drastically increasing the information demands placed on investors.

On this sobering note, recall that the key welfare-economic property of a complete market is full allocational efficiency (FAE), also known as full Pareto efficiency. FAE occurs in incomplete markets only under strong homogeneity conditions—with one exception. This exception is the foundation of the supershare and superfund concepts developed by Hakansson (1977) and is based on a generalization of a property noted by Mossin (1973; 108-109).

Section I summarizes the underlying theory while Section II briefly reviews the principal properties and potential applications of supershares. In essence, while a few hundred supershares need not assure FAE, they do engender a large step in that direction.
I. THEORETICAL FOUNDATION

We consider a pure exchange economy with a single commodity which lasts for two periods under the standard assumptions. That is, at the end of period 1 the economy will be in some state $s$ where $s = 1, \ldots, n$. There are $I$ consumer-investors indexed by $i$, whose probability beliefs over the states are given by the vectors $\pi_i = (\pi_{i1}, \ldots, \pi_{in})$, where, for simplicity, $\pi_{is} > 0$, all $i, s$. The preferences of consumer-investor $i$ are represented by the (conditional) functions $U_{is}(c_i, w_{is})$, where $c_i$ is the consumption level in period 1 and $w_{is}$ is the consumption level in period 2 if the economy is in state $s$ at the beginning of that period. These functions are defined for $(c_i, w_{is}) \geq 0$, all $i, s$, and are assumed to be increasing and strictly concave.

At the beginning of period 1 (time 0), consumer-investors allocate their resources among current consumption $c_i$ and a portfolio chosen from a set $J$ of securities indexed by $j$. Security $j$ pays $a_{js} \geq 0$ per share if state $s$ occurs at the end of period 1 and the total number of outstanding shares is $Z_j$. Let $z_{ij}$ denote the number of shares of security $j$ purchased by investor $i$ at time 0; his or her portfolio $z_i = (z_{i1}, \ldots, z_{ij})$ then yields the payoff

$$w_{is} = \sum_{j \in J} z_{ij} a_{js}$$

available for consumption in period 2 if state $s$ occurs at the end of period 1. Aggregate wealth (and aggregate consumption in period 2) if state $s$ occurs is then given by

$$W_s = \sum_{j \in J} Z_j a_{js}, \text{ all } s.$$
The financial market, as is usual, is assumed to be competitive and
perfect. If the rank of matrix $A = [a_{js}]$ is full (equals $n$), the financial
market will be called complete; if not, it will be called incomplete. The
significance of a complete market is that any payoff pattern $w > 0$ can be
obtained via some portfolio $z$ since the system $zA = w$ will always have a
solution.

Any allocation $(c, z)$ which constitutes a solution to (the usual) system
of equilibrium equations is allocationally efficient with respect to the
market structure $A$—since the marginal rates of substitution for any two
securities are the same across individuals. When $(c, z)$ is allocationally
efficient with respect to all conceivable allocations, whether achieved inside
the existing market $A$ or not, $(c, z)$ will be said to attain full allocational
efficiency (FAE). FAE insures that the marginal rates of substitution of
wealth between any two states are the same for all investors $i$.

Two states $s$ and $s'$ such that $W_s = W_{s'}$, i.e. with equal aggregate
payoffs, are said to belong to the same superstate $t$. It is evident that the
number of superstates is likely to be orders of magnitude smaller than the
number of states.

**Theorem.** If the financial market is complete with respect to the
superstate partition $T$, FAE is attained for arbitrary endowments
if and only if beliefs and preferences satisfy the (rather mild)
requirements

$$\pi_{is}/\pi_{it} = \pi_{is'}/\pi_{it}, \quad \text{all } s \in t, \quad \text{all } i \text{ and } t \quad (1)$$

and

$$U_{is} = U_{is'}, \quad \text{all } s \text{ and } s' \in t, \quad \text{all } i \text{ and } t. \quad (2)$$
Note that (1) and (2) require only that beliefs be homogeneous given the superstate and that preferences be insensitive to states within a superstate—beliefs and preferences with respect to superstates are unrestricted, thus permitting substantial heterogeneity among consumer-investors.

To complete the market with respect to superstates, three simple alternatives are available (Hakansson 1978). The first is a full set of "supershares", each share paying $1 if and only if a given superstate occurs. The second alternative is a full set of (European) call options, and the third a full set of (European) put options, on the market portfolio Z or W. Supershares may be traded in zero net supply, like call and put options, or they may be issued as claims against the assets of a "superfund", which (in the manner of an index fund) passively holds a fraction $\delta < 1$ of the market portfolio Z or W.

II. SUPERSHARE PORTFOLIOS: PROPERTIES AND APPLICATIONS

The following properties and applications of supershares and superfunds may be noted:

1. Under either of the above alternatives, the consumer-investor has complete freedom in shaping his or her payoff patterns across superstates; thus, all currently available superstate payoff patterns can be replicated and a great many genuinely new ones can be created; implicit "borrowing" and "lending" take place at the same rate of interest.

2. Under the conditions of the theorem, short positions (in any security) can be entirely avoided only under the supershare
alternative, and then only if the supershares are issued by a superfund which is sufficiently large (Hakansson 1977).

3. At the practical level, the number of superstates is small. Suppose aggregate wealth could go down as much as 20% or up as much as 30% between now and time t. If we distinguish returns to the nearest .5%, the total number of superstates, which we can then label -20, -19.5, -19, ..., - .5, 0, .5, ..., 29, 29.5, 30, is only 101. If we round to the nearest .1%, the number of superstates becomes 501. In the first case, supershare 11.5, for example, would pay $1 ± ε per share (with ε ≤ .0022) when the return r satisfies 11.25 < r ≤ 11.75, and nothing otherwise. (The fact that most extant markets have many more securities than superstates doesn't mean they are complete with respect to the superstates. They simply don't have the right securities--recall item 1 above.)

4. The previous point implies that supershares cannot be priced via arbitrage analysis alone. However, if additional assumptions are added, such as those underlying the Black-Scholes (1973) formula, supershare prices are readily approximated (Cox and Rubinstein [1985:460-461]; see also Garman [1978]).

5. Supershares may of course be issued in multiple maturities. The resulting family of observable time-superstate prices then provide a direct means for valuing the payoffs from proposed capital budgeting projects (Banz and Miller 1978; see also Breeden and Litzenberger 1978).
6. By use of an inflation index, superstates are readily dominated in real rather than nominal terms (Hakansson 1976). This design generates an observable real interest rate, for example, at which (implicit) borrowing and lending transactions can be consummated.

Chemists tell us that the whole world is made up of somewhat over 100 elements. Similarly, the market portfolio can be pictured as entirely composed of elements or building blocks called supershares, which can be combined into any package that suits the investor's (chemist's) fancy. When issued by a superfund, default is ruled out.

It may be noted that a market in puts and calls on a crude approximation to the United States market portfolio, namely the Standard & Poor's 100 Index, was opened in 1983. These options are now the most actively traded of all (listed) option instruments.

A family of funds based on only four "prototype supershares" is currently near registration. Even with these four prototypes, one can readily shift returns so as to enhance those attained in a down market, in a flat market, or in an up market--or in any paired combination.
References


