Portfolio Policies with Transactions Costs: Discrete Time Model

by

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October 1991
RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

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Finance Working Paper #217

October 1991

The author would like to thank Gerard Gennette and Hayne Leland for their many helpful comments and suggestions. All errors are mine. Financial support from the Berkeley Program in Finance is gratefully acknowledged.
Abstract

The effect of proportional transactions costs on dynamic portfolio strategies is examined in discrete time. The model assumes a single risky asset and a single riskless asset. Adjustments in the dollar holdings of the assets are made through time to maximize the investor's expected utility of terminal wealth. There is no intermediate consumption. The optimal investment policy is described in terms of a no transaction region, where the optimal policy is to refrain from trading if initial portfolio holdings lie within the region, and to transact to the nearest boundary of the region if portfolio holdings lie outside the region. For positively homogeneous utility functions, the boundary is independent of current wealth, but dependent on future investment opportunities. The boundary widens as costs increase and narrows to a width of zero when costs tend to zero. For fixed wealth, the further away the investor's holdings are from the no transaction region, the lower his derived utility of wealth is.
Introduction

The effect of transactions costs on dynamic portfolio strategies is examined in discrete time. The model assumes a single risky asset and a single riskless asset. Adjustments in the dollar holdings of the assets are made through time, with each transaction subject to a trading cost. The cost is taken to be proportional to the size of the trade. In this setting, the paper examines the portfolio strategies of investors who adjust their portfolios through time to maximize the expected utility of their terminal wealth. In the absence of transactions costs, several authors, including Mossin [1968] and Hakansson [1970], showed that expected utility maximizers will adjust the composition of their portfolios at every available trading date. However, in the presence of transactions costs, Kamin [1975], Magill and Constantinides [1976], and Constantinides [1979] demonstrated that such investors may not always adjust their portfolio holdings at each trading date. Thus, unlike the no transactions cost case, the interval between trades is random. In particular, Constantinides [1979] showed that, for investors who maximize the expected utility of lifetime consumption over a finite time horizon, the optimal investment policy is described in terms of a region of no transactions. If the investor’s utility function is additively or multiplicatively separable and positively
homogeneous of degree $\alpha$ in consumption, if portfolio adjustments are subject to proportional costs, and if the return process exhibits constant returns to scale, then the no transaction region is a convex cone characterized by two parameters $a$ and $b$, $a < b$. Thus, if investors find themselves in the no transaction region, the optimal policy is to refrain from trading. The model does not lead to a complete specification of the parameters $a$ and $b$, but Constantinides does infer they may be functions of the ratio of the investor's current stock to bond holdings, the investment opportunity set, and time. In a succeeding paper, Constantinides [1986] looks at the infinite horizon problem from a continuous time perspective and finds that the two parameters are constant through time.

The topic of dynamic strategies under transactions costs has recently attracted substantial interest. Gennette and Jung [1990] apply the theoretical results of this paper and numerically solve for the no transactions region. They demonstrate that (1) the no transaction region narrows as the time horizon lengthens and converges to a constant width as the time horizon increases beyond several trading dates, and (2) converges sooner to a constant width as transactions costs decrease, stock volatility increases, risk aversion decreases, or the number of trading dates increases. Boyle and Vorst [1990] and Hodges and Neuberger [1989] analyze optimal hedging strategies in option pricing. Dumas and
Luciano [1989] were able to derive a closed form solution in the infinite horizon analog of this paper's model.

The optimal investment policy is of the control limit variety, where the parameters $a$ and $b$ are the optimal control limits. If the parameters are constant through time, then the no transaction region is constant through time and completely known at date 0. If the parameters depend on time, then the region varies through time -- varying as the investment opportunities change through time. However, if the parameters are functions of the composition of the investor's portfolio at each date, then the region is dependent on both initial wealth and future investment opportunities.

This paper examines the dependency of the control limits on (1) the composition of the investor's portfolio, (2) the investor's investment opportunity set, and (3) the size of transactions costs. Additionally, horizon effects on the controls are briefly explored. Part I of this paper reviews a specific example in discrete time without transactions costs. A graphical presentation intuitively linking the case with no transactions costs to the case with transactions cost is also presented. In part II, Constantinides' model on optimal portfolio policies in the face of proportional transactions costs is restated for an investor who maximizes the expected utility of his terminal wealth and who has no intermediate consumption. Part III, the heart of the paper,
derives the boundary of the no transaction region. A summary is provided in part IV.
I. Optimal Portfolio Policy without Transactions Costs

The investor's objective at each date, $t = 0, 1, \ldots, T - 1$, is to adjust the composition of his portfolio to maximize the expected utility of his terminal wealth. The portfolio consists of only two assets, a risky stock and a riskless bond. There is no intermediate consumption. The utility function obeys the usual Von Neumann-Morgenstern axioms, and in particular, this paper assumes it is a power function, viz., $U(W_t) = \delta^T w_t^\alpha / \alpha$ for $-\infty < \alpha < 1$, $\alpha \neq 1$, where $\alpha = 1 - \text{relative risk aversion (RRA) coefficient}$. In the absence of trading costs, there is no incentive to hinder portfolio transactions, and in general the investor will adjust his portfolio holdings at each trading date. For an investor with a power utility function, the optimal investment at each date depends only on his current and future investment opportunities.

Let $W_t$ be the investor's wealth at date $t$, and $\omega_t$ and $1 - \omega_t$ be his proportional holdings in the stock and bond, respectively. Let $S_{t+1}/S_t = z_{t+1}$ be the stock return process, where $z_{t+1}$ is stochastic, and $B_{t+1}/B_t = r$ be the bond process, where $r$ is the riskless return. Let $J(W_t, t)$ denote the investor's indirect utility function before an investment is made at date $t$. The investor's objective at this date is to choose a portfolio policy, $\omega_t$, for all $t = t$.

---

1The section follows the presentation of Ingersoll [1987], chapter 11.
t + 1, ..., T - 1, to maximize his expected utility of terminal wealth. That is, 

\[ J(W_t, t) = \max_{\omega_t, \gamma_{t+1}} \mathbb{E}_t[U(W_T, T)] \]

subject to the budget constraint

\[ W_0 > 0 \]

and wealth dynamics,

\[ W_{t+1} = W_t(\omega_t[z_{t+1} - r] + r) = W_t(Z_{t+1}(\omega_t)) \]

where the total return on the portfolio is \( Z_{t+1}(\omega_t) \). Upon following the dynamic programming algorithm, the maximization problem becomes

\[ J(W_t, t) = \max_{\omega_t} \mathbb{E}_t[J(W_{t+1}, t+1)] \tag{1} \]

subject to the boundary condition,

\[ J(W_T, T) = U(W_T, T) \]

Beginning with date \( t = T - 1 \), the investor chooses \( \omega_{T-1} \) to
maximize

\[ J(W_{T-1}, T-1) = \max_{\omega_{T-1}} E_{T-1}\{U(W_t)\} \]
\[ = \max_{\omega_{T-1}} E_{T-1}\left\{ \frac{1}{\alpha} \delta^T W^\alpha_T \right\} \]
\[ = \max_{\omega_{T-1}} \frac{1}{\alpha} \delta^T W^\alpha_T E_{T-1}\left\{ Z^\alpha_T(\omega_{T-1}) \right\} \]

The first order condition (FOC) is

\[ 0 = E_{T-1}\left\{ Z^{\alpha-1}_T(\omega_{T-1}) [z_T - r] \right\} \]
\[ = E_{T-1}\left\{ (\omega_{T-1} [z_T - r] + r)^{\alpha-1} [z_T - r] \right\} \]

and the optimal portfolio policy, \( \omega^*_{T-1} \), is independent of current wealth and only dependent on the investment opportunity set at date T - 1. The envelope condition states that the current marginal utility of wealth is equal to accumulated expected future marginal utility of wealth. This condition is derived by substituting the optimal portfolio policy, \( \omega^*_{T-1} \), into equation (1) and differentiating with respect to wealth. The result is

\[ J_{w_{T-1}} = \delta^T W^\alpha_{T-1} E_{T-1}\left\{ Z^\alpha_T(\omega^*_{T-1}) \right\} \]

The indirect utility function at date T - 1 also follows from equation (1):
\[ J(W_{T-1}, T-1) = \frac{1}{\alpha} \delta^* d_{T-1} W_{T-1}^\alpha > 0 \]

where

\[ d_{T-1} = E_{T-1} \{ d_T Z_T^\alpha (\omega_{T-1}^*) \} > 0, \text{ and } d_T = 1 \]

\[ Z_T = \omega_{T-1}^* (z_T^* - r) + r \]

and where \( d_{T-1} \) is expected future return (viz., future investment opportunities), and \( Z_T \) is total current return.

By induction, similar results hold for all previous periods. At date \( t + 1 \), the indirect utility function is

\[ J(W_{t+1}, t+1) = \frac{1}{\alpha} \delta^{t+1} d_{t+1} W_{t+1}^\alpha \]

where the FOC at date \( t \) is

\[ 0 = E_t \{ d_{t+1} (Z_{t+1})^{\alpha - 1} (z_{t+1} - r) \} \quad (2) \]

Then the optimal portfolio policy, \( \omega^*_t \), is independent of current wealth and dependent only on future investment opportunities (viz., \( Z_{t+1} \) and \( d_{t+1} \)). The corresponding indirect utility function at date \( t \) is
\[ J(W_t, t) = E_t \left\{ \frac{1}{\alpha} \delta^T d_{t+1} W_{t+1}^\alpha \right\} \]
\[ = \frac{1}{\alpha} \delta^T d_t W_t^\alpha \]  

where

\[ d_t = E_t \left\{ d_{t+1} Z_{t+1}^\alpha (\omega_t^*) \right\} > 0 \text{ and } z_{t+1} = \omega_t^* [z_{t+1} - r] + r \, . \]

Thus the indirect utility function is a state-dependent power function.

**Graphical Interpretation**

As an introduction to the transactions costs problem, let \( x_t^0 \) and \( x_t \) denote the pre-investment stock and bond positions at date \( t \), and \( y_t^0 \) and \( y_t \) denote the optimal post-investment stock and bond positions at that date. That is, the investor enters date \( t \) with wealth, \( W_t = x_t^0 + x_t \), and then rebalances his holdings to attain

\[ y_t = \omega_t^* [x_t^0 + x_t] = \omega_t^* W_t \]
\[ y_t^0 = [1 - \omega_t^*] [x_t^0 + x_t] = [1 - \omega_t^*] W_t \, . \]

Because the optimal portfolio policy, \( \omega_t^* \), is independent of the initial bond and stock holdings, \( x_t^0 \) and \( x_t \), the optimal post-investment holdings are linearly homogeneous in \( x_t^0 \) and
\( x_t \). That is, if \( y_t^0 \) and \( y_t \) are optimal for \( x_t^0 \) and \( x_t \), then \( \delta y_t^0 \) and \( \delta y_t \) are optimal for \( \delta x_t^0 \) and \( \delta x_t \) for all \( \delta > 0 \). Therefore, in the \( x_t^0 - x_t \) plane (figure 1), the optimal post-

\[ x_t', y_t \]

Transformation Path
slope = -1

\[ (x_t', x_t^0) \]

\[ y_t = m_t y_t^0 \]

\[ (y_t', y_t^0) \]

\[ x_t^0, y_t^0 \]

Figure 1

investment holdings lie along a ray originating from the origin. The ray, \( y_t = m_t y_t^0 \), has a slope of \( m_t = \omega_t^*/(1 - \omega_t^*) \) independent of \( x_t^0 \) and \( x_t \). Suppose the investor's initial holdings lie above the ray \( y_t = m_t y_t^0 \), i.e., \( x_t/x_t^0 > m_t \). Then the investor's optimal investment decision is to simultaneously reduce his stock position while increasing his bond position. The stock to bond conversion occurs at the transformation rate of \( \partial x_t/\partial x_t^0 = -1 \), and it stops when the resulting holdings lie along the ray \( y_t = m_t y_t^0 \), at which point the stock position has decreased by \( x_t - y_t \), and the bond position has increased by \( y_t^0 - x_t^0 \). The optimum \( (y_t^0, y_t) \) occurs when the investor's expected marginal rate of
substitution (MRS) equals the transformation rate. From the FOC equation (2),

\[ \frac{E_t \{ d_{t+1} (z_{t+1})^{a-1} z_{t+1} \}}{r E_t \{ d_{t+1} (z_{t+1})^{a-1} \}} = 1 \implies \text{MRS}_t = -\frac{\partial x_t}{\partial y_t} . \]

Conversely, if the investor's initial holdings lie below the ray, \( y_t = m_t y_t^0 \), i.e., \( x_t / x_t^0 < m_t \), then the investor's optimal investment strategy is to increase his stock holding while decreasing his bond holding.

If the returns, \( z_{t+1} \), are time independent, then the FOC equation (2) simplifies to

\[ 0 = E_t \{ (z_{t+1})^{a-1} (z_{t+1} - r) \} . \]

From inspection, the optimal portfolio policy, \( \omega^*_t \), depends only on the current investment set, in which case the investor behaves myopically. On the other hand, if the \( z_{t+1} \) are identically distributed, then \( \omega^*_t \) is constant through time. In the first case the slope, \( m_t \), depends only on the current investment set, and in the second case \( m_t \) is constant.

When transactions costs are introduced, there is an analogous graphical argument, provided the linear homogeneity in the post-investment bond and stock holdings, \( y_t^0 \) and \( y_t \), is maintained. Transactions costs will be reflected in the
stock to bond conversion rates and, as section III will illustrate, the optimal investment is characterized by the two rays,

\( y_t = a_t y_t^0 \)
\( y_t = b_t y_t^0 \),

where \( a_t \leq m_t \leq b_t \). As in the case for \( m_t \), \( a_t \) and \( b_t \) are independent of the initial holdings \( x_t^0 \) and \( x_t \). Within the region bounded by the two rays, the optimal investment decision is not to have any transactions, and outside the region the optimal investment decision is to transact to the nearest boundary ((1) or (2)) of the region.

Summary

Investors whose terminal utility function is a power function of wealth have indirect utility functions that are also functions of wealth. Although the optimal portfolio policy is independent of current wealth, it is dependent on the future investment opportunity set. If the stock returns \( (z_t) \) are independent across time, then total current returns \( (Z^*_{t+1}) \) and expected future returns \( (d_{t+1}) \) are uncorrelated, and future investment opportunities drop from the FOC equation (2); thus the indirect utility function is state-independent. On the other hand, if the investment opportunities are constant through time, then the FOC
indicates that the same optimal portfolio will be chosen each date. That is, the investor rebalances his portfolio at each date to maintain a constant proportion of his wealth in the risky stock.
II. Optimal Portfolio Policy with Transactions Costs

The effect of proportional transactions costs on dynamic portfolio strategies is examined in discrete time. With the exception of proportional trading costs, all of the assumptions of section I are maintained. Although trading is permitted on each date \( t = 0, 1, \ldots, T - 1 \), the investor might not adjust his portfolio at each trading date if the costs of the trade exceed the benefits. For an investor who faces convex trading costs and who maximizes his expected utility of lifetime consumption over a finite time horizon, Constantinides (1979) showed that the investor will not always adjust his portfolio at each trading date. Such a portfolio policy can be characterized in terms of a no transaction region.

At date \( t \), and prior to the investment decision at that date, the investor owns \( x_t^0 \) dollars in the bond and \( x_t \) dollars in the stock. After making an investment of \( u_t \) dollars in the stock, he incurs proportional transactions costs of \( k|u_t| \) for \( 0 \leq k < 1 \), which are charged to the bond. After the transaction, the stock and bond holdings become \( y_t = x_t + u_t \) and \( y_t^0 = x_t^0 - u_t - k|u_t| \), respectively. Let \( S_{t+1}/S_t = z_{t+1} \) be the stock process, where \( z_{t+1} \) is stochastic, and \( B_{t+1}/B_t = r \) be the bond process, where \( r \) is the riskless return.

\( J(x_t^0, x_t, t) \) denotes the indirect utility function before an investment is made at date \( t \). The investor chooses \( u_t \), for
all \( t, t + 1, \ldots, T - 1 \), to maximize his expected utility of terminal wealth. That is,

\[
J(x^0_t, x_t, t) = \max_{u_t \in \mathcal{U}_t} \mathbb{E}_t \left[ U(x^0_T, x_T, T) \right]
\]

subject to the budget constraint

\[
x^0_0 + x_0 = \mathbb{N}_0 > 0
\]

and bond and stock wealth dynamics,

\[
x^0_{t+1} = [x^0_t + x_t - y_t - k|u_t|]_+
\]

\[
x_{t+1} = [x_t + u_t]z_{t+1} = y_t z_{t+1}
\]

A feasible portfolio holding at date \( t \) is such that

(i) \( 0 \leq x^0_t < \infty \) and \( 0 \leq x_t < \infty \).

An investment in stock, \( u_t \), is feasible if it is self-financing:

(ii) \( u_t \) is restricted to ensure that post-investment wealth,

\[
W_t = x_t + x^0_t - k|u_t|,
\]

is positive at all dates.
The investor's indirect utility function follows from the
dynamic programming algorithm

\[
J(x_t^0, x_t, t) = \max_{u_t} E_t \left\{ J(x_{t+1}^0, x_{t+1}, t+1) \right\}
\]

(4)

with the boundary condition,

\[
J(x_T^0, x_T, T) = U(x_T^0 + x_T, T).
\]

A sufficient condition for \( J(x_t^0, x_t, t) \) to be monotonically
increasing and concave in \((x_t^0, x_t)\) is for transactions costs to
be convex.\(^2\) This condition also holds for proportional
costs. The first order conditions (FOC) for an interior
solution are

\[
0 = E_t \left\{ J_{x_t^0} z_{t+1} - J_{x_t^0} (1 + k)r \right\} \quad \text{for } u_t \geq 0
\]

(5a)

\[
0 = E_t \left\{ J_{x_t^0} z_{t+1} - J_{x_t^0} (1 - k)r \right\} \quad \text{for } u_t \leq 0.
\]

(5b)

In general, the optimal stock transaction, \( u_t^* \), is a function
of the pre-investment bond and stock holdings, \( x_t^0 \) and \( x_t \), and
time, \( t \). However, \( u_t^* \) can be further specified if

\(^2\)\( J(x_t^0, x_t, t) \) is concave in \((x_t^0, x_t)\): Kamin [1975] lemma 3 and
Constantinides [1979] proposition 1. \( J(x_{t+1}^0, x_{t+1}, t+1) \) is MISC
in \((x_{t+1}^0, x_{t+1})\): Kamin [1975] lemma 4. See also Gennette &
Jung [1990].

\(^3\)Kuhn-Tucker FOC conditions apply for a boundary solution.
restrictions are placed on the investor's utility function. Constantinides [1979] demonstrated that \( u_t^* \) is homogeneous of degree 1 in \( x_t^0 \) and \( x_t \), provided the utility function is positively homogeneous of degree \( \alpha \) in \( x_t^0 \) and \( x_t \), and transactions costs are proportional. Therefore the resulting post-investment bond and stock holdings, \( y_t^0 \) and \( y_t \), are linearly homogeneous in \( x_t^0 \) and \( x_t \), and the optimum lies along one of two rays in the \( x_t^0 - x_t \) plane:

\[
\begin{align*}
1) \quad & y_t = a_t y_t^0, \quad & a_t = \frac{x_t + u_t^*}{x_t - u_t^*(1 + k)} \geq 0 \text{ for } u_t^* \geq 0 \\
2) \quad & y_t = b_t y_t^0, \quad & b_t = \frac{x_t - u_t^*}{x_t^0 + u_t^*(1 - k)} \geq 0 \text{ for } u_t^* \leq 0
\end{align*}
\] (6)

where it will be shown in section (III) that \( a_t \) and \( b_t \) are the boundaries of the no transaction region with \( a_t \leq b_t \).

Because \( u_t^* \) is homogeneous of degree 1 in \( x_t^0 \) and \( x_t \), \( a_t \) and \( b_t \) are homogeneous of degree 0 in \( x_t^0 \) and \( x_t \). Hence, \( a_t \) and \( b_t \) are at most functions of \( x_t / x_t^0 \). Additionally, if an investor enters date \( t \) with portfolio holdings of \( x_t^0 \) and \( x_t \) and earns a realized return of \( x_{t+1}^0 \) and \( x_{t+1} \), then had he entered date \( t \) with portfolio holdings of \( \delta x_t^0 \) and \( \delta x_t \), he would have earned a realized return of \( \delta x_{t+1}^0 \) and \( \delta x_{t+1} \). The following proposition confirms our intuition that as costs increase, the size of the transaction decreases.
Proposition 1

Transactions costs are proportional (i.e., $k|u_t|$), where $u_t$ is the dollar investment in stock), the utility function of terminal wealth is positively homogeneous of degree $\alpha$ in the pre-investment bond and stock holdings, $x_t^0$ and $x_t$, and the return process exhibits constant returns to scale. The optimal stock transaction, $|u_t^*|$, decreases as the transactions costs proportionality rate, $k$, increases.

Proof:

Only $u_t^* \geq 0$ is considered; the proof for $-u_t^* \geq 0$ is identical. Let $f(u_t, k)$ denote the FOC equation (5a) for $u_t \geq 0$, i.e.,

$$f(u_t, k) = E_t \{ J_2 z_{t+1} - J_1 (1 + k) r \}$$

where, for brevity, $J_i(x_{t+1}^0, x_{t+1}, t+1)$ $i = 1, 2$, are first order partial derivatives with respect to the first and second arguments $x_{t+1}^0$ and $x_{t+1}$. Taking the total differential of $f(u_t, k)$, and evaluating it at the optimum, $u_t^*$, produces

$$\frac{du_t^*}{dk} = -\frac{f_k}{f_{u_t}} \bigg|_{u_t^*} \quad \text{for } f_{u_t} \neq 0$$

where,
\[ f_{u_t} = E_t \{ Z'HZ \} \text{ for } Z' = \{ -(1 + k)r \quad z_{t+1} \} \quad \& \quad H = \text{Hessian}[J(\cdot)] \]
\[ f_k = E_t \{ -J_1 r + J_{11} u_t (1 + k)r - J_{12} u_t rz_{t+1} \} \]

From Euler's Theorem, \( J_{12}(\cdot) \) becomes
\[ J_{12} = (\alpha - 1) J_1 \left[ \frac{1}{x_{t+1}} \right] - J_{11} \left[ \frac{x_{t+1}^0}{x_{t+1}} \right]. \]

Substituting this into the equation for \( f_k \) and using the bond and stock wealth dynamics (equation 3) for \( x_{t+1}^0 \) and \( x_{t+1} \), the following is obtained:
\[ f_k = E_t \left\{ -J_1 r \left[ \frac{x_t + \alpha u_t}{x_t + u_t} \right] + J_{11} r^2 u_t \left[ \frac{x_t (1 + k) + x_t^0}{x_t + u_t} \right] \right\} u_t. \]

\( f_{u_t} \) is non-zero and negative if the Hessian of \( J(\cdot) \) is negative definite.\(^4\) Because \( J(x_t^0, x_t, \cdot) \) is monotonically increasing and strictly concave,\(^5\) negative semi-definiteness is implied, although not necessarily negative-

\(^4\)Expanding the Hessian in \( f_{u_t} \) we have,
\[ f_{u_t} = E_t \left\{ J_{11} \left[ (1 + k)r - \frac{J_{12} z_{t+1}}{J_{11}} \right]^2 + z_{t+1}^2 \left[ J_{22} - \frac{J_{12}^2}{J_{11}} \right] \right\}, \]
and upon deriving the envelope condition:
\[ J_1(1 + k) = J_2 \]
where "+" is for \( u_t \geq 0 \), and "-" is for \( u_t \leq 0 \);
differentiating this with respect to its arguments,\( J_{11}(1 + k)^2 = J_{22} \quad \text{and} \quad J_{12}^2 = J_{11} J_{22}; \)
we obtain the result: \( f_{u_t} = 0 \) i.f.f. \( J_{11}, J_{22}, J_{12} = 0 \).
definiteness. For a non-negative utility exponent, $\alpha$, $\alpha \in (0, 1)$, positive stock transaction, $u^*_t$, and positive pre-investment bond and stock holding $(x^0_t, x_t)$, the sign of $f_{u_t}$ and $f_k$ are

$$\text{sign}[f_{u_t}] < 0 \quad \text{and} \quad \text{sign}[f_k] < 0,$$

which implies,

$$\text{sign}\left[\frac{du^*_t}{dk}\right] = \text{sign}\left[-\frac{f_k}{f_{u_t}}\right] < 0.$$

Identical arguments for $u^*_t \leq 0$ show that the optimal $|u^*_t|$ decreases as transactions costs increases.

Region of No Transactions

For a pre-investment bond and stock holding of $(x^0_t, x_t)$ the investor makes a stock transaction of $u_t$ at date $t$ to maximize the single period utility function

$$g(u_t, x^0_t, x_t, t) = E_t\left\{J(x^0_{t+1}, x_{t+1}, t+1)\right\}.$$

If the size of $u_t$ is such that

$$g(u_t, x^0_t, x_t, t) < g(0, x^0_t, x_t, t),$$
then the investor is better off not making the investment, and the optimal investment decision is to have no investment at this date. Correspondingly, let

$$
\psi_t = \left\{ (x_t^0, x_t) \mid g_t(u_t, x_t^0, x_t, t) \leq g_t(0, x_t^0, x_t, t); \forall u_t \right\}
$$

denote the no transaction region, i.e., the region is characterized by the set of \((x_t^0, x_t)\) for which \(u_t^*(x_t^0, x_t) = 0\).

If the investor enters date \(t\) with assets \((x_t^0, x_t)\) contained in \(\psi_t\), then an optimal investment decision is to have no investments at that date. When \((x_t^0, x_t)\) falls outside the region, Constantinides [1979] demonstrated that the optimal investment decision is to transact back to the nearest boundary of the region. His proposition is restated without proof in the context of our model:

**Proposition 2**

Assume proportional transactions costs. The consumer enters date \(t\) with bond and stock holding \((x_t^0, x_t)\). An optimal stock transaction, \(u_t\), is such that

a. after the transaction, the resulting asset holdings lie in the no transaction region, i.e.,

$$\left\{ (x_t^0 - u_t - k|u_t|, x_t + u_t) \right\} \in \psi_t.$$

b. if there exists a \(0 \leq \gamma < 1\), such that

---

6 See also Kamin [1975] lemma 1 and Gennette & Jung [1990].
\[
\{x_t^0 - \gamma u_t - k|\gamma u_t|, x_t + \gamma u_t\} \in \Psi_t,
\]

then \(\gamma u_t\) is also an optimal decision.

Parts (a) and (b) of the proposition imply that if the investor's portfolio holdings lie outside of the no transaction region, then an optimal investment decision is to transact only to the boundary of the region. The region is the closed interval,

\[
a_t \leq x_t/x_t^0 \leq b_t, \text{ for } t = 0, 1, ..., T - 1,
\]

where \(x_t/x_t^0\) is the investor's ratio of stock to bond holding each period, and \(a_t\) and \(b_t\) are parameters representing the boundaries of the region. Constantinides [1979] demonstrated that a necessary and sufficient condition for the closed interval given a positively homogeneous utility function, is for transactions costs to be proportional. He did not specify \(a_t\) and \(b_t\), but equation (6) implies they are only functions of \(x_t/x_t^0\), the investment opportunity set, and time.

Proposition 2 has the following interpretation: the investor should transact to an optimum which is near the corresponding optimum for the no transactions costs case. How close he approaches the latter optimum is determined by his optimization problem (equation 4), i.e., his trades to
attain the no cost optimum will continue until the marginal
benefit equals the marginal cost of the trade. The resulting
portfolio holdings represent the optimum for the transactions
cost case. The spread between the two optimums establishes
the boundary of the no transaction region.\textsuperscript{7} For example, if
the investor's initial holdings \((x_t^0, x_t)\) lie below the region
of no transactions, i.e., \(x_t/x_t^0 < a_t\), then the investor's
optimal investment decision is to convert his bond holdings
into stock at the transformation rate of \(\partial x/\partial x_t^0 = -1/(1 + k)\),
where \(k\) is the transactions cost rate. The process stops
when the resulting portfolio holdings place the investor at
the lower boundary \(a_t\), at which point \(u_t = u_t^*\). Further
increase \((u_t > u_t^*)\) moves the portfolio holdings into the
region of no transactions, which results in lower utility
(see also Appendix A). Similarly, if the investor's initial
holdings lie above the region of no transactions, i.e.,
\(x_t/x_t^0 > b_t\), then the optimal investment decision is to
convert stock into bonds at the rate of \(\partial x/\partial x_t^0 = -1/(1 - k)\),
until the upper boundary \(b_t\) is reached, at which point
\(u_t = u_t^*\). Further reduction \((u_t < u_t^*)\) results in lower
utility. If the initial holdings are contained in the region
of no transactions, i.e., \(a_t \leq x_t/x_t^0 \leq b_t\), first consider the
case in which the holdings are on the boundary \(a_t\), i.e.,
\(x_t/x_t^0 = a_t\). Referring to figure 2, any investment into the
region of no transactions \((u_t > 0)\) leads to lower utility, and

\textsuperscript{7}A discussion on the effect of convex and concave trading
cost on the no transactions region is provided in appendix A.
any feasible investment ($u_t < 0$) leads to movement away from the upper boundary $b_t$, which is the optimum for $u_t < 0$. Therefore, if $u_t$ is anything but zero, the boundary $a_t$ cannot be the optimum for $u_t = u_t^* > 0$. If the initial holdings on are the boundary $b_t$, i.e., $x_t/x_t^0 = b_t$, any investment ($u_t < 0$) into the region of no transactions again leads to lower utility, and any feasible investment ($u_t > 0$) leads to movement away from the lower boundary $a_t$, which is the optimum for $u_t > 0$. If $u_t$ is anything but zero, the boundary $b_t$ cannot be the optimum for $u_t = u_t^* < 0$. Similar arguments apply to non-boundary portfolio holdings within the region of no transactions. Because post-investment positions on the boundary $a_t$ are not feasible for $u_t < 0$, and post-investment positions on the boundary $b_t$ are not feasible for $u_t > 0$, from symmetry, the optimal investment is $u_t^* = 0$.

Figure 2
Although the trading strategy is known, the magnitude of the stock trade \( u_t \) is not, because it is a function of the control variables, \( a_t \) and \( b_t \). The choice of these variables will affect the choice of \( u_t \), which will in turn affect all future investment decisions. Hence, the controls will necessarily impact the terminal value of wealth through the final stock and bond holdings. Given a realization of the controls \( a_t \) and \( b_t \), if the initial holdings lie below the no transaction region, i.e., \( x_t / x_t^0 < a_t \), the size of the trade is

\[
    u_t = \frac{[x_t^0 a_t - x_t]}{(1 + a_t(1 + k))} > 0
\]

(7a)

i.e., the optimal trade is to increase the stock holding to \([x_t^0 + u_t]\) and to decrease the bond holding to \([x_t^0 - (1 + k)u_t]\) until the lower boundary \( a_t \) is reached. If the initial holdings lie above the no transaction region, i.e., \( x_t / x_t^0 > b_t \), \( u_t \) is

\[
    u_t = \frac{[x_t^0 b_t - x_t]}{(1 + b_t(1 - k))} < 0
\]

(7b)

i.e., the optimal trade is to decrease the stock holding to \([x_t^0 + u_t]\) and to increase the bond holding to \([x_t^0 - (1 - k)u_t]\) until the upper boundary \( b_t \). If the holdings are contained in the no transaction region, i.e., \( a_t \leq x_t / x_t^0 \leq b_t \), then \( u_t \)
= 0.

It is easily shown that $u_t$ is monotonically increasing and strictly concave in $a_t$, for all $u_t > 0$, and that $-u_t$ is monotonically increasing and strictly concave in $b_t$ for all $u_t < 0$. The optimal stock trade, $u_t$, at date $t$ is depicted in the figure (3) for fixed $a_t$, $b_t$, and $x_t^0$.

![Graph showing $u$ vs. $x/x^0$ with points labeled $a$, $b$, and origin $0$.]
III. Optimal Control Limits

The boundaries of the no transaction region are the control variables over which the investor's indirect utility of wealth is maximized. The controls are shown to be independent of current holdings, but dependent on current, and possibly future, investment opportunities. The no transaction region costs is also shown to widen as transactions costs are increased.

Notation Convention

Suppose that at time \( t \) the security holdings are such that \( x_t / x_t^0 < a_t \), and a corresponding optimal \( u_t^* \) was selected as per equation (7a), then define \( x_{t+1}^0(a_t) \) and \( x_{t+1}(a_t) \) as next period's bond and stock holdings. Analogous definitions apply to \( x_{t+1}^0(b_t) \) and \( x_{t+1}(b_t) \), for \( x_t / x_t^0 > b_t \). Additionally, let \( x_{t+1}^0 \) and \( x_{t+1} \) be the resulting \( t + 1 \) period bond and stock holdings, given that at time \( t \), \( a_t \leq x_t / x_t^0 \leq b_t \), and \( u_t^* = 0 \) was selected. The investment dynamics for \( t = 0, 1, \ldots, T - 1 \) are

1. If \( x_t / x_t^0 < a_t \), the resulting asset holdings at \( t + 1 \) are

\[
\text{bond: } x_{t+1}^0(a_t) = \left[ x_t^0 - u_t^* (1 + k) \right] r \\
= A_t(a_t) V_t r
\]  

(8a)
stock: \( x_{t+1}(a_t) = \left[ x_t^0 + u_t^* \right] z_{t+1} \)
\[ = a_t A_t(a_t) a V_t z_{t+1} \] (8b)

where \( A_t(a_t) = \left[ 1 + a_t(1 + k) \right]^{-1} \) and \( a V_t = \left[ x_t^0 + x_t(1 + k) \right] \).

\( a V_t \) can be interpreted in terms of the maximum post-adjustment stock position \( y_t = x_t + u_t = a V_t / (1 + k) \) corresponding to a maximum stock adjustment of \( u_t = x_t^0 / (1 + k) > 0 \).

2. If \( x_t / x_t^0 > b_t \), the resulting asset holdings at \( t + 1 \) are

bond: \( x_{t+1}^0(b_t) = \left[ x_t^0 - u_t(1 - k) \right] r \)
\[ = B_t(b_t) b V_t r \] (9a)

stock: \( x_{t+1}^0(b_t) = \left[ x_t^0 + u_t^* \right] z_{t+1} \)
\[ = b_t B_t(b_t) b V_t z_{t+1} \] (9b)

where \( B_t(b_t) = \left[ 1 + b_t(1 - k) \right]^{-1} \) and \( b V_t = \left[ x_t^0 + x_t(1 - k) \right] \).

\( b V_t \) is the liquidation value of the pre-investment holding \( (x_t^0, x_t) \).

3. If \( a_t \leq x_t / x_t^0 \leq b_t \), then \( u_t^* = 0 \), and the asset holdings at \( t + 1 \) are

bond: \( x_{t+1}^0 = x_t^0 r \) (10a)
stock: \( x_{t+1} = x_t z_{t+1} \) \hspace{1cm} (10b)

The following properties and lemma are also useful:

**Properties:**

\[
\frac{\partial A}{\partial a} = -(1 + k)A^2, \quad \frac{\partial B}{\partial b} = -(1 - k)B^2
\]

\[
\frac{\partial aA}{\partial a} = A^2, \quad \frac{\partial bB}{\partial b} = B^2
\]

**Lemma 1:**

Given the bond and stock feasibility constraints, \( 0 \leq x_t^0 < \infty \) and \( 0 \leq x_t < \infty \), for \( t = 0, 1, 2, \ldots, T \), the following results hold,

for \( x_t/x_t^0 < a_t < \infty \),

\[
A_t(a_t) = \left[ 1 + a_t(1 + k) \right]^{-1} > 0
\]

\[
a_t V_t = \left[ x_t^0 + x_t(1 + k) \right] \geq 0
\]

and for \( x_t/x_t^0 > b_t \),

\[
B_t(b_t) = \left[ 1 + b_t(1 - k) \right]^{-1} > 0
\]

\[
b_t V_t = \left[ x_t^0 + x_t(1 - k) \right] \geq 0
\]
Proof:

From the investment dynamics (equation 10a), if 
\[ x_t / x_t^0 \leq a_t < \infty \] then 

\[ u_t = A_t [x_t^0 a_t - x_t] > 0 , \]

and hence \( A_t > 0 \). If \( a_t = \infty \), then \( A_t = 0 \) and a boundary solution exists. The optimal investment is then to be fully invested in the stock. The second part of the lemma follows by noting that \( x_t^0 \geq 0 \), \( x_t \geq 0 \), and \( 0 \leq k < 1 \) imply 

\[ a V_t = [x_t^0 + x_t (1 + k)] \geq 0 . \]

If \( x_t^0 = x_t = 0 \), the investment process stops, and the determination of \( a_t \) is irrelevant. Identical arguments apply for \( b_t \) and \( b V_t \).

The Control Limit Problem\(^8\)

When examining the investment dynamics (equations 8 - 10), it is apparent that the controls \( a_t \) and \( b_t \) do not appear in the set of equations when the initial portfolio holdings fall within the no transaction region (see equation 10). Consequently, the control limit problem exists only for portfolio holdings that lie outside of the region. Because 

\(^8\)For an alternative derivation see Gennotte & Jung [1990].
the investor transacts to the nearest boundary $a_t$ or $b_t$ when his portfolio holdings fall outside of the region, the sign of the optimal stock transaction, $u_t^* < 0$ or $u_t^* > 0$, will determine which of the two boundaries he has transacted to. In particular if $u_t^* > 0$, then proposition 2 implies that the investor has transacted to the lower boundary $a_t$. If on the other hand $u_t^* < 0$, then the investor has transacted to the upper boundary $b_t$. Therefore, solving for an optimal $u_t > 0$ or $u_t < 0$ is equivalent to solving for an optimal $a_t$ or $b_t$. Because the control limit problem is not specified within the no transaction region (i.e., an objective function involving the controls does not exist), its solution cannot be derived from the usual dynamic programming methodologies. The solution procedure at each date $t = 0, 1, \ldots, T - 1$ is

1. obtain the single period objective function

$$\max_{u_t} \{ J(x_{t+1}^0(u_t), x_{t+1}(u_t), t+1) \}.$$ 

2. For the region, $x_t / x_t^0 < a_t$, the objective function is modified, and $a_t$ becomes the control variable. A similar transformation holds for the region, $x_t / x_t^0 > b_t$, where $b_t$ becomes the control variable.

---

$^9$From equations (7a, b),

$$u_t = \begin{cases} 
[x_t^0a_t - x_t]/[1 + a_t(1 + k)] > 0 & \text{for } x_t / x_t^0 < a_t \\
[x_t^0b_t - x_t]/[1 + b_t(1 - k)] < 0 & \text{for } x_t / x_t^0 > b_t
\end{cases}$$
(3) For each region, the corresponding objective function is maximized with respect to the control $a_t$ or $b_t$.

Therefore for portfolio holdings below the no transaction region, viz., $x_t / x^0_t < a_t$, the optimal $a_t$ is a solution of

$$J(x^0_t, x_t, t) = \max_{a_t} \mathbb{E}_t \{J(x^0_{t+1}(a_t), x_{t+1}(a_t), t+1)\},$$

and when portfolio holdings lie above the no transaction region, viz., $x_t / x^0_t > b_t$, the optimal $b_t$ is a solution of

$$J(x^0_t, x_t, t) = \max_{b_t} \mathbb{E}_t \{J(x^0_{t+1}(b_t), x_{t+1}(b_t), t+1)\},$$

where post-investment holdings, $x^0_{t+1}(a_t)$, $x_{t+1}(a_t)$, $x^0_{t+1}(b_t)$, and $x_{t+1}(b_t)$ are given by the bond and stock investment dynamics (equations 8 & 9), and $J(x^0_{t+1}(\cdot), x_{t+1}(\cdot), t+1)$ is given by the investor's maximization problem, equation (4).

Using the homogeneity of $J(\cdot)$, the main theoretical result is derived in the following proposition:

**Proposition 3**

Given proposition 1, the optimal controls $a_t$ and $b_t$ are independent of the pre-investment bond and stock holding $(x^0_t, x_t)$, but dependent on both current and future investment opportunities.
Proof:

If the utility function of terminal wealth is homogeneous of degree \( \alpha \), then by induction \( J(x_{t+1}^0, x_{t+1}, t+1) \) for all \( t \) is also homogeneous of degree \( \alpha \). For \( u_t > 0 \) (i.e., \( x_t / x_t^0 < a_t \)), the indirect utility function is

\[
J(x_{t+1}^0, x_{t+1}, t+1) = J([A(a_t) a_t V_t] r, [a_t A(a_t) a_t V_t] z_{t+1}, t+1)
= a_t^\alpha a_t V_t^\alpha J(r, a_t z_{t+1}, t+1)
\]

and from the first order condition, and noting from lemma 1 that \( a_t V_t > 0 \),

\[
0 = \mathbb{E}_t \left\{ A^{\alpha+1} \begin{bmatrix} -J_{x_{t+1}^0} (r, a_t z_{t+1}, t+1)(1 + k)r + J_{x_{t+1}}(r, a_t z_{t+1}, t+1)z_{t+1} \end{bmatrix} \right\}.
\]

Therefore the solution \( a_t \) is not a function of \( (x_t^0, x_t) \). A similar result holds for \( b_t \).

Given an optimal \( (a_t, b_t) \), the investor's indirect utility function at date \( t \) is
\[ J(x_t^0, x_t, t) \]

\[ = E_t \left\{ J(x_{t+1}^0, x_{t+1}, t+1) \right\} \]

\[ = \begin{cases} 
  a V_t^\alpha A(a_t)^\alpha E_t \left\{ J(r, a_t z_{t+1}, t+1) \right\} & \text{for } \frac{x_t^0}{x_t^r} < a_t \\
  E_t \left\{ J(x_t^0 r, x_t z_{t+1}, t+1) \right\} & \text{for } a_t \leq \frac{x_t^0}{x_t^r} \leq b_t \\
  b V_t^\alpha B(b_t)^\alpha E_t \left\{ J(r, b_t z_{t+1}, t+1) \right\} & \text{for } \frac{x_t^0}{x_t^r} > b_t 
\end{cases} \]

where \( A(a_t) \), \( B(b_t) \), and all terms under the expectation operator for \( x_t/x_t^0 < a_t \) and \( x_t/x_t^0 > b_t \) are independent of \((x_t^0, x_t)\). This implies that for any two portfolios on the same side of the no transaction region, their respective utility functions are proportional to one another. As an example, consider two portfolios (1) and (2) such that \( x_t(2)/x_t^0(2) < x_t(1)/x_t^0(1) < a_t \), i.e., portfolio (2) is furthest from the no transaction region. From the indirect utility function

\[ J(x_t^0(1), x_t(1), t) = \left[ \frac{a V_t(1)}{a V_t(2)} \right]^\alpha J(x_t^0(2), x_t(2), t). \]  

(11)

Therefore \( J(x_t^0(1), x_t(1), t) \) is proportional to \( J(x_t^0(2), x_t(2), t) \), where the proportionality factor is dependent on \((x_t^0, x_t)\).

This leads to the following proposition:
Proposition 4

Given proposition 1, the following are true:

(1) For all portfolio combinations of equal value 
\( w_t = x_t^0 + x_t \), contained in the region, \( x_t/x_t^0 < a_t \), the investor's expected utility of terminal wealth decreases as his portfolio holdings move away from the no transaction region. An identical result holds for all portfolio combinations of equal wealth contained in the region, \( x_t/x_t^0 > b_t \).

(2) For the region, \( x_t/x_t^0 < a_t \), the investor's expected utility of terminal wealth is constant for all portfolio combinations lying on the stock to bond transformation line with slope, \( \partial x_t/\partial x_t^0 = -1/[1 + \kappa] \). An identical result holds in the region, \( x_t/x_t^0 > b_t \), for all portfolio combinations along the stock to bond transformation line with slope, \( \partial x_t/\partial x_t^0 = -1/[1 - \kappa] \).

Proof:

The proof proceeds by comparing the indirect utility function of a single reference portfolio against the indirect utility functions of several other portfolios, when the latter's pre-investment values and compositions are varied. Let the reference portfolio be denoted as portfolio (1). Recall from proposition 3 that the boundaries to the no
transaction region are invariant to changes in the investor's pre-investment stock and bond holdings, and hence they are invariant to changes in his pre-investment wealth levels. Consequently, portfolios of differing initial value and composition may be compared to one another without affecting the boundaries of the region of no transaction.

Proof of Part (1):

Consider two portfolios of equal value, (1) and (2), which lie on the same side of the no transaction region (see figure 4), e.g., the two portfolios lie along the constant wealth line such that \( x_t(2)/x_t^0(2) < x_t(1)/x_t^0(1) < a_t \). The optimal post-investment holdings for portfolios (1) and (2) are indicated in the figure as points (A) and (B), respectively. Because the investor's indifference curves are strictly convex and monotonically increasing, point (B) is inferior to point (A). Formalizing the argument, part (1) of the proposition holds if the investor's expected utility of terminal wealth satisfies \( J(x_t^0(1), x_t(1), t) > J(x_t^0(2), x_t(2), t) \). Using the proportionality relationship,

\[
J(x_t^0(1), x_t(1), t) = \left[ \frac{a V_t(1)}{a V_t(2)} \right]^\alpha J(x_t^0(2), x_t(2), t)
\]

the proposition follows by noting that \( \left[ \frac{a V_t(1)}{a V_t(2)} \right] > 1 \) implies \( J(x_t^0(1), x_t(1), t) > J(x_t^0(2), x_t(2), t) \).
Because portfolios (1) and (2) lie along a constant wealth line, their respective stock and bond holdings, \((x^0_t(1), x_t(1))\) and \((x^0_t(2), x_t(2))\), exhibit the following relationship:

\[
x^0_t(1) + x_t(1) = x^0_t(2) + x_t(2) = W_t,
\]

where

\[
x_t(1) = x_t(2) + \eta \quad \text{for } \eta > 0
\]

\[
x^0_t(1) = x^0_t(2) - \eta
\]

Therefore,

\[
\begin{bmatrix}
aV_t(1) \\
aV_t(2)
\end{bmatrix} = \frac{x^0_t(1) + x_t(1)[1 + k]}{x^0_t(2) + x_t(2)[1 + k]}
\]
\[
\begin{align*}
&= \frac{(x_t^0(2) - \eta) + (x_t^0(2) + \eta)[1 + \kappa]}{x_t^0(2) + x_t^0(2)[1 + \kappa]} \\
&= 1 + \frac{\eta \kappa}{a V_t(2)} \\
&> 1
\end{align*}
\]

The proof for \( x_t^0 / x_t^0 > b_t \) is identical. Thus the proof of part (1) of proposition 4 is complete.

**Proof of Part (2):**

Consider two portfolios, (1) and (3), which lie on the stock to bond transformation line on the same side of the no transaction region, but have different values. Let portfolio (3) have the greater value (see figure 5), e.g., \( x_t(3)/x_t^0(3) < x_t(1)/x_t^0(1) < a_t \) and \( x_t^0(3) + x_t(3) > x_t^0(1) + x_t(1) \).

Because the optimal post-investment holdings for portfolios (1) and (3) are identical (point (A) in figure 5), part (2) of the proposition follows if \( J(x_t^0(3), x_t(3), t) = J(x_t^0(1), x_t(1), t) \).

Using the proportionality relationship,

\[
J(x_t^0(1), x_t(1), t) = \left[ \frac{a V_t(1)}{a V_t(2)} \right]^\kappa J(x_t^0(2), x_t(2), t)
\]

the proposition follows by noting that \( \left[ \frac{a V_t(1)}{a V_t(3)} \right] = 1 \) implies \( J(x_t^0(3), x_t(3), t) = J(x_t^0(1), x_t(1), t) \). Because the two portfolios lie on a line with slope of \(-1/(1 + \kappa)\), the following relationship holds:
\[
\frac{x_t(3) - x_t(1)}{x_t^0(3) - x_t^0(1)} = \frac{1}{1 + k}
\]

\[x_t^0(1) + x_t(1)(1 + k) = x_t^0(3) + x_t(3)(1 + k)\]

\[aV_t(1) = aV_t(3)\]

\[
\begin{bmatrix}
  aV_t(1) \\
aV_t(3)
\end{bmatrix} = 1
\]

Figure 5

The proof for \(x_t/x_t^0 > b_t\) is identical. Thus the proof of part (2) is complete.

Proposition 4 leads to the following observations (figure 6):

a. all portfolio combinations within the region bounded by \(\overline{OCB}\) are inferior to portfolio (1), and all portfolio combinations contained in the region bounded by \(\overline{ACB}\) are
superior to portfolio (1).

![Diagram showing constant wealth line and transformation path with points A, B, C, and D.]

**Figure 6**

b. Although all portfolio combinations within the area bounded by $\overline{AB}$ are of higher value ($W_t = x_t^0 + x_t$) than portfolio (1), the higher wealth is not enough to compensate for the additional cost of trading to an optimum.

c. Although all portfolio combinations within the triangular area $\overline{DCI}$ are of lower value ($W_t = x_t^0 + x_t$) than portfolio (1), the lower wealth is compensated by lower trading costs.

Proposition 4 confirms our intuition that for fixed wealth, the further away the investor is from the no transaction region the higher the transactions costs he will incur and
the lower his expected utility of terminal wealth will be. On the other hand, even if the investor's pre-investment wealth \( W_t = x_t^0 + x_t \) is increased, his expected utility can still drop if the cost of the trading to the optimum exceeds the gain in wealth.

It is easily shown that \( a_t \) is the lower boundary and \( b_t \) is the upper boundary of the no transaction region. From the first order conditions,

\[
\frac{E_t \left\{ r J_{x_t^2+1} \right\} \bigg|_{a_t}}{E_t \left\{ J_{x_t+1} \right\} \bigg|_{a_t}} = \frac{1}{1 + k} \quad \text{and} \quad \frac{E_t \left\{ r J_{x_t+1} \right\} \bigg|_{b_t}}{E_t \left\{ J_{x_t+1} \right\} \bigg|_{b_t}} = \frac{1}{1 - k}.
\]

and noting that in both cases the RHS = 1 when transactions costs are zero (i.e., \( k = 0 \)), the following relationship holds:

\[
\frac{1}{1 + k} < 1 < \frac{1}{1 - k} \quad \text{for } k \neq 0.
\]

Therefore \( a_t < b_t \), and the no cost optimum \( k = 0 \) lies within the no transaction region. If a boundary solution exists when \( a_t = \infty \), then \( b_t \) is irrelevant because \( x_t^0, x_t \geq 0 \) implies \( u_t^* < 0 \) is infeasible. Similarly if \( b_t = 0 \), then \( a_t \) is irrelevant because \( u_t^* > 0 \) is infeasible.
Control Limits and the Size of Transactions costs

Propositions 5 and 6 confirm our intuition that the no transaction region widens as costs increase, and that the region shrinks to a width of zero when costs are zero (i.e., \( k = 0 \)). The second result implies that the investor's problem reduces to finding a single unique point \( x_t / x_t^0 \) for each \( t \).

**Proposition 5**

Given proposition 1, the no transaction region widens with the transactions cost proportionality factor \( k \) (viz., \( k \mid u_t \)).

**Proof:**

From the investment dynamics (equation 7a) for positive stock transactions, \( u_t^* \geq 0 \),

\[
a_t(R_t) = \frac{x_t + u_t}{x_t^0 - [1 + k]u_t}.
\]

Differentiating with respect to \( k \),

\[
\frac{da_t}{dk} = \frac{1}{(x_t^0 - (1 + k)u_t)^2}\left\{\left[x_t^0 + x_t(1 + k)\frac{du_t}{dk}\right] + [x_t + u_t]u_t\right\}.
\]

Because the relation must hold for all optimal stock transactions, \( u_t \), it must hold when the initial portfolio.
holdings \((x_t^0, x_t)\) lie on the boundary \(a_t\), where the optimal investment is \(u_t = 0\). Therefore, from proposition 2 (viz., \(du_t/dk < 0\)),
\[
\frac{da_t}{dk} = \frac{1}{(x_t^0)^2} \left[ x_t^0 + x_t(1 + k) \right] \frac{du_t}{dk} \bigg|_{u_t=0} < 0 .
\]

Similarly for the boundary \(b_t\),
\[
\frac{db_t}{dk} = -\frac{1}{(x_t^0)^2} \left[ x_t^0 + x_t(1 - k) \right] \frac{du_t}{dk} \bigg|_{u_t=0} > 0 .
\]

Hence, the no transaction region widens as \(k\) increases.

**Proposition 6**

Given proposition 1, if costs tend to zero then \(a_t \rightarrow b_t\), and the width of the no transaction region approaches zero.

**Proof:**

For \(k = 0\), the first order conditions are
\[
0 = E_t \left\{ -J_{x_t+1}(a_t) \nu + J_{x_t+1}(a_t)z_{t+1} \right\} \quad \text{for} \quad x_t/x_t^0 < a_t
\]
\[
0 = E_t \left\{ -J_{x_t+1}(b_t) \nu + J_{x_t+1}(b_t)z_{t+1} \right\} \quad \text{for} \quad x_t/x_t^0 > b_t .
\]

Because the terminal utility function is strictly concave, viz., \(U'' < 0\), and because at date T the boundary condition is
\[ J(x^0_T, x^0_T, T) = U(x^0_T + x_T), \] the above expectations are monotonically decreasing in \( a_t \) and \( b_t \). For portfolio holding lying below the no transaction region, viz., \( x_{T-1}/x^0_{T-1} < a_{T-1}, \) the following marginal condition holds

\[
\frac{\partial E_{T-1}\{-U_{x^0_T} x_T + U_{x_T} z_T\}}{\partial a_T} = E_{T-1}\{-U_{x^0_T} x_T \frac{\partial x^0_T}{\partial a_T} + U_{x_T} z_T \frac{\partial x_T}{\partial a_T}\}
\]

\[
= (1 + a_{T-1})^{-2} (x^0_{T-1} + x_{T-1})
\]

\[
E_{T-1}\{U_{x^0_T} x^2_T + U_{x_T} z^2_T\}
\]

< 0

where the last inequality holds for \( x^0_T + x_T > 0 \).

A similar relationship holds for \( x_{T-1}/x^0_{T-1} > b_{T-1} \). Thus \( a_{T-1} = b_{T-1} \), and a unique solution to the FOC exists at \( t = T - 1 \). Because \( U(\cdot) \) is strictly concave in \( x^0_T \) and \( x_T \), \( J(\cdot) \) is also strictly concave in \( x^0_T \) and \( x_T \) for \( t = 0, 1, \ldots, T - 1 \); a similar relationship holds for all earlier times. Therefore, \( a_t \rightarrow b_t \) for all \( t = 0, 1, \ldots, T - 1 \), and the width of the no transaction region approaches zero.
IV. Summary

A discrete time model is used to examine the effects of transactions costs on dynamic portfolio strategies. The model assumes a single risky asset and a single riskless asset. Trading in the two assets is subject to proportional transactions costs, and investors make adjustments in the composition of their portfolios to maximize their expected utility of terminal wealth. The utility function is a power function, viz., $U(W_t) = \delta^2 W_t^\alpha / \alpha$ for $-\infty < \alpha < 1$, $\alpha \neq 1$, and there is no intermediate consumption. The optimal investment policy is defined by a no transaction region, where the optimal policy is to refrain from trading if initial portfolio holdings lie within the region, and to transact to the nearest boundary of the region if portfolio holdings lie outside the region. The boundaries are independent of the investor's current wealth and are dependent on current, and possibly future, investment opportunities. The no transaction region widens as costs increase and narrows to a width of zero when costs tend to zero. For fixed wealth, the further away the investor's holdings are from the no transaction region, the lower his derived utility of wealth is.
APPENDIX A

The No Transaction Region

With

Convex and Concave Costs

With strictly convex costs, the next dollar of trade results in proportionally higher trading costs than the previous dollar of trade, hence the size of the trade should be kept as small as possible. Conversely, with strictly concave costs, the next dollar of trade results in proportionally lower trading costs than the previous dollar of trade, and the size of the trade should increase. The latter implies a narrower no transaction region.

For a stock transaction of $u_t$, let $T(u_t)$ represent either a convex or a concave transactions cost function. Assume $T(u_t)$ is twice differentiable, $T'(u_t) \geq 0$ for all $u_t$, $T(0) = 0$, and $T(h_t) - T(l_t) < h_t - l_t$ for all $h_t > l_t$. The latter requires that the cost of the transaction not exceed the size of the transaction (i.e., $T(u_t) < u_t$). Although, a fix plus concave cost is representative of real world costs, it is not differentiable at $u_t = 0$.

Consider only the case when the optimal stock transaction is positive, viz., $u_t^* = R_t > 0$. The case for $-u_t^* = L_t > 0$ is identical. Because $R_t \geq 0$, an increase in $R_t$ corresponds to an increase in the investor's initial stock holding at date $t$. This is accomplished by converting a portion of his bond holding into additional stock. As previously defined, the
The investor's single period utility function is

\[ g_t(R_t) = E_t \left[ J(x_t^0 - R_t - T(R_t)) \right] I_t \left[ x_t^0 + R_t \right] I_{t+1}, \ t+1 \right) \]

Differentiating with respect to \( R_t \), the following is obtained

\[ g'(0) \leq 0 \quad \text{iff} \quad [1 + T'(0)] \leq \left( \frac{E_t \left[ J x_{t+1}^0 z_{t+1} \right]}{r E_t \left[ x_{t+1}^0 \right]} \right)_{R_t=0} \]

Because the rate of transformation of bond to stock is \( dx_t^0/dR_t = -[1 + T'(R_t)] \), the LHS of the second inequality can be interpreted as the market required cost of increasing the stock holding by $1. The RHS corresponds to the expected marginal rate of substitution (MRS\( _t (R_t) \)) for converting bonds into stock. If the investor's indirect utility function, \( J(\cdot) \), is concave\(^{10} \), then \( dMRS_t/dR_t \leq 0 \). If \( [1 + T'(0)] < MRS_t(0) \), then \( g'(0) > 0 \), and an interior solution \( (R_t > 0) \) exists. Similarly, if \( [1 + T'(0)] > MRS_t(0) \), then \( g'(0) \leq 0 \), and a boundary solution \( (R_t = 0) \) exists.

**Case I: Convex Transactions Costs**

- **Interior Solution**

  If \( [1 + T'(0)] < MRS_t(0) \), the optimal stock transaction is \( R_t^* > 0 \). The investor's willingness to transform bonds into

\(^{10}\)Since convex transactions costs are only sufficient for \( J(\cdot) \) concave, for the sake of the above illustration, suppose there exists concave transactions costs such that \( J(\cdot) \) is concave.
stock exceeds the market cost of such a transaction. Consequently, his utility increases by increasing his stock holding by \( R^*_t \), at which point \([1 + T'(R^*_t)] = MRS_t(R^*_t)\). The scenario holds when the investor’s pre-investment portfolio holdings \((x^0_t, x_t)\) place him outside the no transaction region. The optimal investment is to increase his stock holding by \( R^*_t \), which places his post-investment holdings on the nearest boundary of the no transaction region. Because \( T''(u_t) \geq 0 \) as \( R_t \) increases from 0 to \( R^*_t \), \([1 + T'(R_t)] \) increases, while \( MRS_t(R_t) \) decreases until equality is attained at \( R^*_t \).

However, as the size of the trade continues to increase, \( R_t \) moves beyond \( R^*_t \), and the boundary to no transaction region is broached. Therefore, \([1 + T'(R_t)]\) begins to diverge away from \( MRS_t(R_t) \), and the investor is made progressively worse off.

* Boundary Solution *

If \([1 + T'(0)] \geq MRS_t(0)\), the optimal stock transaction is \( R^*_t = 0 \). The investor’s willingness to convert bonds into stock is no greater than its market cost. Because \( g(R_t) \) is monotonically increasing and concave, any increase in \( R_t \) results in further divergence of \([1 + T'(R_t)]\) from \( MRS_t(R_t) \), and the investor is made progressively worse off. The solution applies when the investor’s pre-investment holdings place him either on the boundary or in the interior of the no transaction region. Because similar arguments hold when \(-u^*_t = L^*_t > 0\), the optimal investment is not to transact at this date.
Case II: Concave Transactions Costs

Because the derivation of the boundary solution is similar to the solution for convex transactions costs, only the interior solution is considered.

- Interior Solution

If \( [1 + T'(0)] < MRS_t(0) \), the optimal stock transaction is \( R_t^* > 0 \). Because \( T''(u_t) \leq 0 \) as \( R_t \) increases from 0, both \( [1 + T'(R_t)] \) and \( MRS_t(R_t) \) decrease. \( T'(R_t) > 0 \) implies \( [1 + T'(R_t)] > 1 \) for all \( R_t \), and figure 1 implies that \( [1 + T'(R_t)] \) can never decrease faster than \( MRS_t(R_t) \). Referring to figure 1, where R1 and R2 denote the optimal stock investment for convex and concave transactions costs, respectively. Because \( R2 > R1 \), in general the no transaction region for concave transactions costs will be narrower than that for convex transactions costs. However, as figure 2 demonstrates, this may not always be the case, because the result is dependent on both the magnitude and rate of change of \( T'(R_t) \).
Key:

R1 = optimum with convex $T(R)$
R2 = optimum with concave $T(R)$
R3 = optimum with $T(R) = 0$

Figure 1

Key:

R1 = optimum with convex $T(R)$
R2 = optimum with concave $T(R)$

Figure 2
REFERENCES


