Continuous Equilibrium in Speculative Markets With Heterogeneous Information

by

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Abstract

Continuous trading, the price impact of trading, and heterogeneous information are central characteristics of market speculation. An example is agricultural commodity markets, where markets around the world are open continuously. For each specific commodity, traders need to take the price impact of their trading into account, because there are only a small number of market makers and big dealers who specialize in that commodity. Each of the traders is seeking to obtain the best information at the earliest possible time, and the traders will get different information at different times.

It has been a great challenge to economists to provide models which include all of these essential characteristics. As an answer to this challenge, this paper proposes and analyzes a model of speculative markets with the following features:
-- Speculators trade continuously in order to optimize their objective intertemporally.
-- Each speculator will take the price impact of his into account.
-- Dynamic information unfolds and resolves continuously.
-- Agents are heterogeneously informed.
-- Information of agents is made up of:
  information from prices
  public information
  private information
-- Agents observe, incorporate and update information continuously.
-- Adjustment costs are included.

Comments welcome

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1. Introduction

Continuous trading, the price impact of trading, and heterogeneous information are central characteristics of market speculation. Take agricultural commodity markets for example. The markets around the world are open continuously. For each specific commodity, there are only a small number of professional traders (i.e., market makers, big dealers, etc.). Therefore, these traders need to take the price impact of their trading into account. Each of the traders is seeking to obtain the best information at the earliest possible time, and the traders will get different information at different times.

It has been a great challenge to economists to provide models which include all of these essential characteristics.

In recent years, economic theories have been developed which recognize that agents are not homogeneously informed.

Grossman & Stiglitz (1980) presents a two period model with two groups of traders, informed and uninformed. The uninformed traders partially learn the information of informed traders from prices. Extending their model into continuous time, Wang (1990) successfully developed a model in which information unfolds and resolves continuously.

The above two models assume that informed traders are a certain proportion of the population and have identical private information. A more realistic assumption is that every trader has an individual observation which is the sum of the true value plus personal observation noise. This is basically the assumption used by Grossman (1977), Hellwig (1980), Bray (1981), Diamond &

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For each specific commodity, there are generally a very large number of sporadic individual market participants (e.g., farmers, businesses, etc.) whose buying and selling orders are generally not synchronized. The lack of synchronization of these order flows creates profit opportunities for a small number of big dealers or market makers who specialize in that market and are able to get speculative profits by taking temporary positions against order imbalances.
Verrecchia (1981), Admati (1985), and Kyle (1989) to build their two period models of market equilibrium with heterogeneous information.

All of the models above, except Kyle (1989), assume that agents behave as price takers. Kyle (1989) argues that since a large proportion of trading activities are concentrated among a few big traders (i.e., dealers, market makers, etc.), it is implausible to assume that each trader will not take the price impact of his trading into account. To deal with these problems, Kyle presents a two period model in which heterogeneously informed agents take the price impact of their trading into account. The fact that a big trader will take the price impact into account is also captured, in a two period model, by Leland (1990) in an investigation of the behavior of a trader who holds inside information.

There are several apparent inadequacies associated with using two period models to capture real world markets, which are continuous or multi-period. First, two period models assume that information will be fully revealed in the second period; while, in actual markets, uncertainties will certainly carry to future periods. Second, two period models assume that information only comes in the first period. This ignores the fact that the entire history of the information matters. Third, the process of information resolution is not clear. In actual markets uncertainty is resolved gradually. But the process of information resolution in two period models happens at once (in the first period). Fourth, two period model cannot capture intertemporal issues, e.g. price dynamics, hedging, etc.

Some research, e.g. Grundy & McNichols (1989), extends the two period model into three periods. But adding one period cannot fundamentally solve the problems mentioned above.

Kyle (1985) extends the two period model by allowing trading to happen between the two periods. In his model, there is a market maker who sets prices equal to the conditional expectation of the final payoff. An informed trader who places a series of market
orders (limit order is not allowed) will gradually release his information between time 0 and T. Because the model still treats time 0 and T as special (i.e., information only comes at 0 and must be totally revealed at T), the model cannot fully solve the problems associated with the two period models mentioned above.

Lu (1991) presents a continuous model in which a single speculator faces the rest of the market which is driven continuously by un-observable state variables. The speculator, who takes the price impact of his trading into account, continuously trades in order to optimize intertemporally with an infinite horizon. To reach intertemporal optimization, the speculator also needs to optimally acquire private information and needs to use the information embedded in the price history and in public information. The model presented here generalizes Lu (1991) by increasing the number of agents from one incompletely informed speculator to N heterogeneously informed speculators.

Lu & Olsder (1990) recognizes that because markets are open continuously and at same time each trader takes the price impact of his trading into account, the speculative market is a stochastic non-zero-sum differential game with asymmetric information. This paper develops a model in which two asymmetrically informed speculators trade continuously and optimally in an intertemporal sense. In this model, the informed speculator, who has a monopoly on the information source, is able to manipulate the speed at which the information is incorporated into the price. On the one hand, the model which will be presented in the following sections simplifies the model of Lu & Olsder by assuming that no single speculator is able to totally monopolize information sources. On the other hand, this model extends Lu & Olsder's model by generalizing the market to one which has N heterogeneously informed speculators and a more realistic information structure.

In summary, this paper studies a model of speculative markets with the following features:

-- Speculators trade continuously in order to optimize their objective intertemporally.
-- Each speculator will take the price impact of his trading into account.
-- Dynamic information unfolds and resolves continuously.
-- Agents are heterogeneously informed.
-- Information of agents is made up of:
    information from prices
    public information
    private information
-- Agents observe, incorporate and update information continuously.
-- Adjustment costs are included.

-- Closed form solutions of the optimal trading policies, the average daily profits and price efficiencies will be derived.

One of the obstacles of modeling dynamic information is the difficulty in capturing the concept of "intrinsic value" in dynamic settings. In a two period model, the intrinsic value is the payoff in the second period, at which time the world is assumed to end. But in real world markets, trading continues indefinitely, and there is no specific date when price is equal to value.

In order to model speculative activity in a dynamic setting, we must have a way to capture, in continuous time\textsuperscript{2}, the conceptual equivalence of the second period payoff in a two period model. This leads me to introduce a concept called benchmark value.

An analogy might help to introduce the definition of benchmark value. Imagine that there are two butterflies flying in the air, one chasing the other. Fig.1.1 illustrates this chasing game. There are three attributes of the butterflies' motion worth noticing. First, the "chasee" moves up and down in a manner somewhat like a random walk. Second, the chaser follows the chasee. Third, the chaser's movements do not exactly mirror those of the chasee.

Now, how does this story relate to economic markets? The instantaneous position of the chaser is analogous to an

\textsuperscript{2}This is also true in multi-period settings.
instantaneous price. The instantaneous position of the chasee is analogous to what the price would be if there were no noise in the market. We call this the benchmark value. Formally, the benchmark value$^3$, $v$, of a good is defined as the market clearing price of a good in the absence of either speculative or noise trading.

If the market is absolutely perfect -- in other words there is no noise caused by order imbalances, there are no liquidity trading activities, and buyers are perfectly synchronized with sellers -- the price should exactly equal the benchmark value. But in real world markets, there is noise-trading in the market. This is because there is liquidity trading activity, and buyers and sellers are not perfectly synchronized. The noise-trading will kick the price away from the benchmark value.

We have portrayed a picture in which the benchmark value moves up and down somewhat randomly because unexpected events of the

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$^3$This concept was called fundamental value when I introduced it in Lu (1990)
economic environment bombard the market continuously. This resembles the somewhat random up and down movements of the chasee butterfly. At the same time, the price roughly follows the benchmark value just as the chaser butterfly roughly follows the chasee butterfly.

This mean-reverting tendency of prices around the benchmark value creates speculative opportunities to buy-low-sell-high. That is, buying when the price is lower than the benchmark value and selling when the price is higher than the benchmark value. But things are not so simple because the benchmark value is not directly observable and it is changing continuously. This imperfect information problem is further complicated by the fact that the speculators are heterogeneously informed about \( v \). This is where information comes into the picture.

The more precisely a speculator can estimate where the benchmark value is the less likely he will be to make mistakes. That is, the less likely he will buy (sell) according to his estimates when he should in fact sell (buy) according to the real benchmark value.

The more private (i.e., exclusive or personal) the information of a speculator, the less competition he will have when he uses that information to speculate.

One way each speculator might trade is to first make an estimate of where the benchmark value is at each instant by best using information available to him, and then conduct the buy-low-sell-high scheme according to the estimate. The following discussion shows that this method is in fact optimal.\(^4\)

Where does a speculator's information come from? Take the example of a market maker in an agricultural commodity market. He is paying close attention to the price tick by tick. This is because the price partially conveys not only the information about the continuously unfolding economic environment but also the information of other traders who use their information when they

\(^4\)See Eq.3.9 and Eq.3.9b for a preview of the result.
trade. At the same time the speculator is observing the public information contained in the news media and in government statistical releases. Furthermore, he receives private information from experts about such things as weather and politics. And he might even receive a few tips from his friends. In summary, there are three sources of information:

-- information from price history
-- public information
-- private information

The plan of this paper is as follows:

Section 2 presents a close form solution of a single-good model with homogeneous information. Extending the model in section 2 into a model with a heterogeneous information structure, section 3 presents a close form solution and then gives a general discussion of the meaning of better information. Section 4 presents a much more comprehensive multi-good model.\(^5\) To make the paper more readable, I have put most of the technical detail in Appendix 1 & 2.

Lu (1990) introduced a network method of studying market structure which is called AGFP (Agent-Good-Flow-Price) network method. Just as in a hydraulically controlled network, where devices are connected by tubes and signal lines with fluid flowing around at equilibrium pressures, the AGFP network method takes the perspective that agents are connected by market and information channels with goods flowing around at equilibrium prices. The network method makes it tractable to study complex multi-agent multi-market economic problems without losing intuition.

In fact, the original motivation of Lu (1990) was to facilitate the model-building process for the problem like the one in this paper. Although this paper is written with those who have not read Lu (1990) in mind, readers who have read Lu (1990) might find that glancing at the AGFP networks in Appendix 3 before reading the next section will save time and provide more insight

\(^5\)Section 4 is not included in this version.
into the models.
2. An Example of Speculation with Homogeneous Information, Model 1\(^6\)

In next two sections, close form solutions will be presented for simple cases of the generalized model which will be introduced in section 4. The emphasis is on tractability and intuition. We will see how each parameter affects the solutions in closed form.

To simplify the presentation and to set up a benchmark, I will assume away heterogeneity in this section by temporarily assuming that the speculators are homogeneously informed. So in this section we will concentrate on stochastic control and imperfect information.

**OBJECTIVE OF THE SPECULATORS:**

average daily profit flow (= capital gain - adjustment cost - carrying cost)

\[
J_n = \max_{f_n(t), \mathcal{F}_n} \mathbb{E}\left[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( g_n dp - \delta f_n^2 dt - \lambda g_n^2 dt \right) \right] \quad n=1, \ldots, N; N \geq 1 
\]  \hspace{1cm} (2.1)

where \( \delta \geq 0, \) and \( \lambda > 0. \)

In general, a speculator likes speculative profits, dislikes adjustment, and dislikes net inventory positions, which are a measure of his risk.

The \( g \) is the inventory held by the speculator. The \( p \) is the instantaneous price. So, the first term in the parentheses, \( gdp, \) represents the instantaneous speculative profit flow made by the speculator.

The second term in the parentheses, \( \delta f^2, \) represents the adjustment cost, which is assumed to be in quadratic form\(^7. \) Here,

\(^6\)See Appendix 3 for the AGFP network.

\(^7\)The quadratic form is chosen mainly to increase the tractability of the model.
the "adjustment cost" includes all costs related to the adjustment of the speculator's inventory position. If a speculator is in a hurry to change his position, he might have to search more aggressively for trading opportunities and will thus be in a disadvantageous position in terms of adjustment costs (e.g., transaction costs, telephone bills, utilization of valuable time of traders, hurried result of bargaining). The case of zero adjustment cost is included because \( \delta \) is allowed to be zero.

The third term in the parentheses, \( \lambda g^2 \), represents the carrying cost which can be thought of as the equivalent risk premium the speculator would require in order to be willing to hold the un-hedged position, \( g \). To capture the general fact that the marginal disagreeableness of the inventory is increasing when the inventory gets larger, the cost is assumed to be quadratic.

One result of Lu (1991) is that when there is a carrying cost, \( \lambda g^2 \), the optimal trading policy of the speculator will have, in addition to the speculative term, a linear inventory reduction term \( \alpha' g \) in which \( \alpha' \) will become infinitesimally small as \( \lambda \) in the carrying cost term becomes infinitesimally small. In other words, when the carrying cost is very small, the effect of the carrying cost on optimal trading policy is negligible. In order to concentrate on the speculative terms, we will assume away the effect of the carrying cost by assuming \( \lambda \) to be infinitesimally small in following analysis.

Note, that the number of speculators, \( N \), is larger than or equal to one. The case of \( N = 1 \) is a special case of Lu (1991).

Here the speculator is only speculating on the spot price. Generally, a forward or future position is equivalent to a spot position plus an arbitrage position. If we assume that any arbitrage opportunity will be taken advantage of until it is so small that we can assume it is zero, then speculating on forward or future prices is equivalent to speculating on spot prices. This observation is in fact supported by empirical data which shows that forward and future prices move almost together with the spot price. This assumption allows us to aggregate the speculative activities
of the forward and spot markets into the spot price speculation only.

**DYNAMICS**

**Relationship of inventory and order flow:**

\[ dg_n = -f_n dt \]  \hspace{1cm} (2.2)

the derivative of the inventory, \( g \), is equal to trading flow, \( f \).

**Dynamic of noise trading flow:**

\[ du = -\phi_u u dt + \sigma_u dB_u, \quad \phi_u > 0 \]  \hspace{1cm} (2.3)

Where \( dB_u \) is a standard Brownian Motion.

Here the noise trading flow, \( u \), is assumed to be an Ornstein-Uhlenbeck process which captures the following facts:

-- The mean of \( u \) is zero.
-- \( u \) is mean-reverting.

**Dynamic of benchmark value:**

\[ dv = \sigma_v dB_v \]  \hspace{1cm} (2.5)

where \( dB_v \) is a standard Brownian Motion.

The benchmark value, \( v \), is assumed to follow a random walk (Brownian Motion) because there should be little predictability for \( v \) in the short term. In fact, prices of a commodity might not follow a random walk exactly but instead have a small drift which is mean-reverting around some very long-term mean. In that case, I have showed in Lu (1991) that the optimal ordering policy and NPV will almost be the same as the zero-drift case.

To introduce the market clearing condition, let's first consider the case without the speculators. Fig.2.1 gives a graphical demonstration of the following concepts:

-- The unfolding (i.e. continuous formation) of the benchmark
value by the continuous interaction of the ever changing demand and supply schedules.

- The continuous formation of the price by adding the effect of noise trading on the benchmark value.

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![Diagram](image)

**Fig. 2.1 Unfolding and Resolving of dynamic information**

We should distinguish between stock concepts, which are measured in terms of units of goods, and flow concepts, which are measured in terms of units of goods per unit of time. In traditional static models, demand, supply and noise trading are considered stock concepts. But in reality, demand, supply and noise trading come continuously and are in fact flow concepts. This is why static models do not work well for modeling intertemporal dynamics. It is worth emphasizing that the concepts in Fig. 2.1 are flow concepts.

Note the inverse of the slope of the residual demand curve at point E is $\varphi_\alpha$, which reflects the responsiveness of the price to noise trading. The inverse of the slope will be defined as the liquidity of the market when there are no speculative activities.

If we also include the contribution of speculative trading, we
get, 
Market clearing law:

\[ \varphi_n(p - v) + \sum_{n=1}^{N} f_n = u \]  

or in words:
residual demand + speculators' trading = noise trading

The first term in market clearing law reflects the residual demand of the good. Note, if both noise trading, \( u \), and speculator's trading, \( f \), are zero, then \( p = v \). In other words, if there were neither noise trading nor speculative trading in the market, the market clearing price would equal the benchmark value, \( v \). This is consistent with the definition of the benchmark value given in the last section.

INFORMATION STRUCTURE:
Public information:

\[ dw = \nu dt + \sigma_w db_w \]  

Public information is a noisy observation of the benchmark value, \( v \).\(^8\)
Information sets of the speculators:

\[ \mathcal{F}_n(t) = \{ p(t), w(t), -\infty < t \leq t \} \quad n = 1, \ldots, N \]  

Note, we use so called information sets to specify "who knows what when". Here, for example, \( \mathcal{F}_n(t) \) means the information available to the speculator \( n \) at time \( t \) which includes the whole history (including the current value) of \( p \) and \( w \).

\(^8\)This kind of observation is the continuous time counterpart to discrete time observation, \( e_n = v_n + n_n \), with \( n_n \) i.i.d. normally distributed.
In the set up of this model, \( b_u, b_v, b_w \) are assumed to be independent standard Brownian Motions.

The central characteristic of speculative activities is that of buying (selling short) something with the intention of selling (buying) it at a higher (lower) price in the future. To concentrate on this characteristic, I will basically assume away other issues like convenience yields, depreciation, cost of capital, delivery, etc. In other words, I assume: 1) zero convenience yield; 2) zero depreciation or appreciation rate of inventory; 3) zero interest rate\(^9\); 4) zero delivery cost, if delivery is ever needed; etc.

In order to present the solution of Model 1, I need first present two propositions.

The first proposition concerns the unconditional distribution of \( u \) and \( v \).

**Prop.2.1 Unconditional variance of \( u \) and \( v \)**

Given the set up of Model 2, the unconditional variance of \( u \) is:

\[
V_u = \frac{\sigma_u^2}{2\Phi_u} \tag{2.7a}
\]

and the variance of \( v \) conditioned only on the current price (hereafter referred to as the unconditional variance of \( v \)) is:

\[
V_v = \Phi_u^{-2} \frac{\sigma_u^2}{2\Phi_u} \tag{2.7b}
\]

**Proof:**

Apply Thm.A2.5 on the dynamics of \( u \), i.e. Eq.2.3. For any

\(^9\)The horizon of a speculative activity such as market making is generally short (i.e., several days at most). Therefore, we are talking about the daily interest rate.
initial condition, \((m_o, \, V_o)\), solving the differential equation Eq.A2.5.3, we get:

\[ m = m_o e^{-\phi s t} \rightarrow 0; \text{as } t \rightarrow \infty \]

From Eq.A2.5.4, the stationary \(V_u\) satisfies the differential equation:

\[ 0 = \dot{V}_u = -2\phi V_u + \zeta_u^2 \]

Solving it, we get Eq.2.7a in the following proposition.

To derive Eq.2.7b, rewrite Eq.2.4 into:

\[ \nu - Z_1 = -\phi_m^{-1} u \]

of which the unconditional variance of both sides should be same because

\[ Z_1 = \rho + \phi_m^{-1} \varepsilon \]

is related with current price only.

The next proposition is about the expression of the conditional distribution of \(v\) conditioned on the information set of the speculator. Note we have a huge amount of data in the information set which includes the whole continuous history of \(p\) and \(w\) from the current instant to \(-\infty\). Fortunately, the conditional mean of \(v\) given \(\mathcal{F}(t)\) is a sufficient statistic. The existence of the sufficient statistic makes it possible to do the calculation recursively using the Kalman Filter.

Prop.2.2 Filtering problem of Model 1

Given the setup of Model 1, the conditional distribution

\[
\begin{bmatrix}
\nu(t) \\
u(t)
\end{bmatrix} | \mathcal{F}_n(t) \sim \text{Normal}
\begin{bmatrix}
\hat{\nu}(t) \\
\hat{u}(t)
\end{bmatrix}, 
\begin{bmatrix}
\Omega_v & -\phi_m \Omega_v \\
-\phi_m \Omega_v & \phi_m^2 \Omega_v
\end{bmatrix}
\]

\[ (2.8) \]
which are perfectly negative correlated, and

\[ \frac{\Omega_v}{\nu_v} = \frac{\Omega_u}{\nu_u} = \frac{2}{1 + \frac{\varphi_m^{-2} \sigma_u^2}{\sigma_v^2} + \frac{\varphi_m^{-2} \sigma_u^2 (\varphi_m^{-2} \sigma_u^2 + \sigma_v^2)}{\sigma_u^2 \sigma_v^2}} \]  \hspace{1cm} (2.9)

\( \hat{\nu} \) satisfies differential equation:

\[ d\hat{\nu} = \frac{\Omega_v \varphi_u + \sigma_v^2}{\sigma_u^2 \sigma_v^2 + \sigma_v^2} [dz_1 - (\varphi_u \hat{\nu} - \varphi_u z_1) \, dt] + \frac{\Omega_v}{\sigma_m^2} [dw - \nu \, dt] \]

\[ = \text{contribution of price information + of public information} \]

\[ \hat{\nu}(-\infty) = \nu(-\infty) \]

where

\[ z_1 = \nu + \varphi_m^{-1} \sum_{n=1}^{N} \mathbf{f}_n \]  \hspace{1cm} (2.11)

and

\[ \varphi_m^{-1} \hat{\nu} + \hat{\nu} = \varphi_m^{-1} u + v = z_1 \]  \hspace{1cm} (2.12)

Proof: See Appendix 1.1

Note the second term in the "\( \sqrt{\cdot} \)" of Eq.2.9 is the contribution of the price information and the third term is the contribution of the non-price information. The effect of these contributions is the reduction of the conditional variance. In other words, the information makes the estimate more precise.

The Eq.2.10 gives the scheme to continuously incorporate new information from the price and the information sources into the conditional mean recursively. The first bracket is the innovation
from \( p \) (note Eq.2.11), and the second bracket is the innovation from the information source, \( w \).

Armed with the two propositions above, we are ready to state the solution of Model 1.

**Solution of Model 1:**

Bayesian-Nash equilibrium trading policy:

\[
\hat{f}_n = \alpha (p - \hat{V}) \quad n = 1, \ldots, N
\]  

(2.13)

where \( \alpha \) is the positive solution of:\(^{10}\)

\[
\alpha = \left[ 2\delta + \left( (N-1)\alpha + \varphi_m \right)^{-1} \right]^{-1}
\]  

(2.14)

And the average daily profit of each player is:

\[
J = J_0 \left( 1 - \frac{\Omega_V}{V_v} \right)
\]  

(2.16)

where

\[
J_0 = \frac{1}{4 \left[ \varphi_m + (N-1)\alpha \right]^2 \left( \varphi_m + (N-1)\alpha \right)^{-1} + \delta} \frac{\sigma_u^2}{2\varphi_u}
\]  

(2.17)

**Proof:** see Appendix 1.2

**Discussions of the solution of Model 1:**\(^{11}\)

The optimal trading policies (Eq.2.13) are flows. This is apparent if we take a close look at the market clearing law,

\[\alpha = \frac{1}{2\delta + \varphi_m}, \quad \text{for } n = 1; \quad \text{and} \]

\[\alpha = \left[ \frac{N-2-2\varphi_m\delta}{(N-1)\delta} \right]^2 + \frac{2\varphi_m}{(N-1)\delta} \frac{1}{2} + \frac{N-2-2\varphi_m\delta}{(N-1)\delta} \quad \text{for } n = 2, 3, \ldots, N\]

\(^{10}\)That is: \( \alpha = 1/(2\delta + \varphi_m) \), for \( n = 1 \); and

\(^{11}\)The discussion of the \( \alpha \) and \( J_0 \) is postponed to the discussion of the solution of Model 2.
Eq.2.4. As soon as a speculator buys or sells, he bids the price against his favor. In order to minimize the price impact of his trading he must buy or sell smoothly. This is captured by the concept of flow.

From Eq.2.13, the equilibrium of trading processes of speculators are as follows:

-- Each speculator updates his estimate of \( v \) continuously by incorporating new data on the price and other observations.\(^{12}\)

-- He conducts the buy-low-sell-high strategy continuously using his estimate of \( v \).

The average daily profit (Eq.2.16) is the product of two terms. The first term, \( J_0 \), represents the average daily profit if the speculator has perfect information. The ratio \( \Omega_v/V_v \) in the parentheses represents the quality of the speculator's information. Recall that \( \Omega_v \) is the conditional variance of \( v \) and \( V_v \) is the unconditional variance of \( v \). So the ratio is some number in \([0, 1]\). If the quality of the information is infinitely good, then \( \Omega_v \) is zero and so is the ratio. In that case, the speculator's profit will reach the highest level. This is not surprising considering \( \Omega_v = 0 \) is another way of saying that \( v \) is perfectly observable. At the other extreme, if the quality of the speculator's information is very bad, the conditional variance \( \Omega_v \) is hardly smaller than the unconditional variance, \( V_v \) (i.e., possession of the information will hardly improve the estimate of \( v \)). In that case, the ratio will be almost as large as one and the expected profit flow will be close to zero.

Because, in fact, there should be some information in the price history observable to everyone, the ratio should never be as large as one. This argument could be made precise by taking a look at the explicit expression of this ratio given in Eq.2.9 in which we can see how each piece of information will contribute to the

\(^{12}\)Their estimates are the same because their information sets are the same.
NPV. Even if the speculator does not have any information other than the price (i.e. the third term is zero), the second term in the square root (which is the contribution of the price information) will make the ratio smaller than one. In actual markets, the speculator will get public information and private information in addition to the price information. In that case, the third term in the square root, which is the contribution of the non-price information, will make the ratio even smaller.

Generally there is a fixed cost associated with speculation which, for example, might be the price of a seat plus and the cost of keeping an active trading staff at the Chicago wheat market. We might consider the problem from the point of view of industrial organization. The number of traders will be just enough so that each trader will make a little more than enough profit to cover his fixed cost.

This point of view over simplifies the industrial organization of speculative markets. First, the adjustment cost and carrying cost are different across the speculators. Speculators that survive will likely be the ones with favorable adjustment and carrying costs. It is a straightforward generalization to make this model take into account this cost divergence. Second, the speculators are heterogeneously informed. It is apparent that "better informed" speculators will get more profit and be more likely to survive. To take this into account, let's extend model 1 to the case of heterogeneous information by adding a private information source to the information set of each speculator.
3. Examples of Speculation with Heterogeneous Information

In this section, I will first present Model 2 which will relax the homogeneity assumption in Model 1 by recognizing the agents are heterogeneously informed. Section 3.2 will investigate a special case of Model 2 to get more insight. Section 3.3 will provide some general discussions of the meaning of better information.

3.1 Model 2, Speculation oligopoly with heterogeneous information

I will introduce the information heterogeneity into the picture by assuming that each speculator has, in addition to the information set of Model 1, a private information source, which is a noisy observation of the benchmark value. Other aspects of the set up are the same as that in the Model 1, and so we can concisely state the model as follows:

OBJECTIVE OF THE SPECULATORS

Average daily profit flow (= capital gain - adjustment cost - carrying cost):

\[ J_n = \max_{f_n} E \left[ \lim_{T \to \infty} \frac{1}{T} \int_0^T (g \cdot dp - \delta f_n^2 dt - \lambda g_n^2 dt) \right] \quad n=1, \ldots, N; N \geq 1 \quad (3.1) \]

where \( \delta \geq 0 \), and \( \lambda = 0^+ \).

DYNAMICS

Relationship of inventory and order flow:

\[ dg_n = -f_n dt \quad n=1, \ldots, N \quad (3.2) \]

Dynamic of market noise:

\[ du = -\varphi_u u dt + \sigma u dB_u, \quad \varphi_u > 0 \quad (3.3) \]
Dynamic of benchmark value:

\[ dv = \sigma_v db_v \quad (3.5) \]

Market clearing law:

\[ \varphi_m(p - v) + \sum_{n=1}^{N} f_n = u \quad (3.4) \]

INFORMATION STRUCTURE

Public information:

\[ dw = v dt + \sigma_w db_w \quad (3.6) \]

Private information:

\[ dz_n = v dt + \sigma_n db_n \quad n = 1, \ldots, N \quad (3.7) \]

Information sets of speculators:

\[ \mathcal{F}_n(t) = \{ p(\tau), w(\tau), z_n(\tau), -\infty < \tau < t \} \quad n = 1, \ldots, N \quad (3.8) \]

The \( b_u, b_v, b_w, b_n, n = 1, \ldots, N \) in Eq.3.3 to Eq.3.8 are standard Brownian Motions with a known covariance matrix. Among them \( b_u \) and \( b_v \) are independent of each other.

Solution of Model 2:

Bayesian-Nash (or rational expectation) equilibrium trading policy:

\[ f_n = \alpha(p - \bar{v}^{(n)}), \quad n = 1, \ldots, N \quad (3.9) \]
where $\alpha$ is the positive solution of:\footnote{That is: $\alpha = 1/(2\delta + \varphi_m)$, for $n = 1$; and $
olinebreak[4]$ $\alpha = [\left(\frac{N-2-2\varphi_m\delta}{(N-1)\delta}\right)^2 + \frac{2\varphi_m}{(N-1)\delta}]^{\frac{1}{2}} + \frac{N-2-2\varphi_m\delta}{(N-1)\delta}$ for $n=2,3,\ldots,N$}

\begin{equation}
\alpha = \left(2\delta + [(N-1)\alpha + \varphi_m]^{-1}\right)^{-1}
\end{equation}

(3.9a)

and

\begin{equation}
\vartheta^{(n)} \triangleq E(v|\mathcal{F}_n).
\end{equation}

(3.9b)

The deviation of $p$ from $v$ is:

\begin{equation}
(p - v) = \frac{1}{\varphi_m + N\alpha} \left[u + \sum_{n=1}^{N} \alpha (v - \vartheta^{(n)})\right]
\end{equation}

(3.10)

The price efficiency\footnote{The price efficiency is defined as standard deviation of $p - v$ for the reason that $v$ is the price if there is no noise in the market.} of the market is:

\begin{equation}
e_p = SD(p - v) = \frac{1}{\varphi_m + N\alpha} \left(E\left[u + \sum_{n=1}^{N} \alpha (v - \vartheta^{(n)})^2\right]\right)^{\frac{1}{2}}
\end{equation}

(3.11)

The average profit flow of player $n$ is:

\begin{equation}
J_n = J_0 \eta_n, \quad n = 1, \ldots, N
\end{equation}

(3.12)

$= [A$ term common for each player$] \times [A$ term depending on the individual information configuration$]$

where\footnote{Note, here $\alpha$, $J_0$ are same as in the Model 1.}

\begin{equation}
J_0 = \frac{1}{4 \left[\varphi_m + (N-1)\alpha\right]^2 \left[\varphi_m + (N-1)\alpha^{-1} + \delta\right]} \frac{\sigma_u^2}{2\varphi_u}
\end{equation}

(3.13a)
and

\[ \eta_n \Delta E \left( \{ \hat{u}^{(n)}_k \} + \sum_{k=1}^{N} \alpha \left[ E(\hat{\varphi}^{(k)} | \mathcal{F}_n) - \hat{\varphi}^{(n)} \right] \right) / \frac{\sigma^2_u}{2 \varphi_u} \]  

(3.13b)

**Proof:** see Appendix A1.3.

**Discussion of the solution of Model 2:**

The optimal trading policy, Eq.3.9, says that the optimal trading process is as follows:

-- Each speculator updates his estimate of \( v \) (i.e., conditional expectation given his information set, \( \hat{\varphi}^{(n)} \)) continuously by incorporating new price data and new non-price data including his private information. When doing so, he rationally recognizes that prices will convey information about the benchmark value through two channel. One channel is residual demand, and the other is the information of other speculators who use their information when they trade. In this sense, the price is a signal of other speculators' information.

-- The speculator conducts the buy-low-sell-high strategy continuously using his estimate of \( v \).

There are three factors which affect how aggressively the speculator trades.

First is how much he thinks the good is mis-priced. The greater the difference between his estimate of the \( v \) and \( p \), the more aggressively he will trade.

Second is the adjustment cost, \( \delta f^2 \). The higher the adjustment cost, \( \delta \), the more aggressively he will trade.

Third is the price impact of his trading. This is because the speculator's trade will bid the price against his favor (see market clearing condition, Eq.3.4). The higher the market liquidity( [(N
he faces, the lower the price impact and the more aggressively he will trade.

The Deviation $\delta$, Eq.3.10, and the price efficiency $\text{SD}(p - v)$, Eq.3.11, are caused by two sources. The first source is the noise trading flow, and the second is the aggregate mistake of the speculators.

Also, the more speculators in the market (larger $N$), the higher the total liquidity in the market ([$N\alpha + \phi_s$]) and the more efficient the price.

The average daily profit flow, Eq.3.12, is the product of two terms.

The first term, $J_0$, is equal to the average daily profit when every speculator has perfect information (in this case, the second term is equal to one, see Prop.2.1). If we take a look at the expression of $J_0$ (Eq.3.13a), it is apparent that the speculator's profit will be higher if:

-- The market he faces is less liquid.
-- The noise order activity of the market is stronger (measured by $V_0$).
-- His adjustment costs are lower.

The second term, $\eta_n(3.13b)$, has two random variables inside the braces. The first one, $\hat{u}^{(u)}$, represents the speculator's opportunity if each other speculator has the same information as he does. The second term is the extra opportunity created by the mistakes (if he can perceive them) made by the other speculators. Note, the difference, $E(\hat{v}^{(k)} | \mathcal{F}_n) - \hat{v}^{(n)}$, is speculator n's best estimate of speculator k's mistake. So it is not only the precision of a speculator's information but also the relationship of his information with the configuration of everyone else's information which determines his profit. To motivate later discussion about this configuration, let's first solve a simple

\[\text{16Which is the inverse slope (see fig.2.1) of the residual demand curve he faces. This comes from two sources: 1) the liquidity contribution of the market public, } \phi_s; \text{ 2) the liquidity contribution of the another } N - 1 \text{ speculators, } (N - 1)\alpha.\]
example where one group has the best private information obtainable and the other has none.

3.2 Model 2a, Speculation with an informed group and an uninformed group:

Let's further specify the private information \( z_n \) as:
player 1, ..., player \( M \) (informed) know \( v \) exactly (\( \sigma_n = 0 \));
player \( M+1, ..., \) player \( N \) (uninformed) do not have private information (\( \sigma_n = +\infty \)). Or we could say, in addition to the set-up of Model 1 in section 2, we will give a group of \( M \) players exact observations of \( v \).

INFORMATION SETS
informed:

\[
\mathcal{F}_i(t) = \{p(t), w(t), v(t), -\infty < t < t\}
\] (3.14)

uninformed:

\[
\mathcal{F}_u(t) = \{p(t), w(t), -\infty < t < t\}
\] (3.15)

To apply the solution of Model 2, we need first solve the filtering problem of the uninformed group.

Prop.3.1 (Filtering problem of Model 2a)

Given the set up of Model 2 from the point of view of the uninformed group:

\[
\begin{bmatrix} v(t) \\ u(t) \end{bmatrix} | \mathcal{F}_u(t) \sim \text{Normal} \left( \begin{bmatrix} \varphi(t) \\ \hat{u}(t) \end{bmatrix}, \begin{bmatrix} \Omega_v & -\varphi_v \Omega_v \\ -\varphi_v \Omega_v & \varphi_v^2 \Omega_v^2 \end{bmatrix} \right)
\] (3.16)

where
\( \varphi_0 = \varphi_m + M \alpha \) (3.16a)

\[
\frac{\Omega}{V} = \frac{\Omega_u}{V_u} = \frac{2}{1 + \sqrt{1 + \frac{\varphi_0^2 \sigma_u^2}{\sigma_v^2} + \frac{\varphi_0^2 \sigma_v^2 (\varphi_0^2 \sigma_v^2 + \sigma_u^2) \varphi_u^2}{\sigma_v^2}}}
\]

(3.17)

\( \tilde{\vartheta} \) satisfies differential equation:

\[
d\tilde{\vartheta} = \frac{\Omega_u \varphi_u + \sigma_v^2}{\varphi_0^2 \sigma_u^2 + \sigma_v^2} [dz_1 - (\varphi_u \tilde{\vartheta} - \varphi_u z_1) dt] + \frac{\Omega}{\sigma_m^2} [dw - \vartheta dt]
\]

(3.18)

\( \tilde{\vartheta}(-\infty) = \varphi(-\infty) \)

where

\[
z_1 = \varphi + \varphi_0^{-1} \sum_{n=M+1}^{N} f_n
\]

(3.19)

and

\[
\varphi_0^{-1} \tilde{\vartheta} + \varphi = \varphi_0^{-1} u + v = z_1
\]

(3.20)

**Proof:**

From the uninformed point of view, the trading flow of the informed \( f_i = \alpha(p - v) \) can be aggregated with residual demand of market public. Then the filtering problem of Model 2a is the same as in Model 1 except the \( \varphi_m \) should be substituted by \( \varphi_m + M \alpha \) which is defined as \( \varphi_0 \). So, we can get filtering solution of uniformed in Model 2a by substituting \( \varphi_m \) in Prop.2.2 by \( \varphi_0 \).

Now, the solution of Model 2a is ready to be presented:
Solution of Model 2a:

\[ f_i = \alpha (p - \nu) \]  \hspace{2cm} (3.20a)

\[ f_u = \alpha (p - \varphi) \]  \hspace{2cm} (3.20b)

and

\[ (p - \nu) = \frac{1}{\varphi_m + Na} \left[ u + \alpha (N-M) (\nu - \varphi^{(n)}) \right] \]  \hspace{2cm} (3.20c)

\[ SD(p - \nu) = \frac{1}{\varphi_m + Na} \left( E\left\{ [u + \alpha (N-M) (\nu - \varphi^{(n)})]^2 \right\} \right)^{\frac{1}{2}} \]  \hspace{2cm} (3.20d)

\[ \eta_u = 1 - \frac{\Omega}{V_\nu} \]  \hspace{2cm} (3.21)

\[ \eta_i = \frac{E\left\{ [u + \alpha (N-M) (\nu - \varphi)]^2 \right\}}{\sigma_u^2} \frac{\varphi_u}{2\varphi} \]  \hspace{2cm} (3.22)

\[ = 1 + \frac{2\varphi_0 \alpha (N-M) + [(N-M) \alpha]^2}{\varphi_0^2} \frac{\Omega}{V_\nu} \]  \hspace{2cm} (3.22')

Proof:

Let's use the terminology of Eq.3.11.

From prop.3.1:

\[ \hat{v}^{(u)} = \hat{v}, \quad \phi^{(u)} = \varphi. \]  \hspace{2cm} (3.23)

Because \( v \) is observed perfectly

\[ \hat{v}^{(f)} = u, \quad \phi^{(f)} = \nu. \]  \hspace{2cm} (3.24)

Now
\[ E(\hat{\nu}^{(j)} | \mathcal{F}_u) = E(\nu | \mathcal{F}_u) = \nu \] (3.25)

the first equality comes from the fact that the informed know \( \nu \) exactly and the second equality comes from definition Eq.3.13.

\[ E(\hat{\nu}^{(u)} | \mathcal{F}_j) = E(\nu | \mathcal{F}_j) = \nu \] (3.26)

The first equality comes from Eq.3.23. The second equality comes from the fact that the informed know \( \hat{\nu} \) exactly because the observation of the informed includes the observation of the uninformed.

Eq.3.2.23, 3.25 \( \rightarrow \) Eq.3.11, and note \( E(\hat{u}^2) = V_u - \Omega_u \), we have:

\[ \eta_u = \frac{E(\hat{u}^2)}{2\varphi_u} = 1 - \frac{\Omega_u}{V_u} \]

Eq.3.24, 3.26 \( \rightarrow \) Eq.3.11, we get 3.22.

Eq.3.20, 2.7a \( \rightarrow \) 3.22, and given \( \hat{u} \) and \( u - \hat{u} \) are orthogonal:

\[ \eta_i = E((\hat{u} + (u - \hat{u})) + \alpha (N-M) (u - \hat{u})/\varphi_u)^2) / V_u \]

\[ = E((\hat{u} + [1 + \frac{\alpha (N-M)}{\varphi_0}] (u - \hat{u})^2) / V_u \]

\[ = (E(\hat{u}^2) + [1 + \frac{\alpha (N-M)}{\varphi_0}]^2 E((u - \hat{u})^2)) / V_u \]

\[ = (E(u^2) + [1 + \frac{\alpha (N-M)}{\varphi_0}]^2 - 1) \Omega_u) / V_u \]

from which 3.22' is the immediate result.

Other formulas are immediate results of the solution of Model 2.
Discussions of the solution of Model 2a:

Let's compare the profit opportunity of uninformed speculators before (Eq. 2.16) and after (Eq. 3.12 & 3.21) giving M player private information. The only difference is on the ratio $\Omega_v/V_v$ (compare Eq. 2.9 for Model 1 with Eq. 3.17 & 3.16a for Model 2a). The fact that $\varphi_m < \varphi_c (= \varphi_m + M\alpha)$ means that the ratio will get smaller. Intuition here is that the existence of M informed traders is equivalent to increasing the liquidity of the market public by $M\alpha$. That, in turn, reduces the contribution of the information.

Let's use another comparison to get some intuition about the profit opportunity of the informed group. If everyone is perfectly informed, then the second term in the braces will disappear. This is because the second term is the extra profit opportunity created by the mistakes of the uniformed group.

An intuitive way to visualize the advantage of informed over uninformed is to see the relationship of the random variables in Hilbert space with the inner product defined as covariance, see fig. 3.1.

Notice the length of OB is the unconditional standard deviation of $u$ and is exogenously given. If we use the length of OB as a unit, then

$$\eta_u = |OA|^2 = \|\hat{u}\|^2$$
$$\eta_i = |OC|^2 = \|u + \frac{\alpha}{\varphi_m + M\alpha} (u - \hat{u})\|^2$$

The first term in the braces reflects the money made from noise order flow. The second term in the braces reflects the
money made from mistakes of the uninformed.

3.3 Some Remarks About the Meaning of Better Information

What do we mean when we say that one speculator has better information than another? It is clear from the discussions of the previous sections that there is more than one attribute which characterizes information.

In the real world, one's information is not made up of one or several numbers as is assumed in the models in most of the literature in information economics. Even if a speculator's information is made up of the information embedded in the price only, it includes an infinite number of data because it contains the whole history of the price. Because of this, it is generally required to keep all historical data in order to make an optimal decision. Therefore, a Markov system will generally lose Markovian properties if the observation is not perfectly precise. In that case, the problem will generally become infinitely dimensional even though the original system is finitely dimensional. This makes it very difficult to deal with information in dynamic settings.

A Hilbert space like fig.3.1 provides a tool to describe the overall information relationships of the speculators. Let's use several more figures of this kind to introduce some points.

In Model 2a, the informed speculators are assumed to have infinitely precise information. It is more realistic to assume instead that they are only relatively more informed but not infinitely informed. Fig.3.2 provides a configuration which is consistent with this assumption. If we orthogonally project the whole structure on \( \mathbb{H}_2 \) which is the sub-Hilbert space spanned by total information available to the speculators of both groups, we get the same triangle as in fig.3.1. In this sense, Model 2a will not lose generality if the informed group is not perfectly informed.

Figures 3.1 and 3.2 are examples of one category of information structure called nested information structure.
Generally, in a nested information structure, it is possible to order the agents according to their information set so that an agent higher in the information hierarchy knows everything which is known to the agents positioned below.

Let's use fig.3.2 to explain the information advantage of a speculator positioned higher in the information hierarchy. If we project orthogonally the error of group 1, \( \|AD\| \), on the sub-Hilbert space spanned by the information set of group 2, \( \mathcal{H}_2 \), we get \( \|DB\| \) which translates into extra profit opportunities for group 2, \( \|BC\| \). But if we project the error of group 2 \( \|AB\| \) on \( \mathcal{H}_1 \), we get only one point, D. This is because if \( \|AB\| \) is orthogonal to \( \mathcal{H}_2 \), it certainly orthogonal to \( \mathcal{H}_1 \) which is nested in \( \mathcal{H}_2 \).

In other words, speculator A can take advantage of the mistake made by another speculator B only if speculator B is positioned...
lower in the information hierarchy than speculator A.

Another category of information structure is overlapping information structure. In overlapping information structure each speculator knows something others do not know. Fig.3.3 is an example of overlapping information structure. Each group in fig.3.3 is able to take advantage of the mistakes of the other group. This is generally true in overlapping information structures.

![fig. 3.3 Overlapping information structure](image)

Let's use fig.3.3 to introduce another attribute of information. It is apparent that the information of group 2 is more precise than that of group 1. But it turns out that group 1 will make more profit than group 2. This is because the number of speculators in group 1 is much smaller than that of group 2. In other words, the information of group 1 is more private compared with the information of group 2.

To summarize the several attributes mentioned above, we could
say that one's information is better if one's information is:
- more precise
- more private
- higher in the information hierarchy

For a linear Gaussian system, if the information structure of a system is nested, it is generally possible to use a finite number of state variables to describe the state of the system even the observations of the agents are infinite. This is because we can use a finite number of sufficient statistics to summarize all the past data. This is in fact the trick we used to get the explicit solutions of Model 1 and Model 2 where \( \hat{v} \) is the sufficient statistic.

But for the cases of overlapping information structure, no one has found this kind of sufficient statistic, as far as I know. That means it is impossible to summarize the current state of the system using a finite number of state variables. In other words, we need to carry a complete data set of the past history in order to do the optimization.

Although the problem is more complicated, the basic concept is still the same, that is, the orthogonal projection of an unknown state variable on the Hilbert space generated by one's information set. Only in this case, the orthogonal projection cannot be done recursively by Kalman Filter but requires functional analysis. Because of this, it seems impossible to get an explicit solution as we get in Model 1 and Model 2a. And even if we were to solve an example, we would probably get no more intuition than we can get from fig.3.3.

**summary**

This paper studies a model of speculative markets with the following features:
-- Bayesian-Nash equilibrium of \( N \) heterogeneously informed speculators who optimally extract information from:
  individualistic private information
public information
information from price
-- Speculators trade continuously in order to optimize their objective intertemporally.
-- Each speculator will take the price impact of his trading into account.
-- Dynamic information unfolds and resolves continuously.
-- Agents observe, incorporate and update information continuously.
-- Adjustment costs are included.
-- Closed form solution of the optimal trading policies and the average daily profits have been derived.

A notion of benchmark value is introduced to capture the "intrinsic value" of a good in continuous time.

It has been shown that speculation is basically an information game, and the information configuration of a speculative market in continuous time can be represented conveniently as a Hilbert space. Using this Hilbert space, we can intuitively show that a speculator's information is better if it is:
- more precise
- more private
- higher in the information hierarchy
Appendix 1 Details of Calculations

This appendix deals with the technical details of the calculations in the main text.

A1.1 Proof of Prop 2.1 (Filtering problem of Model 1)

This proof can be sketched in following two steps:
(a) Write the filtering problem into standard form of Thm.A2.2.
(b) Apply Thm.A2.2.

Rewrite Eq.2.15 into:

\[ \mathcal{P} + \Phi_{-1}^{-1} \sum_{n=1}^{N} f_n = \mathcal{V} + \Phi_{-1}^{-1} U \Delta z_1 \]  

(A1.1.1)

so a linear combination of unknown state variables, \( v \) and \( u \), is observed exactly. This allows us to reduce the order of the filtering problem by one.

Take expectation \( E(\cdot | \mathcal{F}_n) \) on both side of A1.1.1:

\[ \hat{\mathcal{V}} + \Phi_{-1}^{-1} \hat{\Delta} = z_1 \]  

(A1.1.2)

Combine A1.1.1 with A1.1.2, we have Eq.2.12 which can be rewritten into:

\[ \mathcal{V} - \hat{\mathcal{V}} = \Phi_{-1}^{-1} (u - \hat{u}) \]  

(A1.1.3)

from which Eq.2.8 is implied.

rewrite A1.1.2 into

\[ \hat{u} = \Phi_{-1} (z_1 - \hat{\mathcal{V}}) \]  

(A1.1.4)

Therefore we can proceed first to estimate \( \hat{\mathcal{V}} \), then use A1.1.4 to get \( \hat{u} \).

To get \( \hat{\mathcal{V}} \), let's first write the filtering problem into
standard form of Thm.A2.2:
state variable dynamics (from 2.5):
\[ dv = \sigma_v \delta b_v \]  \hspace{1cm} (A1.1.5)

observations
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{w}
\end{bmatrix} =
\begin{bmatrix}
\phi_u & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\eta \\
\omega
\end{bmatrix} dt
+ \begin{bmatrix}
\phi_u \sigma_u \\
\sigma_u \sigma_v
\end{bmatrix}
dv + \begin{bmatrix}
\phi_m^{-1} \sigma_u \\
\sigma_v
\end{bmatrix}
\delta b_u + \sigma_v \delta b_v 
\]  \hspace{1cm} (A1.1.6)

The first row of the observations above comes from substituting Eq.2.3 and 2.5 into A1.1.1:
\[ d\hat{z}_1 = dp + \phi_m^{-1} \sum_{n=1}^{\infty} \hat{f}_n = dv + \phi_m^{-1} du \]  \hspace{1cm} (A1.1.7)

\[ = [-\phi_u (v + \phi_m^{-1} u) + \phi_u v] dt + \phi_m^{-1} \sigma_u \delta b_u + \sigma_v \delta b_v \]

The second row is Eq.2.6.

Before applying Thm.A2.2, it is worth noting that the filtering problem is simplified because:
(i) The parameters are independent of time.
(ii) The initial condition starts at \(-\infty\).
So, it is the steady state solution we are looking for, i.e., only the algebraic Riccati equation rather than the differential form that needs to be solved.

Plug A1.1.5 and A1.1.6 into Thm A2.2, we get Eq 2.10. Solve the Riccati equation:
\[
\Omega_v = \frac{\sigma_u^2}{2 \phi_u \phi_m^2} \frac{2}{1 + \sqrt{1 + \frac{\phi_m^{-2} \sigma_u^2}{\sigma_v^2} + \frac{\phi_m^{-2} \sigma_u^2 (\phi_m^{-2} \sigma_u^2 + \sigma_v^2)}{\phi_u^2 \sigma_m^2}}}  \hspace{1cm} (A1.1.8)
\]
combine Eq.A1.1.8 with, Eq.2.7a and 2.7b, we get Eq 2.9. □

A1.2 Proof of the solution of Model 1

To prove that Eq.2.13 is the Nash equilibrium, we need to prove that from each player's point of view, if other players keep the policy, it is optimal for this player to keep the policy.

Without losing generality, let's consider the problem from the point of view of player N.

From the analysis of Lu (1991), we could ignore the inventory cost term in the objective function with the understanding that the solution we will get should be corrected by an infinitesimal inventory correction term "(0+)\sigma" to avoid the inventory drifting to infinity.

First, let's rewrite Eq.2.1 using Thm.A2.4. Because \( \hat{\nu} \) in Prop.2.2 is a martingale adapted to the information set \( \mathcal{F}_t \), from A2.4.4 of Thm.2.4:

\[
J_N = \max_{f(\cdot, \mathcal{F})} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( g_N dp - \delta f_N^2 \right) dt \\
= \max_{f(\cdot, \mathcal{F})} \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( (p - \hat{\nu}) f - \delta f_N^2 \right) dt + g(t) [p(t) - \nu(t)] \bigg|_0^T \right) \\
= \max_{f(\cdot, \mathcal{F})} \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( (p - \hat{\nu}) f - \delta f_N^2 \right) dt \right) 
\]  

(A1.2.1)

The last equality comes from that \( g(t)[p(t) - \nu(t)] \) is bounded because \( p(t) \) is not far from \( \nu(t) \) and \( g(t) \) is mean reverting to 0. The intuition of this is quite simple, if only the average profit flow is counted and the horizon of the speculator is infinite, the contribution of initial and final condition on the objective
function is infinitely small.

Note the only state variable influenced by $f_n$ is $g_n$. When inventory cost is very small, as assumed in Model 1, the maximization of Eq.A1.2.1 is equivalent to the maximization of the integrant of Eq. A1.2.1 at each moment (See Lu (1991)) for detailed analysis), i.e:

$$\max_{f(\cdot, H)} E[(p - \vartheta) f_N - \delta f_N^2]$$  \hspace{1cm} (A1.2.2)

substitute the first $N-1$ players' ordering policies (Eq. 2.13) into Eq.2.4.

$$\phi_m (p - \vartheta) + \alpha (N-1) (p - \vartheta) + \tilde{f}_N = u$$  \hspace{1cm} (A1.2.3)

take expectation $E(\cdot \mid H)$ on both side of Eq.A1.2.3

$$[\varphi_m + \alpha (N-1)] (p - \vartheta) + \tilde{f}_N = \tilde{\Omega}$$ \hspace{1cm} (A1.2.4)

Eq.A1.2.2 and A1.2.4 are in standard form of Thm.A2.1. Apply Thm.A2.1:

$$f_N^* = (2\delta + [(N-1) \alpha + \varphi_m]^{-1})^{-1} (p - \vartheta)$$

which is Eq.2.13 and 2.14. And

$$J_N(t) = \frac{1}{4[\varphi_m + (N-1) \alpha]^2[\varphi_m + (N-1) \alpha]^{-1} + \delta]} E(\tilde{\Omega}^2)$$ \hspace{1cm} (A1.2.5)

From Thm.A2.3

$$E(\tilde{\Omega}^2) = V_u - \Omega_u = V_u (1 - \frac{\Omega_u}{V_u})$$ \hspace{1cm} (A1.2.6)

Substituting Pro.2.1 into Eq.A1.2.6, we get:
\[ E(\hat{U}^2) = \frac{\sigma_u^2}{2\varphi_u} \left[ 1 - \frac{\Omega_u}{V_u} \right] \quad (A1.2.7) \]

Combining A1.2.5 and A1.2.7, we get Eq.2.16 and 2.17. □

**A1.3 Proof of the solution of Model 2**

This proof is similar to the proof of the solution of model 1, therefore, we will only emphasize the difference.

First, let's rewrite Eq.3.1 using Thm.A2.4. Notice \( \hat{\nu}^{(n)}(t) \) are martingale adapted to the information set \( \mathcal{F}_t \), from A2.4.4 of Thm.2.4:

\[ J_n = \max_{f(\cdot, \mathcal{F}_n)} E \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ \left( p - \hat{\nu}^{(n)} \right) f_n - \frac{\delta f_n^2}{T} \right] dt \right) \quad n = 1, \ldots, N \quad (A1.3.1) \]

If no one can significantly influence \( \hat{\nu}(t) \) of others for a significant time, following the same argument as in Model 1, Eq.A1.3.1 is equivalent to maximization of the integrand, i.e.:

\[ \max_{f(\cdot, \mathcal{F})} E \left[ \left( p - \hat{\nu}^{(n)} \right) f_n - \delta f_n^2 \right] \quad (A1.3.2) \]

To prove that Eq.3.9 is a Nash equilibrium, we need prove that from each player's point of view, if other players keep the policy, it is optimal for this player to keep the policy. Let's look from the point of view of a generic player, say player \( n \). Substitute other \( N-1 \) players' ordering policies (Eq.3.9) into Eq.3.4

\[ \varphi_m(p - \nu) + \sum_{k=1}^{N} \alpha(p - \hat{\nu}^{(k)}) + f_n = u \quad (A1.3.3) \]

Take \( E(\cdot \mid \mathcal{F}_n) \) on both sides of Eq.A1.3.3
\[ \phi_m (\mathbf{p} - \mathbf{\varphi}^{(n)}) + \sum_{k=1}^{N} \alpha_k \left[ \mathbf{p} - E(\mathbf{\varphi}^{(k)} | \mathcal{F}_n) \right] + \mathbf{f}_n = \mathbf{u}^{(n)} \]

rewrite it into

\[ [\phi_m + (N-1) \alpha] (\mathbf{p} - \mathbf{\varphi}^{(n)}) + \mathbf{f}_n = \mathbf{u}^{(n)} + \sum_{k=1}^{N} \alpha \left[ E(\mathbf{\varphi}^{(k)} | \mathcal{F}_n) - \mathbf{\varphi}^{(n)} \right] \]  \hspace{1cm} (A1.3.4)

Eq. A1.3.2 and A1.3.4 are in standard form of Thm A2.1, apply Thm A2.1

\[ \mathbf{f}_n^* = (2\delta + [(N-1) \alpha + \phi_m]^{-1})^{-1} (\mathbf{p} - \mathbf{\varphi}^{(n)}) \]

which is consistent with Eq. 3.9 if Eq. 2.14 is noticed. And:

\[ J_n = \frac{1}{4 [\phi_m + (N-1) \alpha]^2 \left[ (\phi_m + (N-1) \alpha)^{-1} - \delta \right]} E\left( \mathbf{u}^{(n)} + \sum_{k=1}^{N} \alpha \left[ E(\mathbf{\varphi}^{(k)} | \mathcal{F}_n) - \mathbf{\varphi}^{(n)} \right] \right)^2 \]

which is equation Eq. 3.10, 3.11.

Substituting equilibrium trading policy, Eq. 3.9, into the market clearing law, Eq. 3.4, we get:

\[ \phi_m (\mathbf{p} - \mathbf{\varphi}) + \sum_{n=1}^{N} \alpha (\mathbf{p} - \mathbf{\varphi}^{(n)}) = \mathbf{u} \]

Rewriting it, we get Eq. 3.10. Taking standard deviation of both sides of Eq. 3.10, we get Eq. 3.11. \[ \blacksquare \]
Appendix 2 Some Theorems Used In Calculation

In this Appendix, I present some theorems which are used for the calculations of this paper.

I use small letters, \( x, u, \) etc. to represent vectors and capital letters \( A, B, C, \) etc. to represent matrix parameters. The dimension of vectors and matrices can be understood in context and will not be mentioned every time.

**Thm.A2.1** (Maximum profit flow with transaction cost)

Given the constrained optimization problem:

\[
J = \text{Max}(yf-\delta f^2) \\
\{ \begin{array}{c}
(f) \\
s.t. \quad \varphi y + f = u 
\end{array}
\]  

(A2.1.4) 

(A2.1.5)

the solution of which is:

optimal order flow is:

\[
f^* = (\varphi^{-1} + 2\delta)^{-1} y
\]  

(A2.1.1)

and

\[
J = \frac{1}{4\varphi^2(\varphi^{-1} + \delta)} u^2
\]  

(A2.1.2)

**Proof of representation 2**

From A2.1.5

\[
y = \varphi^{-1}(u - f)
\]  

(A2.1.6)

A2.1.6 \( \rightarrow \) A2.1.4

\[
J = \text{Max}_{f} [\varphi^{-1}(u - f) f - \delta f^2]
\]  

(A2.1.7)

which maximized at
\[ f^* = \frac{\varphi^{-1}}{2(\varphi^{-1} + \delta)} u \]  \hspace{1cm} (A2.1.8)

A2.1.8 \rightarrow A2.1.5

\[ \varphi y + \frac{\varphi^{-1}}{2(\varphi^{-1} + \delta)} u = u \]  \hspace{1cm} (A2.1.9)

or

\[ u = \varphi \frac{2(\varphi^{-1} + \delta)}{(\varphi^{-1} + 2\delta)} y \]  \hspace{1cm} (A2.1.10)

A2.1.10 \rightarrow A2.1.8

\[ f^* = (\varphi^{-1} + 2\delta)^{-1} y \]  \hspace{1cm} (A2.1.11)

which is A2.1.1.

To get \( J \), from A2.1.4

\[ J = y f^* - \delta f^* = [ (\varphi^{-1} + 2\delta)^{-1} f^* ] f^* - \delta f^* \]

\[ = (\varphi^{-1} + \delta) f^* = \frac{1}{4\varphi^2(\varphi^{-1} + \delta)} u^2 \]

which is A2.1.2

Thm.A2.2 (Kalman-Bucy Filter)

Given an \( u \) controlled system:

\[ dx = \left[ F_0(t) + F_1(t)x + F_2(t)z + F_3(t)\vartheta + G_x(t)u \right] dt + H(t)dw(t) \]

\[ x(t_0) \sim N(\mu_0, P_0) \]  \hspace{1cm} (A2.2.1)

with imperfect observation:
\[ dz = \left[ M_0(t) + M_1(t) \dot{x} + M_2(t) z + M_3(t) \dot{\varphi} + G_z(t) u \right] dt + dv(t) \] (A2.2.2)

where \( w(t), v(t) \) are Brownian Motion:
\[
\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \sim \mathcal{N} \begin{bmatrix} D(t) & L(t) \\ L^T(t) & R(t) \end{bmatrix}
\] (A2.2.3)

which are independent of \( x(t_0) \). Define information set at \( t \)
\[ \mathcal{F}_t = \{ \mathcal{X}_0, P_0 \text{ and } z(t), u(t), t_0 < t < t \} \] (A2.2.4)

Then, the conditional distribution of \( x \) given \( \mathcal{F}_t \) is:
\[ (x(t) | \mathcal{F}_t) \sim \text{Normal} [ \mathcal{X}(t), P(t) ] \] (A2.2.5)

where \( P(t) \) satisfies Riccati equation:
\[
\dot{P}(t) = F_1 P + PF_1^T + HDH^T - (PM_1^T + HL) R^{-1} (PM_1 + HL)^T + K(t) K(t)^T \quad P(t_0) = P_0
\] (A2.2.6)

and \( \hat{x}(t) \) is the solution of the following differential equation:
\[
\dot{\hat{x}} = \left[ F_0(t) + F_1(t) \hat{x} + F_2(t) z + G_x(t) u \right] dt
\]
\[
\begin{cases}
\{ \dot{z} - \left[ M_0(t) + M_1(t) \hat{x} + M_2(t) z + G_z(t) u \right] dt \\
\hat{x}(t_0) = x_0
\end{cases}
\] (A2.2.7)

where
\[ K(t) = (PM_1^T + HL) R^{-1} \] (A2.2.8)

and innovation process,
\[ dy = dz - \left[ M_0(t) + M(t) \hat{x} + M_2(t) z + G_z(t) u \right] dt \] (A2.2.9)
is a Brownian Motion adapted to the information set $\mathcal{F}_t$ with
\[ [y(t) | \mathcal{F}(t)] - BM[R(t)] \]  \hspace{1cm} (A2.2.10)

Remark:
This is a straightforward extension of standard representation of Kalman Filter. The reason to put $z, \hat{z}, u$ in both system and observation equations is to emphasize that all these variables are intertwined into the system because of the complexity of multi-agent control. The standard representation of Kalman Filter can be found in, for example, Jazwinski('70).

Thm.A2.3 (Orthogonal projection in Hilbert space)
Given the set up of Thm.A2.2
\[ \text{Var}(\hat{x}) = \text{Var}(x) - P \] \hspace{1cm} (A2.3.1)

Proof:
Without losing generality, consider the case where $E(x) = 0$.
\[ \text{Var}(x) = E(x^2) = E[E(x^2 | \mathcal{F}_t)] \] \hspace{1cm} (A2.3.2)
\[ = E(E[[(\hat{x} + (x-\hat{x}))^2 | \mathcal{F}_t]] \] 
\[ = E[E(\hat{x}^2 | \mathcal{F}_t)] + E[E((x-\hat{x})^2 | \mathcal{F}_t)] + \] 
\[ + 2E[E(\hat{x}(x-\hat{x}) | \mathcal{F}_t)] \] 
\[ = E(\hat{x}^2) + E[P] \] \hspace{1cm} (A2.3.2')
\[ = \text{Var}(\hat{x}) + P \] \hspace{1cm} (A2.3.2'')
which is Eq.A2.3.1.

Eq.A2.3.2 comes from theorem of iterative of expectations. Eq.A2.3.2 comes from Eq.A2.2.5 and orthogonality of \( \hat{x} \) and \( (x - \hat{x}) \) under \( \mathcal{F}_t \). And Eq.A2.3.3" comes from noting \( P \) is not a random variable. 

**Thm.A2.4 (Equivalent representation theorem)**

Assume

\[
dg = -fdt + rgdt
\]  

(A2.4.1)

the following four representations are equivalent

(1) discount profit flow representation:

\[
E\left( \int_{t_0}^{t_f} e^{-rt} g^T dp \right)
\]  

(A2.4.2)

(2) discount cash flow representation:

\[
E\left[ \int_{t_0}^{t_f} e^{-rf} \tau pdt + [e^{-rt} g^T(t)p(t)]^{t_f}_{t_0} \right]
\]  

(A2.4.3)

(3) discount potential profit flow representation:

\[
E\left( \int_{t_0}^{t_f} e^{-rt}(p - v) dt + [e^{-rt} g^T(t)[p(t) - v(t)]]^{t_f}_{t_0} \right)
\]  

(A2.4.4)

where \( v(t) \) is any martingale process.

(4) discount price deviation representation:

\[
E\left( \int_{t_0}^{t_f} e^{-rt} f^T (p - m) dt + [e^{-rt} g^T(t)[p(t) - m]]^{t_f}_{t_0} \right)
\]  

(A2.4.5)
where \( m \) is any constant.

**Proof:**

Because constant is an martingale, case(4) and case(2) (zero is a constant) are special cases of case(3). Now let's prove case(1) \( \iff \) case(3). Note if \( v \) is martingale, then

\[
E(\int_{t_0}^{t_f} e^{-rt} g \, dv) = 0 \quad \text{for any } g
\]  

(A2.4.6)

so

\[
E(\int_{t_0}^{t_f} e^{-rt} g \, dp) = E[\int_{t_0}^{t_f} e^{-rt} g \, d(p - v)]
\]

(A2.4.7)

\[
= E(\int_{t_0}^{t_f} f^r(p - v) \, dt + e^{-rt} g^r(t) [p(t) - v(t)]_{t_0}^{t_f})
\]

the last equality comes from integration by part and substitution of Eq.A2.4.1. \[ \square \]

**Thm.A2.5 (Solution of linear stochastic differential equations)**

Given a system:

\[
dx = F(t) x \, dt + E(t) \, dB(t)
\]

\[
\{ x(t) \sim \text{Normal}(m_0, Q_0)
\]  

(A2.5.1)

where \( B(t) \) is a vector Brownian Motion:

\[
B(t) \sim BM[Q(t)]
\]  

(A2.5.2)

Then, the conditional distribution of \( x(t) \) given \( (m_0, V_0) \) only is:

\[
(x(t) \mid (m_0, V_0)) \sim \text{Normal}(m, V)
\]
where m is the solution of:

\[
\begin{align*}
\dot{m} &= F(t)m \\
m(0) &= m_0
\end{align*}
\]  

(A2.5.3)

and V is the solution of:

\[
\begin{align*}
\dot{V} &= FV + VF^T + EE^T \\
V(0) &= V_0
\end{align*}
\]  

(A2.5.4)

Remark: This is a well established result of stochastic system theory. The reader can find a formal proof from, for example, Davis (1977).
Appendix 3  AGFP Networks of the Models in This Paper

This appendix collects the AGFP networks of the Models in this paper. Please refer to Lu (1990) for the explanations of the symbols.

Model 1  Homogeneous informed speculation oligopoly
Model 2  Heterogeneous informed speculation oligopoly
Reference


Davis . H. A., (1977) Linear Estimation and Stochastic Control


