Efficient Consumption-Portfolio Policies

by

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Abstract

We give necessary and sufficient conditions for a given consumption-portfolio policy to be optimal for some time-additive utility function. These conditions involve a set of consistency and state independent conditions and a partial differential equation. As a part of the sufficient conditions, we show how one recovers the utility function that supports the given policy. These necessary and sufficient conditions can also be utilized to answer the question: Does there exist a price process and a utility function for which a given consumption-portfolio policy is optimal? A series of examples is given to demonstrate our results.
1 Introduction

The study of optimal consumption and portfolio policies in continuous time under uncertainty has been a central topic in financial economics; see, for example, Merton (1973), Cox and Huang (1989a,b), and He and Pearson (1991). The main question addressed in this literature is: Does there exist an optimal consumption-portfolio policy for an economic agent represented by a time-additive utility function and what are the properties of his/her optimal policy if it indeed exists?

We address in this paper a related but different question: For a given price process, what are the necessary and sufficient conditions for a given consumption-portfolio policy to be optimal for some time-additive utility function? For brief, these conditions will be termed the necessary and sufficient conditions for efficiency.

Cox and Leland (1982) is the first to investigate issues related to this question in an economy where the stock price follows a geometric Brownian motion, also see Black (1988) for related work. Our contribution here lies in giving a complete characterization of the necessary and sufficient conditions for efficiency when the stock price follows a general diffusion process. These conditions necessarily only involve the given policy. As a part of the sufficient conditions, we also demonstrate how to recover the utility function for which the given policy is optimal.

Since these necessary and sufficient conditions are derived for a general representation of the price process, we can also use them to answer a more general question: Can a given consumption-portfolio policy be optimal for some utility function and some price process?

The interest in our results is twofold. First, there are many trading rules used in practice; see Black and Perold (1989), Dybvig (1981), and Leland (1980) for examples of trading rules. It is thus desirable to be able to see whether these rules are consistent with utility maximization. In addition, accumulating empirical evidence has suggested that the stock price processes, and especially the price processes for portfolios, are not best described by a geometric Brownian motion; see, for example, Black (1976) and Lo and MacKinley (1988). Consequently, there is need for the necessary and sufficient conditions for efficiency to allow for price processes more general than the geometric Brownian motion.

Second, besides the necessary and sufficient conditions for efficiency, we also have some characterizations of an efficient consumption-portfolio policy that are of independent interest. For example, it is shown that an efficient consumption-portfolio policy must make the risk tolerance of the indirect utility function in units of the riskless asset a positive local martingale\footnote{For now, a local martingale is a process that is a martingale locally. A formal definition will be given later.} (or a martingale under some additional conditions) under the so-called risk neutral probability (to be defined...
formally later). This implies in particular that, when the price process is a geometric Brownian motion, the present value of the dollar amount invested in the stock in the future is less than (or equal to in the case of martingale) the dollar amount currently in the stock. This holds true for all efficient consumption-portfolio policies.

Our work is related to the characterization of efficient portfolios in a one-period setting due to Peleg (1975), Peleg and Yarri (1975), and Dybvig and Ross (1981). Because of the single period setting, dynamic rules are not considered in this literature. Our work is also related to Dybvig (1981), who discusses the recoverability of utility functions from dynamic trading strategies in a binomial model.

The rest of this paper is organized as follows. Section 2 formulates a dynamic consumption-portfolio problem and derives necessary conditions satisfied by an indirect utility function. Section 3 shows how to express these necessary conditions solely in terms of a given consumption-portfolio policy and presents the necessary conditions for efficiency. This section also gives some examples to demonstrate how to use the necessary conditions for efficiency, and some characterizations of an optimal consumption-portfolio policy that are of independent interest. Section 4 shows that the necessary conditions for efficiency are sufficient conditions under some regularity conditions. Section 5 contains more examples illustrating our results and Section 6 has some concluding remarks.

2 The setup

Consider a securities market economy with a finite horizon $[0, T]$ in which there is one stock and one bond available for trading. The bond price grows exponentially at a constant rate $r$, the riskless interest rate. The stock does not pay dividends and has a diffusion price process whose dynamics is described by the following stochastic differential equation

$$dS(t) = \mu(S(t), t)S(t)dt + \sigma(S(t), t)S(t)dw(t), \quad t \in [0, T],$$

where $\mu(\cdot, t)$ and $\sigma(\cdot, t)$ are twice continuously differentiable functions of $S$, and $w$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Investors are assumed to have access only to the information contained in historical prices, that is, the information the investors have at time $t$ is the sigma-field generated by $\{S(s); 0 \leq s \leq t\}$. We assume that the stock price is strictly positive with probability one. For brevity, we will sometimes simply use $\mu(t)$ and $\sigma(t)$ to denote $\mu(S(t), t)$ and $\sigma(S(t), t)$, respectively.

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2In applications, one can take the risky asset to be an index portfolio.
3Nothing will be affected if the stock pays dividends at rates as functions of the stock price.
4Implicit in this is the hypothesis that a solution to the stochastic differential equations exists.
5We will use weak relations throughout. For example, positive means nonnegative, concave means weakly concave, and so forth. For strict relations, we use, for example, strictly increasing, strictly concave, and so forth.
We assume that there exists an equivalent martingale measure, or a risk neutral probability\(^6\) \(Q\) for the price processes. Given our current setup, this equivalent martingale measure must be uniquely represented by

\[ Q(A) = \int_A \xi(\omega, T) P(d\omega) \quad \forall A \in \mathcal{F}, \]

where

\[ \xi(t) = \exp \left\{ \int_0^t \dot{\kappa}(S(s), s) dw(s) - \frac{1}{2} \int_0^t \kappa^2(S(s), s) ds \right\} \quad (1) \]

and

\[ \dot{\kappa}(S(t), t) \equiv - \frac{\mu(S(t), t) - r}{\sigma(S(t), t)}. \]

The stock price dynamics becomes

\[ dS(t) = rS(t)dt + \sigma(t)S(t)dw^*(t), \quad t \in [0, T], \]

where \(w^*\) is a standard Brownian motion under \(Q\).

A dynamic consumption and portfolio policy \((C, A)\), is a pair of consumption and investment functions, \((C(W, S, t), A(W, S, t))\), denoting the consumption rate and the dollar amount invested in the risky asset at time \(t \in [0, T]\), respectively, when the wealth is \(W\) and the price of the risky asset is \(S\). For simplicity, we will often use \(C(t)\) and \(A(t)\) to denote the consumption function and the investment function at time \(t\), respectively. The policy \((C, A)\) is said to be efficient, if there exist a utility function for intermediate consumption and a utility function for final wealth, \(u(x, t) : \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R} \cup \{-\infty\}\) and \(V(x) : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}\), respectively, which are twice continuously differentiable, increasing and concave in \(x\), and either \(u(x, t)\) for almost all \(t\) or \(V(x)\) is strictly concave in \(x\) such that \((C, A)\) is the solution to the following dynamic consumption and portfolio problem:\(^7\)

\[
\begin{align*}
&\sup_{C \geq 0, A} \mathbb{E}_t \left[ \int_t^T u(C(s), s) ds + V(W(T)) \right] \\
&\text{s.t. } dW(s) = [rW(s) + A(s)(\mu(s) - r) - C(s)] ds + A(s)\sigma(s) dw(s), \quad s \in [t, T], \\
&W(s) \geq 0, \quad s \in [t, T], \\
&W(t) = x,
\end{align*}
\]

for any \(x > 0\) and any \(t \in [0, T]\), where \(\mathbb{E}_t[\cdot]\) is the expectation conditional on the information at \(t\).

Note that a dynamic consumption and portfolio policy that satisfies the first condition of (2) is a self-financing strategy; see Merton (1971) and Harrison and Kreps (1979), for example. The second condition is a positive wealth constraint that rules out the possibility of creating something out of nothing; see Dybvig and Huang (1989).

\(^6\)See Harrison and Kreps (1979) for the former and Cox and Ross (1976) for the latter.

\(^7\)Imposing strict concavity on \(u\) ensures that the consumption function is continuous in \(W\), which we will assume later.
Note also that there is a vast literature on the existence and the characterization of an optimal consumption and portfolio policy for a given pair of utility functions \((u,V)\); see Merton (1971), Cox and Huang (1989a,b), and He and Pearson (1991), for example. Our purpose here is different from that of this literature. We take a consumption and portfolio policy \((C,A)\) as given and ask:
What are the necessary and sufficient conditions for it to be an optimal policy for some pair of utility functions \((u,V)\)? As we don't know the utility functions to begin with, these necessary and sufficient conditions necessarily only involve the given policy \((C,A)\).

Now suppose that \((C,A)\) is efficient. Then there exists \((u,V)\) so that \((C,A)\) solves (2). Let
\[J(W,S,t)\]
be the value of the objective function of (2), or the indirect utility function, given that the wealth and the risky asset price at time \(t\) are \(W\) and \(S\), respectively. By the monotonicity and the strict concavity of either \(u(x,t)\) for almost all \(t\) or \(V(x)\) in \(x\), \(J\) must be increasing and strictly concave in \(W\). We will restrict our attention to efficient policies \((C,A)\) so that the following conditions hold:

A1. the parameters of the dynamics of \(W\) satisfy a linear growth condition and a local Lipschitz condition;\(^8\)

A2. \(A(W,S,t)\) is continuous and \(C(W,S,t)\) is continuously differentiable in \(W\) and \(S\) for \(W > 0\), \(S > 0\);

A3. for \((W,S,t) \in [0,\infty) \times (0,\infty) \times [0,T]\), \(R(W,S,t) \equiv -\frac{J_W(W,S,t)}{J_W(W,S,t)}\) and \(H(W,S,t) \equiv -\frac{J_S(W,S,t)}{J_W(W,S,t)}\), and for \((W,S,t) \in (0,\infty) \times (0,\infty) \times [0,T]\), \(N(W,S,t) \equiv \frac{J_W(W,S,t)}{J_W(W,S,t)}\) are well-defined; and for \((W,S,t) \in (0,\infty) \times (0,\infty) \times [0,T]\), \(R\) and \(H\) are twice continuously differentiable in \((W,S)\) and once continuously differentiable in \(t\), and \(N\) is continuously differentiable in \((W,S)\), where the subscripts denote partial derivatives;

A4. the wealth never reaches zero before time \(T\).

The interpretations of the terms defined in A3 will be given later. Henceforth, subscripts denote partial derivatives unless mentioned otherwise.

Note that Condition A1 above is a sufficient condition for the stochastic differential equation governing the evolution of wealth to have a unique solution. Condition A2 requires the policy \((C,A)\) be sufficiently smooth. Condition A3 allows us to work with many derivatives of \(J\). Condition A4 needs some explanation. Cox and Huang (1989, proposition 3.1) have shown that the optimally

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\(^8\)A function \(f: \mathbb{R}^n \times [0,T] \rightarrow \mathbb{R}\) is said to satisfy a linear growth condition if \(|f(z,t)| \leq K(1 + |z|)\) for all \(z\) and \(t\), where \(|z|\) denotes the Euclidean norm of \(z\) and \(K\) is a strictly positive scalar. A function \(f: \mathbb{R}^n \times [0,T] \rightarrow \mathbb{R}\) satisfies a local Lipschitz condition if for any \(M > 0\) there is a constant \(K_M\) such that for all \(y,z \in \mathbb{R}^n\) with \(|y| \leq M\) and \(|z| \leq M\) and \(t \in [0,T]\), we have \(|f(y,t) - f(z,t)| \leq K_M |y - z|\), where \(|y|\) denotes the Euclidean norm.
invested wealth for any \((u, V)\) must not reach zero before \(T\) when the stock price follows a geometric Brownian motion. Merton (1990, theorem 16.2) generalizes this result to any diffusion price process obeying two regularity conditions satisfied by most of the processes that financial economists have worked with. Thus Condition A4 can be viewed as a necessary condition for \((C, A)\) to be efficient.

Later in this section, we will add another regularity condition A5 that \((C, A)\) satisfies. Until then, an efficient \((C, A)\) will be understood to satisfy Conditions A1–A4.

Now let \((C, A)\) be efficient with corresponding utility functions \((u, V)\). Then \(J\) must satisfy the following Bellman's equation

\[
0 = \max_{C \geq 0, A} \left\{ u(C, t) + J_t + [rW + A(\mu - r) - C]J_W + \mu SJ_S + \frac{1}{2} \sigma^2 A^2 J_{WW} + \sigma^2 SAJ_{WS} + \frac{1}{2} \sigma^2 S^2 J_{SS} \right\},
\]

for all \((W, S, t) \in (0, \infty) \times (0, \infty) \times (0, T)\), with the boundary conditions

\[
\lim_{t \to T} J(W, S, t) = V(W), \quad \lim_{W \to 0} J(W, S, t) = \int_t^T u(0, s) ds + V(0),
\]

where we have suppressed the arguments of \(J, C,\) and \(A\).

Note that the second boundary condition is a consequence of the positive wealth constraint and it necessitates immediately that

\[
C(0, S, t) = 0, \quad \text{and} \quad A(0, S, t) = 0.
\]

The first order necessary conditions for the dynamic consumption and portfolio problem (2) are, for all \((W, S, t) \in (0, \infty) \times (0, \infty) \times (0, T),\)

\[
u_c(C(W, S, t), t) \leq J_W(W, S, t), \quad \text{if} \quad C(W, S, t) = 0,
\]

\[
u_c(C(W, S, t), t) = J_W(W, S, t), \quad \text{if} \quad C(W, S, t) > 0;
\]

\[
A(W, S, t) = \left( \frac{\mu(S, t) - r}{\sigma^2(S, t)} \right) R(W, S, t) + SH(W, S, t),
\]

where we have used the notation define in Condition A3. Note that the first term on the right-hand-side of (7) is an instantaneous mean-variance efficient portfolio and \(R(W, S, t)\) is the Arrow-Pratt measure of the absolute risk tolerance of the indirect utility function, and the second term is the "hedging demands against adverse changes in the consumption/investment opportunity set".\(^9\) By Condition A3, \(A\) must be twice continuously differentiable in \((W, S) \in (0, \infty) \times (0, \infty)\) and is once continuously differentiable in \(t \in (0, T)\).

Furthermore, (6) and the chain rule of differentiation implies that \( u_\varepsilon C_W = J_{WW} \) when \( C > 0 \). Since \( J_{WW} < 0 \), we must have \( C_W > 0 \) when \( C > 0 \). That is, strictly positive consumption can only occur when the marginal propensity to consume is strictly positive.

The first order necessary conditions and the Bellman equation place strong restrictions on \((C, A)\). But these restrictions are expressed in terms of the partial derivatives of \( J \), which is not known (as \((u, V)\) is unknown). However, it will be shown in the next section that these necessary conditions can be transformed to be expressed solely in terms of \((C, A)\). Therefore, one can directly check whether a given pair \((C, A)\) is a candidate for an efficient policy without any other information. Furthermore, the necessary conditions for \((C, A)\) to be efficient become sufficient conditions after adding some regularity conditions.

Clearly, for all \((W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T)\),

\[
\frac{\partial}{\partial S} \left( \frac{1}{R} \right) = - \frac{\partial}{\partial W} \left( \frac{H}{R} \right),
\]

\[
\frac{\partial N}{\partial S} = \frac{\partial}{\partial t} \left( \frac{H}{R} \right),
\]

\[
\frac{\partial N}{\partial W} = \frac{1}{R^2} \frac{\partial R}{\partial t}.
\]

In addition, for all \( W > 0, S > 0, \) and \( t \in [0, T] \) define

\[
Q(W, S, t) = \int_{W_0}^{W} \frac{1}{R(z, S, t)} \, dz,
\]

\[
X(S, t) = \int_{S_0}^{S} \frac{H(W_0, \eta, t)}{R(W_0, \eta, t)} \, d\eta,
\]

\[
Y(t) = \int_{0}^{t} N(W_0, S_0, r) \, dr,
\]

where for simplicity we have used \( W_0 \) and \( S_0 \) to denote \( W(0) \) and \( S(0) \), respectively, and have used the convention that \( \int_{a}^{b} = -\int_{b}^{a} \) when \( a > b \).

Note that, for all \( W > 0, S > 0, t \in [0, T) \), by the continuity of \( R, H, \text{ and } N \),

\[
\ln \mathcal{U}(W, S, t) = -Q(W, S, t) + X(S, t) + Y(t)
\]

\[
= \ln J_W(W, S, t) - \ln J_W(W_0, S_0, 0)
\]

\[
\begin{align*}
&= \ln u_z(C(W, S, t)) - \ln J_W(W_0, S_0, 0), & \text{if } C(W, S, t) > 0; \\
&\geq \ln u_z(0) - \ln J_W(W_0, S_0, 0), & \text{if } C(W, S, t) = 0.
\end{align*}
\]

For \( t < T \), since \( C_W > 0 \) for all \( C > 0 \), we can write, for all \( z > 0 \),

\[
\ln \mathcal{U}(C^{-1}(z, S, t), S, t) = \ln u_z(z, t) - \ln J_W(W_0, S_0, 0),
\]

where \( C^{-1} \) denotes the inverse of \( C \) with respect to its first argument. From this relation, we conclude that its left-hand-side must be independent of \( S \) when \( C(W, S, t) > 0 \) as the utility
function is state independent. Furthermore, by the continuous differentiability of \(u(x,t)\) in \(x\) and (6), we must have, for all \(W > 0, S > 0\), and \(t \in (0,T)\) so that \(C(W,S,t) = 0,\)

\[
U(W, S, t) \geq \lim_{t \to 0} U(C^{-1}(x, \dot{S}, t), \dot{S}, t) \quad \forall \dot{S} > 0.
\] (16)

Relations (14)—(16) involve \(J\) and its derivatives in the interior of their domain. We now proceed to derive some conditions at the boundary of their domain. For this purpose, we use a result of Cox and Huang (1989a), which states that there exists a scalar \(\lambda > 0\) so that

\[
J_W(W(t), S(t), t) \begin{cases} = \lambda \xi(t) e^{-rt}, & t \in [0,T) \\ \leq \lambda \xi(T) e^{-rT} & t = T, \end{cases}
\] (17)

where the inequality holds as an equality when \(W(T) > 0\) and we recall the definition of \(\xi\) in (1). Note that the \(\lambda\) above is a Lagrangian multiplier and \(\xi(t) e^{-rt}\) is the Arrow-Debreu state price for time \(t\) consumption per unit of probability \(P\). Since \(W, S,\) and \(\xi\) are processes with continuous paths, (17) implies that \(J_W\) is continuous except possibly when \(W = 0\) at \(T\) and

\[
\lim_{t \uparrow T} J_W(W, S, t) \geq V'(0), \quad \forall S > 0.
\] (18)

Given the above discussion, we now impose one more regularity condition on \((C, A)\):

A5. \(R(W, S, t), H(W, S, t), N(W, S, t),\) and their derivatives are continuous functions of \(t\) at \(t = T\)

except possibly at \(W = 0\).

This condition together with Condition A3 accomplishes two things. First, (14)—(16) can extended to \(t = T\) for \(W > 0\) and \(S > 0\), and we have

\[
\ln U(W, S, T) = \ln V'(W) - \ln J_W(W_0, S_0, 0), \quad W > 0, S > 0,
\]

\[
\lim_{t \uparrow T} U(W, S, t) \geq \ln V'(0) - \ln J_W(W_0, S_0, 0) = \lim_{t \uparrow T} \ln U(W, \dot{S}, T), \quad \forall S > 0, \dot{S} > 0,
\] (19)

where we note that the first relation indicates that \(\ln U(W, S, T)\) is a function of \(W\) only, and the equality in the second relation follows from the fact that \(V\) is continuously differentiable. In addition, \(Q, X,\) and \(N\) are twice continuously differentiable in \((W, S)\) and once continuously differentiable in \(t\) for \(W > 0, S > 0,\) and \(t \in [0, T]\).

Second, we conclude by continuity and \(H(W, S, T) = 0\) that

\[
\lim_{t \uparrow T} H(W, S, t) = 0, \quad W > 0, S > 0,
\] (20)

and

\[
A(W, S, T) \left( \frac{\mu(S, T) - r}{\sigma^2(S, T)} \right)^{-1} = \lim_{t \uparrow T} \left( R(W, S, t) + S H(W, S, t) \left( \frac{\mu(S, T) - r}{\sigma^2(S, T)} \right)^{-1} \right)
\]

\[
= R(W, S, T) = -\frac{V'(W)}{V''(W)}, \quad W > 0, S > 0,
\] (21)
which is a function of \( W \) only.

We will term relations (8)-(10) as consistency conditions; and relations (15), (16), (19), (20), and (21) as state independent conditions. Note that if we defined efficiency more broadly to include state-dependent utility functions, then obviously the state independent conditions need not be satisfied. We will denote henceforth the set of efficient policies satisfying Conditions A1–A5 by \( \mathcal{E} \). For brevity, \( (C, A) \) is said to be efficient if it is an element of \( \mathcal{E} \).

Before leaving this section, we record one well-known fact about \( (C, A) \in \mathcal{E} \), namely, that the current wealth plus the cumulative past consumption, both in units of the bond, is a martingale under \( Q \).

**Proposition 1** Let \( (C, A) \in \mathcal{E} \). Then \( W(t)e^{-rt} + \int_0^t C(W(s), S(s), s)e^{-rs}ds \) is a martingale under \( Q \).

**Proof.** See, for example, Dybvig and Huang (1989).

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3 Necessary Conditions for Efficiency

We derive in this section necessary conditions for a given \( (C, A) \) to be efficient. In this process, we also have some characterizations of \( (C, A) \in \mathcal{E} \) that are of independent interest. For example, we show that when there is no intermediate consumption and when the hedging demand multiplied by the price of the risky asset is decreasing over time and the risk premium per unit of variance is positive and increasing over time, one expects an efficient portfolio policy to invest less in the stock over time in present value terms normalized by the risk premium per unit of variance. When the hedging demand is increasing and the risk premium per unit of variance is increasing over time, the converse is true under an additional condition.

Thereafter, we will use \( \kappa(S, t) \), or simply \( \kappa(t) \) to denote \( (\mu(S, t) - r)/\sigma^2(S, t) \), which is the risk premium on the stock per unit of the variance on its rate of return. Assume \( \kappa(S, t) \neq 0 \) except possibly on set of \( S \) and \( t \) that is of Lebesgue (product) measure zero.

We begin by giving a lemma and a proposition. The lemma expresses the function \( N \) in terms of \( R, H, C, A, \) and the derivatives of \( R \) and \( H \). Then we show in the proposition that \( R \) must satisfy a linear partial differential equation.

**Lemma 1** Let \( (C, A) \in \mathcal{E} \). Then, for \( (W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T] \),

\[
N = \frac{1}{2} \sigma^2 A^2 \left( \frac{1}{R} \right)_W + \sigma^2 AS \left( \frac{1}{R} \right)_S + \frac{1}{2} \sigma^2 S^2 \left( \frac{H}{R} \right)_S
\]

\[
+ (rW + A(\mu - r) - C) \frac{1}{R} - \mu S \frac{H}{R} - \frac{\kappa^2 \sigma^2}{2} - r.
\]

(22)
That is, $N$ can be expressed in terms of $R$, $H$, $C$, $A$, and the derivatives of $R$ and $H$.

**Proof.** We will prove (22) for $(W, S, t) \in (0, \infty) \times (0, \infty) \times (0, T)$. The assertion then follows from continuity.

Differentiating Bellman's equation with respect to $W$ and simplifying the resulting equation using the first order conditions (6) and (7), we get

$$0 = J_{W_t} + rJ_W + J_{WW}(rW + A(\mu - r) - C) + J_{WS} \mu S + \frac{1}{2} \sigma^2 A^2 J_{WWW} + A \sigma^2 S J_{WWS} + \frac{1}{2} \sigma^2 S^2 J_{WSS},$$

where we have used the fact that by the continuous differentiability hypothesis on $C$ with respect to $W$, $C_W = 0$ on the set $\{(W, S, t) : C(W, S, t) = 0\}$. This equation implies that the drift of $dJ_W$ is $-rJ_W$. Since (7) implies that the diffusion term of $dJ_W$ is $-\kappa \sigma J_W$, we conclude that

$$dJ_W = -rJ_W dt - \kappa \sigma J_W dw(t).$$

Hence, the drift of $d \ln J_W$ must be equal to $-r - \frac{\sigma^2 \sigma^2}{2}$. We get (22) and this completes our proof. \qed

**Proposition 2** For $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T]$, the function $R$ must satisfy the following partial differential equation:

$$\frac{1}{2} \sigma^2 A^2 R_{WW} + A \sigma^2 S R_{WS} + \frac{1}{2} \sigma^2 S^2 R_{SS} + (rW - C)R_W + rS R_S + R_t + C_W R - rR = 0. \tag{23}$$

Equivalently, $R(W(t), S(t), t)e^{-rt} + \int_0^t C_W(s) R(s)e^{-rs} ds$ is a positive local martingale\(^{10}\) and thus is a supermartingale\(^{11}\) under the equivalent martingale measure $Q$ on $[0, T)$ and on $[0, T]$ if $W(T) > 0$ a.s.

**Proof.** The first assertion can be proved by differentiating (22) with respect to $W$ and simplifying terms using (7), (8), and (10).

For the second assertion, we first show that it is implied by the first. From Condition A4, we know that the wealth never reaches zero before $T$. For any $t \in [0, T)$, apply Itô's lemma to $R(t)e^{-rt}$ and use (23) to get

$$R(W(t), S(t), t)e^{-rt} + \int_0^t C_W(s) R(s)e^{-rs} ds = R(W_0, S_0, 0) + \int_0^t e^{-r(s)}(R_W(s)A(s)\sigma(s) + R_S(s)\sigma(s)S(s))dw^*(s), \quad t \in [0, T).$$

\(^{10}\) The process $X$ is a local martingale under $Q$ if there exists a sequence of stopping times $\tau_n$ with $\tau_n \to T$ $Q$-a.s. so that $\{X(t \wedge \tau_n), t \in [0, T]\}$ is a martingale under $Q$ for all $n$. For the definition of a stopping time see, for example, Liptser and Shiryaev (1977, p.25).

\(^{11}\) The process $X$ is a supermartingale under $Q$ if $\mathbb{E}[X(s) | \mathcal{F}_t] \leq X(t)$ $Q$-a.s.
Since $R > 0$ for all $W > 0$ and $C_W \geq 0$, the left-hand-side is strictly positive. The right-hand-side is a local martingale under $Q$ since it is an Itô integral. It is well-known that a positive local martingale is a supermartingale; see, for example, Dybvig and Huang (1989, lemma 2). Thus $R(W(t), S(t), t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is a supermartingale on $[0,T]$. In the case that $W(T) > 0$ a.s., the above arguments apply to the entire interval $[0,T]$. We have thus shown that the second assertion is implied by the first.

To see that the first assertion is implied by the second, we again apply Itô's lemma to $R(t)e^{-rt}$ on $[0,T]$ under $Q$. Since a local martingale cannot have any time drift, we get (23) for $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0,T)$. For (23) at $t = T$, we use Condition A5.

Given $C_W \geq 0$, the above proposition states that an efficient $(C, A)$ must make the risk tolerance in units of the bond a supermartingale under $Q$ on $[0,T]$—one expects the risk tolerance in units of the bond, on the average according to $Q$, to go down in the future, except possibly at $t = T$. In the case where $W(T) > 0$ a.s., this supermartingale result holds also at $t = T$. We record an immediate corollary of Proposition 2.

**Corollary 1** Let $(C, A) \in \mathcal{E}$. Then

$$\frac{A(t) - S(t)H(t)}{\kappa(t)} e^{-rt} + \int_0^t C_W(s) \frac{A(s) - S(s)H(s)}{\kappa(t)} e^{-rs}ds$$

is a positive local martingale and thus is a supermartingale under $Q$ on $[0,T)$ and on $[0,T]$ if $W(T) > 0$ a.s.

**Proof.** The assertion follows directly from (7) and Proposition 2.

Several implications of Corollary 1 deserve attention. First, consider the special case of a geometric Brownian motion stock price with $\mu > r$. Note that in this case the hedging demand $H$ is zero, $\kappa$ is a constant, $A = \kappa R > 0$ when $W > 0$, and Condition A4 is satisfied. Corollary 1 implies that

$$A(t)e^{-rt} + \int_0^t C_W(s)A(s)e^{-rs}ds$$

is a supermartingale under $Q$ on $[0,T)$ and on $[0,T]$ if $W(T) > 0$ a.s.; equivalently, for all $(W, t) \in (0, \infty) \times [0,T)$,

$$\frac{1}{2}\sigma^2 A^2 Aw + (rW - C)A W - rA + C W A + A_t = 0.$$

This is just proposition 3 of Cox and Leland (1982).12 Since $C_W \geq 0$, these imply that $A(t)e^{-rt}$ is a positive supermartingale before the wealth reaches zero. When $W(T) = 0, A = 0$ by (5).

---

12For the reader familiar with Cox and Leland, he will have noted a difference between the reported result here and that in Cox and Leland. Here $A(t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is shown to be a supermartingale under $Q$ instead of a martingale, and this supermartingale result holds for $[0,T]$ when $W(T) > 0$ a.s.. Indeed, with additional regularity conditions, $A(t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ becomes a martingale; see Corollary 2 below.
Thus $A(t)e^{-rt}$ is a supermartingale on all of $[0,T]$. That is, the present value of the dollar amount invested in stock in the future is less than the current amount invested in the stock. This, however, does not necessarily mean that one expects to shift value over time from the stock to the bond. To see this we recall from Proposition 1 that

$$W(t)e^{-rt} + \int_0^t C(s)e^{-rs}ds \quad t \in [0,T]$$

is a martingale under $Q$. This implies that

$$(W(t) - A(t))e^{-rt} + \int_0^t (C(s) - C_W(s)A(s))e^{-rs}ds \quad t \in [0,T]$$

is a submartingale under $Q$. Note that the difference between $W(t)$ and $A(t)$ is the dollar amount invested in the bond. If $C - C_W A < 0$, $(W - A)e^{-rt}$ is also a submartingale under $Q$ on $[0,T]$ and the optimal policy shifts value away from the stock to the bond and to consumption, except possibly at $t = T$. When $C - C_W A > 0$, however, $(W(t) - A(t))e^{-rt}$ can indeed be a supermartingale under $Q$. In such case, the policy shifts value away from both the stock and the bond to consumption, except possibly at $t = T$.

The interpretation of Corollary 1 in the general case is a bit more complicated as there are now hedging demands. Assume for example that $\kappa(t)$ is a positive increasing process, that is, the risk premium per unit of variance increases over time, and $C_W = 0$. When the hedging demand multiplied by $S$ is a decreasing process (given that $H(W,S,T) = 0$ and $S$ is strictly positive, this implies that the hedging demand is positive), $\kappa e^{-rt}/\kappa$ is a supermartingale under $Q$ (on all of $[0,T]$ as when $W(T) = 0$, $A(T) = 0$). The present value of one's optimal investment in the stock in the future, per unit of $\kappa$, is lower than one's current investment in the stock, per unit of $\kappa$. Interpretations similar to the special case above can be made when $C_W \neq 0$ but with everything here normalized by $\kappa$.

Under certain regularity conditions, Proposition 2 can be strengthened so that $R(t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is not only a supermartingale but is indeed a martingale under $Q$. We record this result below in the second corollary of the proposition:

Corollary 2 Suppose that $W(T) > 0$ a.s. and $R$ and $C_W$ satisfy a polynomial growth condition.\(^{13}\) Then $R(t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is a martingale on $[0,T]$ under $Q$.

PROOF. The assertion is a consequence of Feynman-Kac representation; see, for example, Karatzas and Shreve (1988, theorem 5.7.6). \(\blacksquare\)

\(^{13}\)A function $f : \mathbb{R}^N \times [0,T] \rightarrow \mathbb{R}$ is said to satisfy a polynomial growth condition if $|f(z,t)| \leq K(1 + |z|^2 \gamma)$ for all $z$ and $t$, where $|z|$ denotes the Euclidean norm of $z$ and $K$ and $\gamma$ are two strictly positive scalars.
With the conditions of Corollary 2, we have a sharper interpretation of the intertemporal behavior of \( R \) as well \( A \). For example, in the geometric Brownian motion special case discussed above, when \( C_W = 0 \) and \( W(T) > 0 \) a.s., \( A(t) e^{-rt} \) becomes a martingale under \( Q \) and thus there is no shift of value over time away from or into the stock. This is the Cox-Leland result. Other interpretations are left to the reader. We now proceed to complete our derivation of the necessary conditions for \((C,A)\) to be efficient.

First, consider the special case that \( C(W,S,t) > 0 \) for all \((W,S,t) \in (0,\infty) \times (0,\infty) \times (0,T)\). Since the first order condition (6) holds as an equality, the chain rule of differentiation and Conditions A2 and A3 imply that

\[
H(W,S,t) = -\frac{C_S(W,S,t)}{C_W(W,S,t)} \forall W > 0, S > 0, t \in [0,T).
\]  

(24)

Conditions A2 and A5 immediately necessitate that

\[
H(W,S,T) = \lim_{t\to T} -\frac{C_S(W,S,t)}{C_W(W,S,t)} = 0, \quad W > 0.
\]  

(25)

Note that (25) places nontrivial restrictions on consumption policies. For example, any consumption policy that is time separable in that \( C(W,S,t) = c(W,S)f(t) \) for all \( W > 0, S > 0, \) and \( t \in (0,T) \) for some functions \( c \) and \( f \) can never be an efficient consumption policy.

Now substituting (24) into (7) and using Conditions A2, A3, and A5 give

\[
R = \left[A + S \frac{C_S}{C_W}\right]/\kappa, \quad \forall W > 0, S > 0, t \in [0,T].
\]  

(26)

We have thus expressed \( R \), and \( H \) solely in terms of \( C \), \( A \), and their derivatives over their domain. Define \( N \) by (22). The \( R \), \( H \), and \( N \) so defined must satisfy the necessary conditions stipulated in Proposition 2 and the consistency conditions and the state-independent conditions derived in Section 2.

The following theorem summarizes the above discussion.

**Theorem 1** Let \((C,A) \in \mathcal{E}\) with \( C(W,S,t) > 0 \) for all \( W > 0, S > 0, \) and \( t \in (0,T)\). Define \( R \), \( H \), \( N \) as in (26), (24), (25), and (22), respectively, and \( Q, X, Y, \) and \( \mathcal{U} \) as in (11), (12), (13), and (14) respectively. We must have

1. \( A(0,S,t) = C(0,S,t) = 0, \quad S > 0, \quad t \in [0,T]; \)
2. \( R(W,S,t) > 0 \) for \( W > 0; \)
3. \( C_W(W,S,t) \geq 0, \) and \( C_W(W,S,t) > 0 \) for \( W > 0; \)
(4) the state-independent conditions: \( U(C^{-1}(x,S,t),S,t) \) is independent of \( S \) for all \( z > 0 \) and \( t < T \), \( U(W,S,T) \) is independent of \( S \) for all \( W > 0 \), \( \lim_{t \to T} U(W,S,t) \geq \lim_{t \to T} U(\bar{W},\bar{S},T) \) for all \( S > 0 \) and \( \bar{S} > 0 \), \( \lim_{t \to T} C_{w}^{S}(W,S,t) = 0 \), and \( A(W,S,T)/\kappa(S,T) \) is a function of \( W \) only for all \( W > 0 \);

(5) the consistency conditions (8), (9), and (10) hold; and

(6) \((C,A)\) satisfies the PDE (23).

Second, consider the general case that \( C \) can be zero at strictly positive wealth levels. For example, when \( u = 0 \), then \( C = 0 \) for all wealth levels. In this case \( H \) cannot be expressed in terms of \( C \) and \( A \) directly using (6) in the region where \( C = 0 \). The following proposition is instrumental for expressing \( R \) and \( H \) in terms of \( C, A \), and their derivatives generally.

**Proposition 3** Let \((C,A)\) \( \in \mathcal{E} \). For \( W > 0 \), \( S > 0 \), and \( t \in [0,T] \),

\[
\Gamma_{1}R + \Gamma_{2}H = \sigma K(W,S,t),
\]  

where

\[
\Gamma_{1} = -\frac{1}{2} \left[ \kappa_{SS} \sigma^{2} S^{2} + 2 \kappa_{S} \mu S + 2 \kappa_{t} \right],
\]

\[
\Gamma_{2} = \frac{1}{2} S \left[ \sigma_{SS} \sigma^{2} S^{2} + 2 \sigma_{S} \sigma^{2} S + 2 \sigma_{S} \sigma S + 2 \sigma_{t} \right],
\]

\[
K(W,S,t) = \frac{1}{2} \sigma^{2} A_{2} A_{W} + \sigma^{2} S A_{A} A_{S} + \frac{1}{2} \sigma^{2} S^{2} A_{S} + \left( \sigma_{S} \sigma S + r W \right) A_{W}
\]

\[
+ \left( \sigma_{S} \sigma S^{2} + 2 \sigma_{S} S \right) A_{S} + \frac{1}{2} \left( \sigma_{SS} \sigma S^{2} + 2 \sigma_{S} \sigma S \sigma - 2 \sigma_{S} \sigma S / \sigma - 2 \sigma_{S} \sigma S \right) A_{S}
\]

\[
+ A_{t} + \left( C_{w} A - C_{w} A \right) + SC_{S}.
\]

**Proof.** We will prove the assertion for \( t \in (0,T) \). At \( t = 0 \) and \( T \), the assertion follows from Conditions A2, A3, and A5 by continuity.

For any function \( f \) of \( W, S \) and \( t \) that is twice continuously differentiable in \( (W,S) \) and continuously differentiable in \( t \), define the differential generator \( \mathcal{L} \) under \( Q \):

\[
\mathcal{L}(f) = \frac{1}{2} \sigma^{2} A_{2} \frac{\partial^{2} f}{\partial W^{2}} + \sigma^{2} S A \frac{\partial^{2} f}{\partial W \partial S} + \frac{1}{2} \sigma^{2} S^{2} \frac{\partial^{2} f}{\partial S^{2}} + (r W - C) \frac{\partial f}{\partial W} + r S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t}.
\]

Direct computation shows, for \( W > 0 \), \( S > 0 \), and \( t \in (0,T) \),

\[
\mathcal{L} \left( \frac{1}{R} \right) = - \left[ \sigma^{2} A_{W} \left( \frac{1}{R} \right) \right]_{W} + \sigma^{2} A_{W} S \left( \frac{1}{R} \right)_{S} + (r - C_{W}) \left( \frac{1}{R} \right),
\]

(28)
\[ \mathcal{L} \left( \frac{H}{R} \right) = \hat{\kappa}S - \kappa - C_s \left( \frac{1}{R} \right) - r \left( \frac{H}{R} \right) + \frac{1}{2} (\sigma^2 A^2) s \left( \frac{1}{R} \right) + (\sigma^2 A) s \left( \frac{1}{R} \right) + \frac{1}{2} (\sigma^2 S^2) s \left( \frac{H}{R} \right), \]  

where we have used the fact that

\[ \sigma A_{W} R = R_s \sigma S + R_{W} A \sigma, \]  

which is a consequence of (7) and (8), and (23) for (28), and (7), (9), (10), and (22) for (29).

Following the definition of \( K \), we have

\[ \mathcal{L}(\sigma A) = \sigma K \rightleftharpoons r \rightleftharpoons \sigma A C_{W} - \sigma S C_s \]  

Since \( \frac{\sigma A}{R} = -\hat{\kappa} + \sigma S \frac{H}{R} \),

\[ \mathcal{L} \left( \frac{\sigma A}{R} \right) = -\mathcal{L}(\hat{\kappa}) + \mathcal{L} \left( \sigma S \frac{H}{R} \right). \]

Applying \( \mathcal{L}(XY) = Y \mathcal{L}(X) + X \mathcal{L}(Y) + \sigma^2 A X W Y W + \sigma^2 A S (X S Y W + X W Y S) + \sigma^2 S^2 X S Y S \) and substituting (28), (29) and (31) into the above equation, we get (27).

Proposition 3 plays an important role. It allows us to express the unknown functions \( R \) and \( H \) in terms of \( A \) and \( C \) and their derivatives. This is done as follows. If \( \Gamma_1 \sigma S + \Gamma_2 \kappa \neq 0 \), we can solve from (7) and (27) for \( R \) and \( H \) as functions of \( A \) and its derivatives, except when \( W = 0 \) at \( T \):

\[ R = \frac{\sigma^2 S K - \sigma A \Gamma_2}{\sigma S \Gamma_1 + \hat{\kappa} \Gamma_2}, \]

\[ H = \frac{\sigma A \Gamma_1 + \hat{\kappa} \sigma K}{\sigma S \Gamma_1 + \hat{\kappa} \Gamma_2}. \]

Once this is done, (23) becomes a PDE in \( A \) and \( C \). In addition, substituting (32) and (33) into (22) expresses \( N \) solely in terms of \( C \), and \( A \) and their derivatives.

If \( \Gamma_1 \sigma S + \Gamma_2 \kappa = 0 \), we cannot solve for \( R \), \( H \), and \( N \) in terms of \( C \) and \( A \). Nevertheless, we still get a PDE that \( C \) and \( A \) must satisfy. We take two cases. First, \( \Gamma_1 = \Gamma_2 = 0 \) except possibly when \( \kappa = 0 \). (Here we note that if \( \kappa \neq 0 \), \( \Gamma_1 = 0 \) only if \( \Gamma_2 = 0 \). In addition, \( \Gamma_2 = 0 \) implies \( \Gamma_1 = 0 \).) Then we have \( K = 0 \) except possibly when \( \kappa = 0 \). Second, \( \Gamma_1 \neq 0 \) and \( \Gamma_2 \neq 0 \) except possibly when \( \kappa = 0 \). Then (7) and (27) imply that

\[ \frac{-\Gamma_1}{\kappa} = \frac{\Gamma_2}{\sigma S} = \frac{K}{A} \]  

except possibly when \( \kappa = 0 \). In both cases, we have an equation solely in terms of \( A \) and \( C \) to verify. We now collect all the necessary conditions any \((C, A) \in \mathcal{E}\) have to satisfy:

**Theorem 2** For \((C, A) \in \mathcal{E}\), we must have
(1) \( A(0, S, t) = C(0, S, t) = 0, S > 0 \) and \( t \in [0, T] \); and \( C_W(W, S, t) > 0 \) when \( C > 0, W > 0, S > 0, t \in [0, T] \);

(2) Suppose \( \Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0 \). Define \( R, H, \) and \( N \) as in (32), (33), and (22), respectively; and define \( Q, X, \) and \( Y \) as in (11), (12), and (13), respectively. Then

(a) \( R(W, S, t) > 0 \) for \( W > 0 \);

(b) the state-dependent conditions: \( U(C^{-1}(x, S, t), S, t) \) is independent of \( S \) for all \( x > 0 \) and \( t < T \), \( U(W, S, t) \geq \lim_{\tau \to 0} U(C^{-1}(x, \hat{S}, \tau), \hat{S}, \tau) \) for all \( W > 0, S > 0, \) and \( t \in [0, T] \) such that \( C(W, S, t) = 0 \) and all \( \hat{S} > 0 \), \( \lim_{\tau \to T} U(W, S, t) \geq \lim_{\tau \to T} U(W, \hat{S}, T) \) for all \( S > 0 \) and \( \hat{S} > 0 \), \( H(W, S, T) = 0 \) for all \( (W, S) \), and \( A(W, S, T) / \kappa(S, T) \) is a function of \( W \) only;

(c) the consistency conditions (9), (9), and (10) hold;

(d) \( (C, A) \) satisfies the PDE (29);

(3) Suppose that \( \Gamma_1 = \Gamma_2 = 0 \) except possibly when \( \hat{\kappa} = 0 \). Then \( K = 0 \) except possibly when \( \hat{\kappa} = 0 \);

(4) Suppose that \( \Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0 \) and \( \Gamma_1 \neq 0 \) and \( \Gamma_2 \neq 0 \) except possibly when \( \hat{\kappa} = 0 \). Then (34) holds except possibly when \( \hat{\kappa} = 0 \).

Note that in Theorems 1 and 2, (23) is the most substantive necessary condition. Other conditions are either consistency conditions or the state-independent conditions.

We now present two examples, one to demonstrate the necessary condition for efficiency and another to demonstrate a price process where \( \Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0 \). More examples can be found in Section 5.

Example 1 It is well-known that the pair

\[
C(W, S, t) = f(t)W, \\
A(W, S, t) = \frac{\mu(S, t) - r}{\sigma^2(S, t)}W,
\]

with \( f(t) > 0 \), is the optimal consumption-portfolio policy for the log utility function with certain time preferences captured by \( f(t) \). Since \( C > 0 \) for all \( W > 0 \) and \( S > 0 \), this policy must satisfy the conditions of Theorems 1 and 2.

First, the consumption policy implies that \( H(W, S, t) = 0 \) and thus the portfolio policy implies \( R(W, S, t) = W \). Direct computation using (22) gives \( N(t) = -f(t) \). One can easily verify that all the conditions of Theorem 1 are satisfied.
Next we turn our attention to Theorem 2. Note that this theorem applies generally independently of whether \( C > 0 \) for all \( W > 0 \). Direct computation shows that \( K(W, S, t) = \Gamma_1(S, t) W / \sigma(S, t) \).

We take cases. Case 1. Suppose that \( \Gamma_1 \sigma S + \Gamma_2 \bar{k} \neq 0 \). By (32) and (33), we have

\[
R = \frac{\sigma^2 \Gamma_1 / \sigma + \bar{k} \Gamma_2}{\sigma \Gamma_1 + \bar{k} \Gamma_2} W = W,
\]

\[
H = \frac{-\bar{k} \Gamma_1 + \bar{k} \sigma \Gamma_1 / \sigma}{\sigma \Gamma_1 + \bar{k} \Gamma_2} W = 0.
\]

These are consistent with our calculation above while using Theorem 1. Then it is straightforward to verify that (2a)–(2d) of Theorem 2 are satisfied.

Case 2. Suppose that \( \Gamma_1 = \Gamma_2 = 0 \). Then \( K = 0 \) and (9) of Theorem 2 is satisfied.

Case 3. Suppose that \( \Gamma_1 \sigma S + \Gamma_2 \bar{k} = 0 \) and \( \Gamma_1 \neq 0 \) and \( \Gamma_2 \neq 0 \). Then \( K/A = -\Gamma_1 / \bar{k} \) and (4) of Theorem 2 is satisfied.

The following example gives a scenario where \( \Gamma_1 = \Gamma_2 = 0 \).

Example 2 Suppose that \( \Gamma_1 \sigma S + \Gamma_2 \bar{k} = 0 \) and \( \sigma \) is a constant. Then \( \Gamma_2 = 0 \). This implies that \( \Gamma_1 = 0 \). Note that \( \Gamma_1 \) is the drift term of \( d\bar{k} \). Thus \( \bar{k} \) must be a local martingale. Since \( \sigma \) and \( r \) are constant, this implies that \( \mu \) is a local martingale.

Conversely, given that \( \sigma \) is a constant and \( \mu \) is a local martingale, \( \Gamma_1 = \Gamma_2 = 0 \). In this case,

\[
K = \frac{1}{2} \sigma^2 A^2 A_W W + \sigma^2 S A A_W S + \frac{1}{2} \sigma^2 S^2 A_S S + r W A_W + r S A_S - r A + A + C_W A + S C_S - C A_W = 0.
\]

Using the same arguments as in the proof of Proposition 2 we have that

\[
A(t)e^{-rt} + \int_0^t (C_W(s)A(s) + S(s)C_S(s) - C(s)A_W(s))ds, \quad t \in [0, T)
\]

is a local martingale under \( Q \), and is a martingale under \( Q \) with similar regularity conditions as in Corollary 2. In particular, if \( C = 0, W(T) > 0 \) a.s. and if \( A \) satisfies a growth condition stipulated in Corollary 2, we know \( A(t)e^{-rt} \) must be martingale under \( Q \). Thus the present value of the future investment in the stock must be equal to the current investment, a property obeyed by any optimal policy satisfying the same regularity conditions in the geometric Brownian motion case.

Besides checking whether a given pair of policies satisfy the necessary conditions for efficiency for a fixed stock price process, Theorems 1 and 2 can also be utilized to answer a more general question: Is a given pair of policies efficient for some stock price process? The following familiar example demonstrates this.

For example, it is well-known that when the stock price follows a geometric Brownian motion and if the utility function exhibits a constant Arrow-Pratt measure of relative risk aversion, then
the optimal portfolio policy is a constant mix policy, that is, \( A(t) = \alpha W \) for some \( \alpha \), and the optimal consumption policy is a linear policy \( C(t) = f(t)W \) for some strictly positive function \( f(t) \). On the other hand, it seems quite likely that for this pair of linear policies to be optimal it would be necessary that the stock price process is a geometric Brownian motion and the utility function must exhibit a constant Arrow-Pratt measure of relative risk aversion. Using Theorem 1, we show in the following example that these linear policies only necessitate that \( \kappa \) be a constant and \( \mu \) and \( \sigma \) be independent of \( S \), except in the case where \( \alpha = \kappa \) and we are back to the case in Example 1.

**Example 3** Let \( C(W,S,t) = f(t)W \) and \( A(W,S,t) = \alpha W \), where \( f(t) > 0 \) and \( \alpha \neq 0 \). Since the consumption policy is independent of \( S \), \( H = 0 \). This implies that \( R(W,S,t) = \alpha W/\kappa(S,t) \) and

\[
N = (\mu - r)\frac{\kappa - \alpha}{2} + \frac{r\kappa}{\alpha} - f(t)\frac{\kappa}{\alpha} - r + \sigma^2 S \kappa_S.
\]

Thus

\[
\ln U(x/f(t),S,t) = \frac{\kappa(S,t)}{\alpha} \left[-\ln (x/f(t)) + \ln W_0\right] + Y(S_0,t). \tag{35}
\]

By the state-independence of the utility function, the right-hand-side must be independent of \( S \). Thus \( \kappa \) is a function only of \( t \), and hence, \( N = (\mu - r)(\kappa - \alpha)/2 + (r - f(t))\kappa/\alpha - r \).

Assume without loss of generality that \( \kappa(t) \neq 0 \). Then \( R(W,S,t) = \alpha W/\kappa(t) \) must satisfy (29). This implies \( \kappa'(t)\alpha W = 0 \) for all \( W > 0 \) and \( \kappa(t) \) must be independent of \( t \) and is a constant. Relation (35) thus shows that the utility function exhibits a constant relative risk aversion equal to \( \kappa/\alpha \).

The fact that \( \kappa \) is a constant does not necessarily mean that \( \mu \) and \( \sigma \) are constants. Now note that the consistency condition (9) implies that

\[
N_S = \mu_S \frac{\kappa - \alpha}{2} = 0.
\]

Suppose \( \kappa \neq \alpha \). Then \( \mu_S = 0 \) and \( \mu \) is independent of \( S \). Consequently, \( \sigma \) is independent of \( S \).

In summary, \( \kappa \) is a constant and the utility function must exhibit a relative risk aversion \( \kappa/\alpha \). In addition, \( \mu \) and \( \sigma \) are functions of time only if \( \kappa \neq \alpha \).

4 Sufficient Conditions for Efficiency

We give two sets of sufficient conditions in this section for a given \( (C,A) \) that satisfy Conditions A1, A2, and A4 of Section 2 to be efficient. First, for a \( C \) such that \( C(W,S,t) > 0 \) for all \( W > 0 \), the necessary conditions recorded in Theorem 1 together with the hypothesis that the \( R, H, \) and \( N \) defined in (26), (24), and (22), respectively, satisfy Conditions A3 and A5 are sufficient for \( (C,A) \) to be efficient. Second, in the case where consumption is not always strictly positive for strictly
positive wealth and where $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$, the necessary conditions of Theorem 2 together with
the same conditions on $R$, $H$, and $N$, are also sufficient. Our proof for these two sets of sufficient
conditions are through construction: we construct a pair of utility functions $(u, V)$ so that the
policy $(C, A)$ solves (2).

Since any efficient policy must be such that $C(0, S, t) = 0$ and $A(0, S, t) = 0$, that is, whenever
the wealth reaches zero there will be neither investment nor consumption afterwards, we will restrict
our attention to this kind of policies. For any one of these policies, it follows from Dybvig and
Huang (1989) that we must have, for all $t$,

$$E_t^* \left[ \int_t^T C(W(s), S(s), s)e^{-r(s-t)}ds + W(T)e^{-r(T-t)} \right] \leq W(t), \quad (36)$$

that is, the present value of future consumption and final wealth must be less than the current
wealth. Cox and Huang (1989) show that the sufficient conditions for $(C, A)$ to be a solution to (2)
are that i) (36) holds with equality; ii) there exists a strictly positive scalar $\lambda > 0$ so that, for all
t < T,

$$u_c(C(W(t), S(t), t), t) \begin{cases} = \lambda \xi(t)e^{-rt} & \text{if } C(W(t), S(t), t) > 0, \\ \leq \lambda \xi(t)e^{-rt} & \text{if } C(W(t), S(t), t) = 0; \end{cases} \quad (37)$$

$$V(W(T)) \begin{cases} = \lambda \xi(T)e^{-rT}, & \text{if } W(T) > 0, \\ \leq \lambda \xi(T)e^{-rT}, & \text{if } W(T) = 0, \end{cases} \quad (38)$$

where we recall the definition of $\xi$ in (1) and its interpretation as the Arrow-Debreu price for time
t consumption per unit of probability $P$. The following is our first set of sufficient conditions:

**Theorem 3** Let $S\sigma$ satisfy a linear growth condition and let $(C, A)$ satisfy Conditions A1, A2, and
A4, and $C > 0$ for all $W > 0$. Define $R$, $H$, $N$ as in (28), (24), and (22), respectively, and $Q$, $X$, $Y$, and $U$ as in (11), (12), (13), and (14) respectively. Suppose that

(1) $R$, $H$, $N$ satisfy the continuity and differentiability conditions of Conditions A3 and A5;

(2) conditions (1)–(6) of Theorem 1 are satisfied.

Then $(C, A) \in \mathcal{E}$ and the utility functions correspond to $(C, A)$ are

$$\ln u_c(x, t) = \begin{cases} U(C^{-1}(x, S, t), S, t) & x > 0, \\ \lim_{x \downarrow 0} U(C^{-1}(x, S, t), S, t) & x = 0; \end{cases}$$

$$\ln V'(x) = \begin{cases} U(x, S, T) & x > 0, \\ \lim_{x \downarrow 0} U(x, S, T) & x = 0. \end{cases}$$
5 FURTHER EXAMPLES

PROOF. See Appendix.  

When $C$ is not always strictly positive for strictly positive wealth, $R$ cannot be defined through (26) and we have the second set of sufficient conditions. Note however that this set of conditions applies generally whenever $\Gamma_1^{\sigma}S + \Gamma_2^{\hat{k}} \neq 0$, independent of whether $C(W, S, t) > 0$ for all $W > 0$.

**Theorem 4** Let $\sigma$ satisfy a linear growth condition and let $(C, A)$ satisfy Conditions A1, A2, and A4, and $\Gamma_1^{\sigma}S + \Gamma_2^{\hat{k}} \neq 0$. Define $R$, $H$, and $N$ as in (32), (33), and (22), respectively; and define $Q$, $X$, $Y$, and $U$ as in (11), (12), (13), and (14), respectively. Suppose that

1. $R$, $H$, and $N$ satisfy Conditions A3 and A5;

2. conditions (1), and (2a)-(2d) of Theorem 2 are satisfied.

Then $(C, A) \in \mathcal{E}$ and the utility functions correspond to $(C, A)$ are

$$\ln u_c(x, t) = \begin{cases} 
U(C^{-1}(x, S, t), S, t) & x > 0, \\
\lim_{z \to 0^+} U(C^{-1}(x, S, t), S, t) & x = 0;
\end{cases}$$

$$\ln V'(x) = \begin{cases} 
U(x, S, T) & x > 0, \\
\lim_{z \to 0^+} U(x, S, T) & x = 0.
\end{cases}$$

PROOF. See Appendix.  

Using Theorem 3 one easily shows that the linear policies in Example 3 are indeed optimal for the stock price process identified there and the utility function exhibiting a constant relative risk aversion equal to $\kappa/\alpha$.

Before leaving this section, we point out that the necessary and sufficient conditions established in Sections 3 and 4 can be readily extended to infinite horizon problems, i.e., $T = \infty$ and $V = 0$. In this case, in Theorems 1 and 2, all conditions relating to time $T$ should be removed. In addition, in Theorems 3 and 4, one needs to add that the present value of the future wealth goes to zero when the future extends to infinity; that is, $E^\pi[W(t)e^{-rt}] \to 0$ as $t \to \infty$.\textsuperscript{14}

5 Further Examples

We present in this section two more examples to demonstrate our results.

**Example 4** Consider a pair of consumption and investment functions

$$C(W, S, t) = f(t)W^{\alpha(t)}S^{1-\alpha(t)}$$

$$A(W, S, t) = \left( \frac{\mu(S, t) - \rho W^\beta(t)S^{1-\beta(t)}}{\sigma(S, t)^2} \frac{1 - \alpha(t)}{\alpha(t)} \right)$$

\textsuperscript{14}See Huang and Pagès (1991).
where \( f, \alpha, \) and \( \beta \) are strictly positive and deterministic functions of \( t \) satisfying \( \lim_{t \to T} \alpha(t) = 1 \), and \( \rho \) is a strictly positive constant. The parameters of the wealth process generated by \((C, A)\) may not satisfy a linear growth and a local Lipschitz condition for some functions \( \alpha \) and \( \beta \). We will ignore this problem for now and proceed with other necessary conditions.

We will show that for this pair of policies to be efficient, it is necessary that \( \alpha = \beta = 1 \) for all \( t \) and either \( \rho = 1 \) or \( (\mu - r)/\sigma \) is independent of \( S \).

To begin, the consumption policy determines the hedging demand function

\[
H(W, S, t) = \frac{C_S(W, S, t)}{C_W(W, S, t)} = -\frac{1 - \alpha(t)}{\alpha(t)} \frac{W}{S},
\]

which in turn determines the risk tolerance function

\[
R(W, S, t) = \frac{1}{\rho} W^{\beta(t)} S^{1 - \beta(t)}.
\]

Since \( \alpha(t) \to 1 \) as \( t \to T \), we have \( H(W, S, t) \to 0 \) as \( t \to T \), which is (25) and is part of the state-independence condition.

Next, for (8) to be satisfied, we need

\[
\rho (\beta(t) - 1) W^{-\beta(t)} S^{\beta(t) - 2} = -\rho (\beta(t) - 1) \frac{1 - \alpha(t)}{\alpha(t)} W^{-\beta(t)} S^{\beta(t) - 2}.
\]

This implies that \( \beta(t) = 1 \) for all \( t \) and thus

\[
R(W, S, t) = \frac{1}{\rho} W, \quad A(W, S, t) = \left( \frac{\mu(S, t) - r}{\sigma(S, t)^2 \rho} - \frac{1 - \alpha(t)}{\alpha(t)} \right) W.
\]

Now, define \( N \) according to (22):

\[
N = \frac{1}{2} \sigma^2 A^2 \left( -\frac{\rho}{W^2} \right) - \frac{1}{2} \sigma^2 S^2 \left( \frac{1 - \alpha(t)}{\alpha(t)} \right) \frac{\rho}{S^2} + (rW + A(\mu - r) - C(t) W^{\alpha(t)} S^{1 - \alpha(t)}) \frac{\rho}{W} + \mu S \left( \frac{1 - \alpha(t)}{\alpha(t)} \frac{\rho}{S} \right) - \frac{\kappa^2 \sigma^2}{2} - r.
\]

Since \( NW = R_t / R^2 = 0 \) by (10), we deduce that

\[
-f(t) \rho (\alpha(t) - 1) W^{\alpha(t) - 1} S^{1 - \alpha(t)} = 0
\]

for all \( W > 0, S > 0, \) and \( t \in [0, T] \). This cannot be true unless \( \alpha(t) = 1 \) for all \( t \). Consequently, the hedging demand must be zero and that

\[
N = \frac{1}{2} \left( 1 - \frac{1}{\rho} \right) \kappa^2 \sigma^2 + r\rho - r - f(t)\rho.
\]

Consequently, (9) implies that

\[
N_S = \frac{1}{2} \left( 1 - \frac{1}{\rho} \right) (\kappa^2 \sigma^2)_S = (H/R)_t = 0,
\]
and we must either have \( \rho = 1 \) or \((\kappa^2 \sigma^2)_S = 0\). Note that in the former case the parameters of the wealth process satisfy a local Lipschitz and a growth condition if \((\mu - r)^2/\sigma^2\) does, and in the latter case the parameters of the wealth process is purely deterministic.

In summary, in order for the pair of policies to be efficient for some stock price process, it must be the case that \(\alpha\) and \(\beta\) be constant and equal to one, and either \(\rho = 1\) or \((\mu - r)/\sigma\) is independent of \(S\). Indeed, when \(\rho = 1\), the log utility function supports \((C, A)\), and when \(\kappa^2 \sigma^2\) is independent of \(S\), the utility function that exhibits a constant coefficient of relative risk aversion equal to \(\rho\) supports \((C, A)\).

Note that in the case where \(\kappa^2 \sigma^2\) is independent of \(S\), \(\kappa\) is independent of \(S\). From Cox and Huang (1989), there will be no hedging demand. The optimal consumption policy will be a function only of wealth, and the optimal portfolio policy can be calculated as an explicit integral and can be represented generally as

\[
A(W, S, t) = \frac{\kappa}{\sigma(S, t)} g(W(t), t)
\]

for some function \(g\).

**Example 5** Consider a pair of consumption and investment functions for an infinite horizon problem,

\[
C(W, S, t) = \gamma SW,
\]

\[
A(W, S, t) = f(S)W,
\]

where \(f(S) = A_3 - r/(\alpha \sigma^2) - \sqrt{A_1 + A_2 S}\), where \(\alpha > 0\), \(\gamma > 0\), \(\sigma > 0\), \(A_1 \geq 0\), \(A_2 > 0\), and \(A_3 > 1\) are constants. Note that the marginal propensity to consume is proportional to the stock price and the proportion of the wealth invested in the stock is a decreasing function of the stock price and is positive when the stock price is low (for sufficiently large \(A_3\)) and is negative when the stock price is high. Note also that with \((C, A)\) defined above, the parameters of the wealth dynamics may not satisfy a linear growth condition. We will ignore this problem and proceed with other necessary conditions. We ask: Can \((C, A)\) be a pair of optimal policy for some utility function and for some stock price process? We will see that the answer is affirmative for the following price process:

\[
dS(t) = \alpha \sigma^2 (A_3 + 1 - \sqrt{A_1 + A_2 S(t)})S(t)dt + \sigma S(t)dw(t),
\]

provided that there exists a solution to this stochastic differential equation, the equivalent martingale measure exists for this price process on any finite interval \([0, t]\), the parameters of the price process
satisfy certain restrictions, and the wealth never reaches zero. By Itô’s lemma,
\[ d\ln S(t) = \alpha \sigma^2 \left( A_3 + 1 - \frac{1}{2\alpha} - \sqrt{A_1 + A_2 S(t)} \right) dt + \sigma dw(t). \]
Thus the log price process follows a mean reversion process if \( A_3 + 1 - 1/2\alpha > 0 \). Note first that the consumption policy implies that \( H = -W/S \). Thus
\[ R = \frac{A - SH}{\kappa} = \frac{f + \frac{1}{\kappa} W}{\kappa} = \frac{1}{\alpha} W. \]
Using relation (22) we write
\[
N = -\frac{\alpha \sigma^2}{2} \left( A_3 - \frac{r}{\alpha \sigma^2} - \sqrt{A_1 + A_2 S} \right)^2 - \frac{1}{2} \alpha \sigma^2 \\
+ \alpha \left( r + \alpha \sigma^2 \left( A_3 - \frac{r}{\alpha \sigma^2} - \sqrt{A_1 + A_2 S} \right) \right) \left( A_3 - \frac{r}{\alpha \sigma^2} + \frac{A_2 S}{\sqrt{A_1 + A_2 S}} + 1 - \gamma S \right) \\
+ \alpha^2 \sigma^2 \left( A_3 + 1 - \sqrt{A_1 + A_2 S} \right) - \frac{\alpha^2 \sigma^2}{2} \left( A_3 - \frac{r}{\alpha \sigma^2} + 1 - \sqrt{A_1 + A_2 S} \right)^2 - r. \\
= -\frac{\alpha \sigma^2}{2} \left( A_3 - \frac{r}{\alpha \sigma^2} - \sqrt{A_1 + A_2 S} \right)^2 + \frac{\alpha^2 \sigma^2}{2} \left( A_3 - \frac{r}{\alpha \sigma^2} + 1 - \sqrt{A_1 + A_2 S} \right)^2 \\
- \alpha \gamma S - \frac{1}{2} \alpha \sigma^2 + 2\alpha r - r.
\]
Let
\[
\gamma = \frac{1}{2} A_2 \sigma^2 (\alpha - 1), \\
\alpha = \frac{A_3 + r/\sigma^2 + \sqrt{(A_3 + r/\sigma^2)^2 - 4(A_3 + 1)r/\sigma^2}}{2(A_3 + 1)}.
\]
It is clear that \( \alpha > 0 \) for sufficiently large \( A_3 \). But we also need \( \gamma \) to be strictly positive. Assume therefore that the parameters of the price processes are such that \( \alpha > 1 \), which can be achieved if \( r > \sigma^2 \) and \( A_3 \) is sufficiently large. This implies that \( \ln S \) follows a mean reversion process. For the \( \gamma \) and \( \alpha \) defined above, \( N \) is independent of \( S \) and is a constant. The consistency and the state-independence conditions are easily seen to be satisfied.

Next, one verifies that (23) is satisfied and, by Theorem 3, \((C, A)\) is efficient provided if all the regularity conditions stated above can be verified. The utility function that supports \((C, A)\) exhibits a constant coefficient of relative risk aversion equal to \( \alpha \), which is strictly greater than one.

6 Concluding Remarks

We have derived the necessary and sufficient conditions for a given consumption-portfolio policy to be optimal for some time-additive utility function in an economy with one risky and one riskless

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15 We ignore the condition that the expected discounted wealth under the equivalent martingale measure goes to zero. These conditions may be checked by simulation.
asset. The risky asset price follows a general diffusion process and the riskless interest rate is a constant. Using identical arguments, we can generalize the results reported to cases where there are more than one risky asset and the interest rate is stochastic. We leave this generalization to the interested reader.

The technique we have exploited to derive our results is dynamic programming. We have not investigated but it seems plausible that our method might also be generalized to allow for non-additive utility functions as long as the optimal consumption-portfolio problem for these utility functions can be analyzed by dynamic programming. This may be a fruitful direction for future research given the increasing interest in non-additive utility functions.
7 References


Appendix

A Proofs

Proof of Theorem 3. First we show that (36) holds as an equality. Under $Q$, the discounted wealth process and the discounted stock price process become

\[
\begin{align*}
    d(W(t)e^{-rt}) &= -C(t)e^{-rt}dt + A(t)\sigma(t)e^{-rt}dw^*(t), \quad t \in [0,T], \\
    d(S(t)e^{-rt}) &= S(t)e^{-rt}\sigma(t)dw^*(t),
\end{align*}
\]

where we recall that $w^*$ is a Brownian motion under $Q$. By the hypothesis that $A\sigma$ and $S\sigma$ satisfy a linear growth condition (see Condition A1 for the former), Friedman (1975, theorem 5.2.3) shows that there exist constants $L_m$ so that $E^*[|W(t)|^{2m}] \leq (1 + |W_0|^{2m})e^{L_mt}$ and $E^*[|S(t)|^{2m}] \leq (1 + |S(0)|^{2m})e^{L_mt}$, for all integers $m = 1, 2, \ldots$. Given this, we can easily show that

\[
    E^*\left[\int_0^T |A(t)\sigma(t)e^{-rt}|^2 dt\right] < \infty.
\]

Hence $W(t)e^{-rt} + \int_0^t C(s)e^{-rs}ds$ is a square integrable martingale under $Q$; see, for example, Liptser and Shiryayev (1977, §4.2). Consequently, (36) holds as an equality.

Next, by condition (4) of Theorem 1, $u$ and $V$ are well-defined and are state-independent; and by condition (2) of Theorem 1, $u$ and $V$ are strictly increasing concave. We remain to verify that there exists a $\lambda > 0$ such that the first order conditions (37) and (38) hold. For any function $f$, define the operator $\bar{L}$:

\[
\bar{L}(f) = \frac{1}{2} \sigma^2 A^2 \frac{\partial^2 f}{\partial W^2} + \sigma^2 S^2 \frac{\partial^2 f}{\partial W \partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (rW - C) \frac{\partial f}{\partial W} + rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} - rf + C_W f.
\]

Condition (5) of Theorem 1 can be written as $\bar{L}(R) = 0$. This implies that

\[
\bar{L}\left(\frac{1}{R}\right) = \frac{\sigma^2 A^2 R_W^2}{R^3} + 2\sigma^2 S^2 \frac{R_W R_S}{R^3} + \sigma^2 S^2 \frac{R_S^2}{R^3} - \frac{2r}{R} + \frac{2C_W}{R},
\]

where the second equality follows from (30), which is a consequence of the definition of $R$ and $H$ in (26) and (24), respectively, and the consistency condition (8). By Condition A4, the wealth never reaches zero before $T$. For any $t \in [0,T)$, Itô's lemma implies that

\[
\begin{align*}
    dQ(W(t), S(t), t) &= \left(\mathcal{L}(Q) + Q_t\right)dt + \frac{1}{R(W(t), S(t), t)} \sigma A dw(t) \\
    &\quad + \left(\int_{W_0}^{W(t)} \left(\frac{1}{R(z, S(t), t)}\right) dz\right) \sigma S(t) dw(t),
\end{align*}
\]
where $\mathcal{L}$ is the differential generator of $W$ and $S$ under $P$. By the consistency condition (8), the diffusion term of $dQ$ becomes

$$
\frac{1}{R(W(t), S, t)} \sigma A - \left( \int_{W_0}^{W(t)} \left( \frac{H(z, S, t)}{R(z, S, t)} \right)_z dz \right) \sigma S
$$

$$
= \left( \frac{1}{R(W(t), S, t)} \sigma A - \frac{H(W(t), S, t)}{R(W(t), S, t)} \sigma S + \frac{H(W(t), S, t)}{R(W(t), S, t)} \sigma S \right)
$$

$$
= \left( -\dot{\kappa}(S) + \frac{H(W(t), S, t)}{R(W(t), S, t)} \sigma S \right).
$$

The drift of $dQ$ is

$$
\mathcal{L}(Q) + Q_t = \mu S \int_{W_0}^{W(t)} \left( \frac{1}{R} \right)_s dz + (rW(t) + A(\mu - r) - C) \frac{1}{R(W(t), S, t)}
$$

$$
+ \frac{1}{2} \sigma^2 S^2 \int_{W_0}^{W(t)} \left( \frac{1}{R(z, S, t)} \right)_{ss} dz + \frac{1}{2} \sigma^2 A^2 \left( \frac{1}{R(W(t), S, t)} \right)_w
$$

$$
+ \sigma^2 A S \left( \frac{1}{R(W(t), S, t)} \right)_t + \int_{W_0}^{W(t)} \left( \frac{1}{R(z, S, t)} \right)_t dz.
$$

Using integration by parts, we have

$$
\mathcal{L}(Q) + Q_t = \int_{W_0}^{W(t)} \mu S \left( \frac{1}{R(W(t), S, t)} \right)_s dz + \int_{W_0}^{W(t)} (rz + A(\mu - r) - C) \left( \frac{1}{R(z, S, t)} \right)_z dz
$$

$$
+ \int_{W_0}^{W(t)} \left( r + A_x(\mu - r) - C_z \right) \frac{1}{R} dz + \int_{W_0}^{W(t)} \frac{1}{2} \sigma^2 S^2 \left( \frac{1}{R(z, S, t)} \right)_{ss} dz
$$

$$
+ \int_{W_0}^{W(t)} \sigma^2 A^2 \left( \frac{1}{R(z, S, t)} \right)_{xx} dz + \int_{W_0}^{W(t)} \sigma^2 A z \left( \frac{1}{R(z, S, t)} \right)_z dz
$$

$$
+ \int_{W_0}^{W(t)} \sigma^2 A S \left( \frac{1}{R(z, S, t)} \right)_t dz + \int_{W_0}^{W(t)} \sigma^2 A S \left( \frac{1}{R(z, S, t)} \right)_s dz
$$

$$
+ \int_{W_0}^{W(t)} \frac{1}{R(z, S, t)} dz + h_1(S, t),
$$

where

$$
h_1(S, t) = (rW_0 + A(W_0, S, t)(\mu - r) - C(W_0, S, t)) \frac{1}{R(W_0, S, t)}
$$

$$
+ \frac{1}{2} \sigma^2 A^2(\mu - r) \left( \frac{1}{R(W_0, S, t)} \right)_w + \sigma^2 A(W_0, S, t) S \left( \frac{1}{R(W_0, S, t)} \right)_s.
$$

Using the definition of $F$, we get

$$
\mathcal{L}(Q) + Q_t = \int_{W_0}^{W(t)} \mathcal{L} \left( \frac{1}{R(z, S, t)} \right) dz + \int_{W_0}^{W(t)} (\mu - r) S \left( \frac{1}{R(z, S, t)} \right)_s dz
$$

$$
+ \int_{W_0}^{W(t)} (\mu - r) A \left( \frac{1}{R(z, S, t)} \right)_z dz + \int_{W_0}^{W(t)} (\mu - r) A_x \left( \frac{1}{R(z, S, t)} \right)_x dz
$$
\[
+ \int_{W_0}^{W(t)} \sigma^2 A_z A \left( \frac{1}{R(z, S, t)} \right)_z dz + \int_{W_0}^{W(t)} \sigma^2 A_z S \left( \frac{1}{S} \right)_z dz \\
+ \int_{W_0}^{W(t)} 2r \frac{2 C_z(z, S, t)}{R(z, S, t)} dz - \int_{W_0}^{W(t)} \frac{2 C_z(z, S, t)}{R(z, S, t)} dz + h_1(S, t) \\
= \int_{W_0}^{W(t)} \sigma^2 A_z^2 \frac{1}{R(z, S, t)} dz + \int_{W_0}^{W(t)} \sigma^2 A_z \left( \frac{1}{R(z, S, t)} \right)_z dz \\
+ \int_{W_0}^{W(t)} \sigma^2 A_z S \left( \frac{1}{R(z, S, t)} \right)_z dz + h_1(S, t) \\
= h_1(S, t)
\]

where both the second and the third equalities follow from (30), which as we mentioned before is a consequence of the definition of \( R \) and \( H \) in (28) and (24), respectively, and the consistency condition (8). We can thus write

\[
dQ(W(t), S(t), t) = h_1(S(t), t) dt - \frac{H(W_0, S(t), t)}{R(W_0, S(t), t)} \sigma S(t) dw(t) - \kappa dw(t).
\]

Similarly, Itô's lemma implies that

\[
dx(S(t), t) = \frac{H(W_0, S(t), t)}{R(W_0, S(t), t)} \sigma S(t) dw(t) + h_2(S(t), t) dt,
\]

where

\[
h_2(S, t) = \frac{H(W_0, S, t)}{R(W_0, S, t)} \mu s + \frac{1}{2} \sigma^2 s^2 \left( \frac{H(W_0, S, t)}{R(W_0, S, t)} \right)_s + \int_{S_0}^{S} \left( \frac{H(W_0, \eta, t)}{R(W_0, \eta, t)} \right)_t d\eta.
\]

Using the consistency condition (9) and the definition of \( N \) in (22), we have

\[
-h_1(S, t) + h_2(S, t) = -N(W_0, S_0, t) - \frac{\kappa^2}{2} - r.
\]

Finally, since \( dY(t) = N(W_0, S_0, t) dt \),

\[
\ln \mathcal{U}(W(t), S(t), t) = -Q(W(t), S(t), t) + X(S(t), t) + Y(t) = \ln \xi(t) - rt.
\]

Since \( C(W, S, t) > 0 \) for all \( W > 0 \) and the wealth never reaches zero before \( T \) by the hypothesis, we thus have

\[
\ln u_c(C(W(t), S(t), t), t) = \ln \lambda + \ln \xi(t) - rt, \quad a.s. \quad t \in [0, T)
\]

with \( \lambda = 1 \). At \( T \), on the set where \( W(T) > 0 \), by the continuity of \( \ln \mathcal{U}(W, S, t) \) except when \( W = 0 \) at \( T \), we have

\[
\ln \xi(T) - rt = \lim_{t \uparrow T} \ln \mathcal{U}(W(t), S(t), t) = \ln \mathcal{U}(W(T), S(T), T) = \ln V'(W(T)).
\]

On the other hand, on the set where \( W(T) = 0 \), we have

\[
\ln \xi(T) - rt = \lim_{t \uparrow T} \ln \mathcal{U}(W(t), S(t), t) \geq \lim_{W \downarrow 0} \ln \mathcal{U}(W, S(T), T) = V'(0).
\]

We have thus shown that \((C, A)\) is efficient. \( \square \)
PROOF OF THEOREM 4. The only thing different in this case is that the consumption may not be strictly positive at nonzero wealth levels. But we still have

$$\ln U(W(t), S(t), t) = \ln \lambda + \ln \xi(t) - rt, \quad \text{a.s. } t \in [0, T)$$

with $\lambda = 1$. Naturally, if $C > 0$, $\ln u_c(CW(t), S(t), t) = \ln \xi(t) - rt$. If $C = 0$, condition (2b) shows that

$$\ln u_c(0, t) \leq \ln \xi(t) - rt.$$  

Finally, arguments identical to those used in the proof of Theorem 3 show that

$$\ln V'(W(T)) \leq \ln \xi(T) - rT$$

with equality holds for $W(T) > 0$. These prove the assertion.