RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

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Exotic Options
by Mark Rubinstein and Eric Reiner
Introduction

by

Mark Rubinstein and Eric Reiner

May 3, 1992

When one of us first read "Path-Dependent Options: Buy at the Low, Sell at the High" (by Goldman, Sozin and Gatto) in 1979, he doubted (as the authors must also have doubted) that anyone would ever take a serious practical interest in the forms of options they were inventing. Their proposed options were designed to meet every investor’s dream: to buy (sell) a security at its bottom (top) in a tour-de-force of perfect market timing. While the proposed options sounded cute and did exemplify some interesting theoretical problems in options valuation, surely they were not to be taken seriously by men and women of practical affairs.

How different the world is ten years later. Today, these options are known as "lookbacks" and have a very active over-the-counter market. This is just one example of the extraordinary development of markets for highly customized options. One recent advertisement promised to supply markets in "min-max-zeros, range forwards, cylinder options, reverse forward options, quantos, zero cost collars, compound options, targets, scouts, flying hedges, moon rockets, the almost impossible to understand option," etc., and ended with the promise, "We’ll write it. You name it."

This series of essays proposes to survey the wide variety of these "exotic options," briefly describing what kinds of exotics exist, how they can be used for hedging or speculative purposes, tricks used to create valuation techniques, unusual valuation and hedging implications, and examples of option values and sensitivities to the factors that determine these values. Each essay provides both a concise summary of the published literature as well as some new results.

Each essay has been written so that it can be read independently of the others. In that way, if the reader is interested in a particular type of exotic option, he can read only about that type without concerning himself about the others. Despite this, readers interested in chooser options may find it helpful to read first about compound options, and readers interested in binary options should consider first reading about barrier options.

Unless otherwise stated, all valuation results are embedded in an economic context which is consistent with that required for the Black-Scholes formula for valuing standard options. The first essay on standard options provides a brief review of option pricing theory for standard options. Results in this essay are derived in more detail than subsequent essays to provide an introduction to the mathematical techniques used in the subsequent essays. Readers with little background in modern option pricing theory should read this essay first.

The exotics surveyed here fall conveniently into eleven classes:

(1) packages

options which are equivalent to a portfolio containing only standard European calls and possibly cash and the underlying asset itself

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1 Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates, and Eric Reiner is a vice-president of Leland O'Brien Rubinstein Associates.
(2) *forward-start options*  
options which are paid for in the present but which are only received at a prespecified future date

(3) *compound options*  
options whose underlying assets are themselves options

(4) *chooser options*  
options which are paid for in the present but which at some prespecified future date are chosen to be either a put or a call

(5) *barrier options*  
options whose payoff depends not only on the price at expiration of the underlying asset but also on whether or not the underlying asset has previously reached some other "barrier" price

(6) *binary options*  
options with binary and discontinuous payoff patterns

(7) *lookback options*  
options whose payoff depends not only possibly on the price at expiration of the underlying asset but also on the minimum or maximum price experienced by the underlying asset during at least some portion of the life of the option

(8) *Asian* options  
options whose payoff depends not only possibly on the price at expiration of the underlying asset but also on the average price experienced by the underlying asset during at least some portion of the life of the option

(9) *exchange options*  
options to exchange one asset for another which can be exercised early

(10) *currency-translated options*  
options whose underlying asset or striking price is denominated in a foreign currency at a random or prespecified exchange rate

(11) *two-color "rainbow* options*  
options on two risky assets which cannot be interpreted as if they were options on a single underlying asset

* versions of these essays have appeared in RISK
### Notation

- $d$ = one plus annualized discrete payout rate of an underlying asset (one plus annualized discrete foreign interest rate in a foreign exchange context)
- $h$ = annualized sampling frequency (only needed for average price/strike options)
- $k$ = dollar striking price of a compound option (only needed for compound options)
- $n$ = number of steps in a binomial tree
- $r$ = one plus annualized discrete interest rate
- $t$ = time-to-expiration of an option in years
- $t_0$ = time before averaging period begins in years (only needed for average price/strike options)
- $t_1$ = time since averaging period begun in years (only needed for average price/strike options)
- $t_2$ = time remaining for averaging in years (only needed for average price/strike options)
- $A$ = geometric dollar average of underlying asset price experienced so far during the averaging period (only needed for average price/strike options)
- $E$ = expectation operator
- $H$ = dollar knock-out or knock-in boundary (needed only for barrier options)
- $K$ = dollar striking price of an option
- $M$ = current dollar minimum or maximum price of underlying asset experienced so far during the life of an option (only needed for lookback options)
- $N[.]$ = area under the standard normal distribution function
- $N_2[.]$ = area under the standard bivariate normal distribution function
- $PV[.]$ = present value after time $t$ of quantity in brackets
- $R$ = dollar rebate (only needed for barrier options)
- $S$ = current dollar price of an underlying asset
- $S'$ = dollar price of an underlying asset at the expiration time of an option
- $T$ = time-to-expiration of underlying option in years (needed only for compound options)
- $X$ = predetermined payoff from a all-or-nothing option
- $\alpha$ = upside capture (only needed for portfolio insurance and generalized payoffs)
- $\beta$ = upside gain (only needed for portfolio insurance)
- $\lambda$ = slope of payoff pattern (only needed for generalized payoffs)
- $\rho$ = correlation of natural logarithms of one plus rates of return of two underlying assets (only needed for options on two underlying assets)
- $\sigma$ = volatility of an underlying asset (annualized standard deviation of the natural logarithm of one plus its rate of return)
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Exotic Options Classification

(1) Packages: options which are equivalent to a portfolio containing only standard European calls and possibly cash and the underlying asset itself

- collar
  \[ \min \{ \max(S', K_1), K_2 \} \text{ where } 0 < K_1 < K_2 \]
- zero-cost options:
  - break forward [Boston option, cancelable forward]
    \[(F - K) + \max(0, S' - F) \text{ where } F < K \text{ and } K \text{ set such that } C = 0 \]
  - range forward [zero-cost collar, flexible forward, cylinder option, option fence]
    \[(S' - F) + \max(0, K_1 - S') \cdot \max(0, S' - K_2) \text{ where } K_1 < F < K_2 \text{ set such that } C = 0 \]
- portfolio insurance:
  - upside capture portfolio insurance
    \[ \max(K, \alpha S') \text{ where } 0 < \alpha < 1 \]
  - upside gain portfolio insurance
    \[ \max(K, S + \beta(S' - S)) \text{ where } 0 < \beta < 1 \]
- simple choosers:
  - buyer’s choice
    \[ \max[C(S_i, K, T-t), P(S_i, K, T-t); t] \]
  - dealer’s choice
    \[ \min[C(S_i, K, T-t), P(S_i, K, T-t); t] \]

(2) Forward-start options: options which are paid for in the present but which are only received at a prespecified future date

- employee stock option
  \[ \max[0, S_t - \alpha S_i] \text{ where } \alpha > 0 \text{ and } t < T \]

(3) Compound options: options whose underlying assets are themselves options

- ordinary compounds:
  - call/put on a call
    \[ \max(0, C(S_i, K_2, T-t) - K_i; t) \]
  - call/put on a put
    \[ \max(0, P(S_i, K_2, T-t) - K_i; t) \]
- complex choosers (strikes or expiration dates not equal):
  - buyer’s choice
    \[ \max[C(S_i, K_1, T_1-t), P(S_i, K_2, T_2-t); t] \]
  - dealer’s choice
    \[ \min[C(S_i, K_1, T_1-t), P(S_i, K_2, T_2-t); t] \]
- extendible calls/puts:
  - buyer’s choice
    \[ \max(0, (S_i - K_i) \cdot C(S_i, K_2, T-t) - X; t) \]
  - writer’s choice
    \[ \max((S_i - K_i) \cdot (S_i \geq K_1) + C(S_i, K_2, T-t) \cdot (S_i < K_1); t) \]

(4) Dual-date payoff options: options with alternative dates to determine payoffs

- cliquet call/put (one click)
call: max[0, (S_i - K), (S_T - K)] where t < T is preset
☐ shout call/put
call: max[0, (S_s - K), (S_T - K)] where s < T is set by buyer on date s

(5) **Power options:** options whose payoff depends on the underlying asset price raised to a power

☐ power call/put
call: max[0, (S^λ - K)] where λ > 0
☐ self-quanto call/put [turbo option]
call: S^max[0, S^λ - K]

(6) **Barrier options:** options whose payoff depends not only on the underlying asset price at expiration but also on whether or not the underlying asset has previously reached some other "barrier" price [inside barrier options]

☐ down-and-out or up-and-out call/put (with/without rebate) [knock-out option, drop-out option]
  down-and-out call: S > H: max[0, S - K] if for all t < T, S_t > H
☐ down-and-in or up-and-in call/put (with/without rebate) [knock-in option, birth option]
  down-and-in call: S > H: max[0, S - K] if for some t < T, S_t < H
☐ cap call (up-and-out call where rebate = barrier - strike)
  S < H: max[0, S - K] if for all t < T, S_t < H where R = H - K
☐ cap put (down-and-out put where rebate = strike - barrier)
  S > H: max[0, K - S] if for all t < T, S_t > H where R = K - H
☐ double barrier call/put (in/out) [double knock-out, double knock-in, corridor option, range option]
  out-call: H_1 < S < H_2: max[0, S - K] if for all t < T, H_1 < S < H_2

(7) **Binary options:** options with binary or discontinuous payoff patterns [digital options, bet options]

☐ cash-or-nothing or call/put
  call: 0 if S ≤ K or X if S > K
☐ asset-or-nothing call/put
  call: 0 if S ≤ K or S - K if S > K
☐ gap options:
  • general gap call/put
    call: 0 if S ≤ K or S - K if S > K
  • pay-later call/put [contingent premium option]
    call: 0 if S ≤ K or S - K - X if S > K where X > 0 is set such that C = 0
  • money-back call/put [contingent premium option]
    call: 0 if S ≤ K or S - K - C if S > K
☐ range binaries:
  • cash-or-nothing option
    X if and only if K_1 ≤ S ≤ K_2
  • asset-or-nothing option [supershare]
    S/K if and only if K_1 ≤ S ≤ K_2
☐ barrier-dependent cash- or asset-or-nothing call/put

(8) **Lookback options:** options whose payoff depends not only possibly on the underlying asset price at expiration but also on the minimum or maximum price experienced by the underlying asset during at least
some portion of the life of the option

- floating-strike lookbacks:
  - call with minimum as strike \([\text{buy-at-the-low option}]\)
    \[ S' - \text{MIN}' \]
  - put with maximum as strike \([\text{sell-at-the-high option}]\)
    \[ \text{MAX}' - S' \]
- fixed-strike lookbacks:
  - call with maximum as asset
    \[ \max[0, \text{MAX}' - K] \]
  - put with minimum as asset
    \[ \max[0, K - \text{MIN}'] \]
- lookback straddle \([\text{range option}]\)
  \[ \text{MAX}' - \text{MIN}' \]
- lookback spread call/put \([\text{double lookback}]\)
  call: \[ \max[0, \text{MAX}' - \text{MIN}'] - K \]
- partial lookback call/put
  \[ \max[0, S' - \alpha \text{MIN}'] \ (\alpha > 1) \quad \text{or} \quad \max[0, \alpha \text{MAX}' - S'] \ (\alpha < 1) \]
- ladder call/put
  call: \[ \max[0, \max\{\min(S_1, \text{MAX}'), \min(S_2, \text{MAX}'), \min(S_3, \text{MAX}'), \ldots\} - K] \quad \text{where} \quad S_1 < S_2 < S_3 < \ldots \]
- do-nothing call/put
  call: \[ \max[0, \alpha S'/(\text{MAX}' - \text{MIN}'+ X)] - K \]

(9) "Asian" options: options whose payoff depends not only possibly on the price at expiration of the underlying asset but also on the average price experienced by the underlying asset during at least some portion of the life of the option

- arithmetic or geometric average price call/put \([\text{average rate option}]\)
  \[ \max[0, A' - K] \quad \text{or} \quad \max[0, K - A'] \]
- arithmetic or geometric average strike call/put
  \[ \max[0, S' - A'] \quad \text{or} \quad \max[0, A' - S'] \]

(10) Exchange options: options to exchange one asset for another

- option to exchange one asset for another
  \[ \max[0, S_2 - S_1] \]
- option on the better of two assets \([\text{out-performance option}]\)
  \[ \max[S_1', S_2'] \]
- option on the worse of two assets \([\text{under-performance option}]\)
  \[ \min[S_1', S_2'] \]
- composite calls/puts:
  - ratio call/put \([\text{relative performance option}]\)
    call: \[ \max[0, S_2' / S_1' - K] \]
  - product call/put
    call: \[ \max[0, S_1' S_2' - K] \]

(11) Currency-translated options: options whose underlying asset or striking price is denominated in a foreign currency but translated back to the domestic currency at a random or prespecified exchange rate

- foreign equity call/put struck in foreign currency
  call: \[ \max[0, S' X' - K X] \]
- foreign equity call/put struck in domestic currency
(12) Two-color "rainbow" options: options on two risky assets which cannot be interpreted as if they were options on a single underlying asset

- Option delivering better of two risky assets and cash
  \[ \max[S_1, S_2, K] \]

- Call or put on maximum or minimum of two risky assets
  \[ \max[0, \max(S_1, S_2)] - K \]

- Spread call/put
  \[ \max[0, (S_2^* - S_1^*) - K] \]

- Portfolio call/put [basket option]
  \[ \max[0, (n_1 S_1^* + n_2 S_2^*)] - K \]

- Dual-strike call/put
  \[ \max[0, S_1^* - K_1, S_2^* - K_2] \] where \( K_1 \neq K_2 \)

- Dual-asset choosers (call-call, call-put, put-call, put-put):
  - Buyer's choice
    \[ \max[C(S_1, K, T-t), C(S_2, K, T-t); t] \]
  - Dealer's choice
    \[ \min[C(S_1, K, T-t), C(S_2, K, T-t); t] \]

- Defined exercise call/put (exercise permitted only if condition on second asset is satisfied)
  \[ \max[0, (S_1^* - K) \times (S_2 > X)] \]

- Dual-asset barriers options: [dynamite warrants]
  - Outside barrier up/down-in/out-call/put (barrier determined by second asset price)
    \[ \max[0, \max(S_1^*, S_2^*) - K] \] if for all \( t \leq T \), \( S_2 > H \)
  - Dual-asset, double barrier up/down-in/out-call/put
    \[ \max[0, \max(S_1^*, S_2^*) - K] \] if for all \( t \leq T \), \( S_1 > H_1 \) and \( S_2 > H_2 \)

- Dual-asset binary option (binary where condition for positive cash flow depends on second asset)
  \[ \text{asset-or-nothing call: } 0 \text{ if } S_2 < K \text{ or } S_1 > K \text{ if } S_2 > K \]

- Dual-asset lookbacks:
  - Call/put on the maximum
    \[ \max[0, (\max(S_2^* - \max(S_1^* - K)] \text{ or } \max[0, K - (\max(S_2^* - \max(S_1^* - K)] \]
  - Call/put on the minimum
    \[ \max[0, (\min(S_2^* - \min(S_1^* - K)] \text{ or } \max[0, K - (\min(S_2^* - \min(S_1^* - K)] \]
  - Semi-dual-asset lookback call/put
    \[ \max[0, \max(S_2^* - S_1^*)] \text{ or } \max[0, S_1^* - \min(S_2^*)] \]

(13) Multi-color "rainbow" options: options on many risky assets which cannot be interpreted as if they were options on one or two underlying assets

- Portfolio call/put [basket option]
  \[ \max[0, (n_1 S_1^* + n_2 S_2^* + n_3 S_3^* + \ldots + n_m S_m^*) - K] \]
Notation

\( A^* \) = geometric or arithmetic average underlying asset price through expiration date

\( C \) (\( P \)) = current value of call (put)

\( F \) = forward price of underlying asset for delivery on option expiration date

\( H \) = barrier price of underlying asset

\( H_1, (H_2) \) = barrier price of underlying asset 1 (2)

\( K \) = option striking price

\( K_1, (K_2) \) = first (second) striking price

\( K' \) = striking price denominated in foreign currency

\( S \) = current underlying asset price

\( S_1, (S_2) \) = current underlying asset price of asset 1 (2)

\( S_t, (S_T) \) = underlying asset price after elapsed time \( t \) (\( T \))

\( S' \) = underlying asset price on option expiration date

\( S_1', (S_2') \) = underlying asset price of asset 1 (2) on option expiration date

\( S'' \) = underlying asset price on option expiration date denominated in foreign currency

\( X \) = constant

\( X_o \) = current exchange rate (domestic/foreign)

\( X' \) = exchange rate on option expiration date (domestic/foreign)

\( n_1, (n_2) \) = number of shares held in underlying asset 1 (2)

\( t, T \) = elapsed time from current date to some date in the future

\( T_1, (T_2) \) = first (second) time-to-expiration

\( \alpha, \beta \) = multiplicative constants

\( \lambda \) = exponential constant

\( \text{MAX}' \) = maximum underlying asset price through expiration date

\( \text{MIN}' \) = minimum underlying asset price through expiration date
Standard Options

by
Mark Rubinstein
November 25, 1995

The payoff at expiration of a standard option can be described mathematically as:

\[ \max[0, \phi S^* - \phi K] \]

where \( K \) is the option striking price, \( S^* \) is the price of the underlying asset at the expiration date of the option, and \( \phi \) is a binary variable set equal to 1 for a call and -1 for a put. The "max" notation symbolizes that the payoff will be one of the two quantities in the brackets, whichever is larger. If the payoff can occur only at expiration immediately after elapsed time \( t \), the option is said to be "European." If the payoff

\[ \max[0, \phi S(\tau) - \phi K] \quad \text{for all} \quad 0 \leq \tau \leq t \]

can additionally be realized at any earlier time \( \tau \), then the option is instead said to be "American." The act of realizing a positive payoff is termed "exercising" the option, and is solely at the discretion of the buyer. It will always be assumed here that throughout the life of the option the buyer exercises the option so as to maximize its present value (at \( \tau = 0 \)).

Our principal objective will be to find ways to measure this present value denoted as \( C \). Occasionally, to distinguish a put, \( P \) will be used for its present value.

We will always assume that the options, their underlying assets and a riskless security trade in a "perfect" financial market -- one of no transactions frictions of any kind, including no penalties to short selling and no commissions and taxes, and one in which there are no riskless opportunities to make arbitrage profits.

In this book, we will use several different techniques to value exotic options, typically choosing the one that seems most convenient at the time. Here are some of the approaches we will use:

- restate the option payoff as a portfolio of standard call options
- work backwards recursively from the expiration date using a binomial tree
- risk-neutral discounting of the expiration date payoff with an explicit solution
- reinterpret the input variables in a binomial tree or in the Black-Scholes formula
- risk-neutral discounting of the expiration date payoff with numerical integration
- monte-carlo simulation

In this chapter, we will illustrate each of these to derive the value and hedging parameters of standard options.

I. Restate the Option Payoff as a Portfolio of Standard Call Options

In a market with no transactions frictions and no riskless arbitrage opportunities, the present value of an option of interest can often be deduced from the present value of a portfolio of standard call options by

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2 This assumption explains why we have written the payoff of the option as we have.
showing that the two have identical payoffs in every situation. The most fundamental relation of this sort is known as “put-call parity.” Observe that the payoff of a standard European put is equal to the payoff of an otherwise identical call, a short position in the underlying asset and a loan. Stating this precisely,

$$\max[0, K - S^*] = \max[0, S^* - K] = S^* + K$$

To prove this, in the two relevant possibilities, $S^* \leq K$ and $S^* > K$, the above equation becomes:

- If $S^* \leq K$:
  $$K - S^* = 0 - S^* + K$$

- If $S^* > K$:
  $$0 = S^* - K - S^* + K$$

Therefore, the present value of an otherwise identical put $P = C - Sd^1 + Kr^1$, where $S$ is the current price of the underlying asset, $r$ is one plus the riskless rate of interest, and $d$ is one plus the payout rate of the underlying asset. The portfolio contains one otherwise identical call, $d^1$ units of the underlying asset, and a loan of $Kr^1$ dollars.³

II. Work Backwards Recursively from the Expiration Date Using a Binomial Tree⁴

The binomial approach to option pricing grew out of a conversation between Nobel laureate William Sharpe and one of the authors that occurred at a conference in 1975 held in Ein Bokek, Israel, bordering the Dead Sea. With nothing to do during the breaks (except to take a dip in the sea), the 20 or so attending financial economists, including Harry Markowitz and Merton Miller who recently shared Sharpe’s Nobel prize, had little to do but discuss provocative questions in financial theory. At one point, we wondered how it was that the then two-year old Black-Scholes approach to valuing options could recreate a riskless payoff using only the option and its underlying asset. It was then that Sharpe said, “I wonder if it’s really that there are only two states of the world, but three securities, so that any one of the securities can be replicated by the other two.” We immediately thought that insight coupled with our intuitive understanding of the central limit theorem of probability theory could potentially lead to the Black-Scholes formula as the limiting form of successive two-state movements of the underlying asset price. Subsequently, Sharpe used this insight in his textbook, Investments (Prentice-Hall), to explain modern option pricing theory. John Cox, Stephen Ross, and Mark Rubinstein worked out the model in detail in their article, “Option Pricing: A Simplified Approach,” which was published at the invitation of the editor, Michael Jensen, in the Journal of Financial Economics in September of 1979.

A simple approach to valuing standard options is to suppose that the underlying asset price follows a binomial random process over time. For an option expiring at the end of two periods, assume that the price

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³ The present value of $S^*$ is not $S$, but $Sd^1$, to account for the fact that, like exchange-traded options, the call and put are not protected against payouts.

of its underlying asset moves binomially each period as follows:

\[
\begin{align*}
S & \quad uuS & \Rightarrow C_{uu} = \max(0, \phi uuS - \phi K) \\
& \quad udS & \Rightarrow C_{ud} = \max(0, \phi udS - \phi K) \\
S & \quad duS & \Rightarrow C_{du} = \max(0, \phi duS - \phi K) \\
& \quad ddS & \Rightarrow C_{dd} = \max(0, \phi ddS - \phi K)
\end{align*}
\]

where \( u \) and \( d \) are the one plus the possible rates of growth of the underlying asset price over each binomial period after payouts. This setup leads us to think of the stock process of a multiplicative random walk. Of course, the price paths \( udS \) and \( duS \) will end up at the same point, but our subsequent argument will be clearer if we temporarily keep these two paths distinct.

The terms \( C_{uu}, C_{ud}, C_{du}, \) and \( C_{dd} \) represent the value of the option if held on its expiration date, corresponding to each of the four possible paths.

For example, suppose \( S = K = 100, \phi = 1, \ u = 1.2 \) and \( d = 1/u = .8333... \). After the path \( uu \), if the call were held to expiration, it would finish in-the-money and be worth \( \max(0, (1.2^2 \times 100) - 100) = 44 \); while after path \( ud \), the call would finish at-the-money and be worth nothing.

While it is quite easy to write down a formula for the expiration value of the option, it is more difficult to derive a formula for its value prior to maturity. Taking the next simplest situation, suppose we were at the beginning of the second period after the underlying asset price had just moved from \( S \) to \( uS \). From this point, we will try to replicate the payoff from the option if held one more period by just holding the underlying asset and cash.

Let \( \Delta_u \) and \( B_u \) be the number of units of underlying asset and the dollar investment in cash, respectively, held in the replicating portfolio during the second period. Then,

\[
\begin{align*}
C_{uu} &= uu \delta \Delta_u + rB_u \\
C_{ud} &= ud \delta \Delta_u + rB_u
\end{align*}
\]

where \( \delta \) is one plus the payout rate of the underlying asset and \( r \) is one plus the riskless interest return earned on cash over each binomial period. Since we can interpret the conditions above as two independent equations in two unknowns \( (\Delta_u, B_u) \), it is possible to choose \( \Delta_u \) and \( B_u \) to replicate the payoff from the option if held to expiration.

Solving these simultaneous equations for \( \Delta_u \) and \( B_u \):

\[
\begin{align*}
\Delta_u &= (C_{uu} - C_{ud})/[(u-d)\delta S] \quad (1) \\
B_u &= (uC_{ud} - dC_{uu})/(u-d)r \quad (2)
\end{align*}
\]
Consistent with our assumption of no riskless arbitrage, if two investments made at the same date have identical values one period later, then they must have same cost. Therefore, the unexercised value of the option at the beginning of the second period must be equal to the cost at that time of constructing the replicating asset/cash portfolio:

\[ uS\Delta_u + rB_u \]

Considering the potential of early exercise at the beginning of the second period, the value \( C_u \) of the option at that time must then be:

\[ C_u = \max\{\phi uS - \phi K, (uS\Delta_u + rB_u)\} \tag{3} \]

since the buyer will exercise the option and receive \( \phi uS - \phi K \) if this exercisable value exceeds the holding value \( uS\Delta_u + rB_u \).

Using equations (1) and (2) to substitute for \( \Delta_u \) and \( B_u \), a little algebra shows that (3) can be rewritten as:

\[ C_u = \max\{\phi uS - \phi K, [pC_{u reports} + (1-p)C_{u discount}] / r\} \quad \text{where} \quad p = \frac{(r/\delta) - d}{u - d} \tag{4} \]

Similarly, when the underlying asset falls during the first period:

\[ C_u = \max\{\phi dS - \phi K, [pC_{d reports} + (1-p)C_{d discount}] / r\} \tag{5} \]

To value the option at the beginning of the first period, again try to replicate the value of the option one period later with a position in the asset and cash by choosing \( \Delta \) units of the asset and \( B \) dollars of cash such that:

\[ C_u = u\delta S\Delta + rB \]
\[ C_d = d\delta S\Delta + rB \]

As before, solving these equations now for \( \Delta \) and \( B \):

\[ \Delta = (C_u - C_d) / [(u-d)\delta S] \tag{6} \]
\[ B = (uC_u - dC_d) / (u-d)r \tag{7} \]

Considering again the potential of early exercise now at the beginning of the first period, the value \( C \) of the option at this time must be:

\[ C = \max\{\phi S - \phi K, [S\Delta + rB]\} \tag{8} \]

Using equations (6) and (7) to substitute for \( \Delta \) and \( B \) in (8):

\[ C = \max\{[\phi S - \phi K], [pC_u + (1-p)C_d] / r\} \tag{9} \]

To summarize, we have valued a standard option by working backwards on the binomial tree of underlying asset prices. In particular, with two periods, first calculate \( C_{u reports}, C_{d reports}, C_{u discount}, C_{d discount} \), the possible values of the option at expiration, then work backwards and derive expressions for \( C_u \) and \( C_d \) (equations 4, 5), and then derive the equation for the current option value \( C \) (equation 9).

Having completed our formal argument, we now make several observations which will allow us to generalize or, under certain conditions, simplify the valuation procedure.
First, the pair of coefficients \((p, 1-p)\) which appears in (4) and (9) is a mathematical probability measure. To preclude riskless arbitrage using only the underlying asset and cash, we must have \(d < \frac{r}{\delta} < u\) which implies \(0 < p < 1\). Moreover, \(p\) is frequently termed a "risk neutral" probability. That is, it is the probability which causes the expected return of the underlying asset to equal the interest return, since the above equation for \(p\) can be rewritten:

\[ p \delta u + (1-p) \delta d = r. \]

Second, since the sizes of the up and down moves are constant throughout the tree, the payoff from all permutations of the same set of moves is the same. For example, the payoff from move sequence \(ud\) is the same as the payoff from move sequence \(du\). This results in substantial computational simplification since

\[
\begin{align*}
\text{uS} & \Rightarrow C_{uu} = \max[0, \phi uuS - \phi K] \\
\text{dS} & \Rightarrow C_{dd} = \max[0, \phi ddS - \phi K] \\
\text{us} & \Rightarrow C_{ud} = C_{du} = \max[0, \phi udS - \phi K]
\end{align*}
\]

Generalizing this to many periods, \(i = 0, 1, 2, 3, \ldots, n\), at the end of the last period this simplification reduces the number of nodes from \(2^n\) to \(n+1\). In this case, the binomial tree is said to be "recombining".

Third, if early exercise is either precluded (as in the case of European options) or never optimal\(^6\), then equations (4), (5), and (9) can be combined into a single simple formula for the current value of the option:

\[ C = \left[p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}\right]/\delta^2 \]

This permits a shortcut technique for valuing standard European options. For each possible path, (1) calculate the payoff at expiration, (2) weight this by the risk-neutral probability of the path, (3) add the resulting terms, and (4) discount this back to the present by the interest rate. This simplification, of course, is not available for an American option for which early exercise may be advisable.

Generalizing this to many periods, the binomial valuation formula for a call then becomes:

---

\(^6\) In our binomial framework, it is easy to show that if payouts are zero \((\delta = 1)\), then it will never be optimal to exercise a call option early.
\[
C = \frac{\sum_{j=0}^{n} c(j, n) p^j (1-p)^{n-j} \max[0, u^j d^{n-j} - K]}{r^n}
\]

where \( c(j, n) = \frac{n!}{j! (n-j)!} \)

We can simplify this further if we let \( a \) be the smallest nonnegative integer such that \( u^a d^{-a} > K \). Therefore, \( a \) is the smallest nonnegative integer greater than \( \log(K/Sd^a)/\log(u/d) \). Then, the above formula can be written:

\[
C = Sd^{-a} \sum_{j=a}^{n} c(j, n) \bar{p}^j (1-\bar{p})^{n-j} - Kr^{-a} \sum_{j=a}^{n} c(j, n) p^j (1-p)^{n-j}
\]

where \( \bar{p} = \frac{u}{r/S} \)

**Fourth.** Suppose in place of the binomial inputs \( u \) and \( d \), we start only with an estimate of the mean, \( \mu \), and the standard deviation, \( \sigma \), of the natural logarithm of one plus the annualized rate of return of the underlying asset. We can infer a consistent \( u \) and \( d \) which improves as \( n \) is increased. Let \( q \) be the subjective probability of a binomial up move. Then, to obtain consistency we choose \( u, d \) and \( q \) such that as \( n \to \infty \):

\[
E[\log(S'/S)] = Mn = [q(\log u) + (1-q)(\log d)]n = \mu t
\]

and

\[
\text{Var}[\log(S'/S)] = \left[q(\log u) - M\right]^2 + (1-q)[(\log d) - M]^2]n = \sigma^2 t
\]

One possibility, among others, is

\[
u = e^{\sqrt{t/n}} \quad d = 1/u \quad q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma) \sqrt{t/n}
\]

since with these choices

\[
E[\log(S'/S)] = \mu t \quad \text{for any } n
\]

\[
\text{Var}[\log(S'/S)] = (\sigma^2 - \mu^2 t/n)t \to \sigma^2 t \quad \text{as } n \to \infty
\]

Therefore, starting with an estimate of \( \sigma \) and given a tree size \( n \), we can calculate consistent values for \( u \) and \( d \) using the definitions in the above binomial option valuation formulas.

**Fifth.** For American options it is interesting to calculate the risk-neutral expected life of the option, known as the “fugit.” This can also be calculated by using a binomial tree:

1. set the fugit of all nodes at the end of the tree equal to \( i = n \)
2. work backwards recursively:

---

8 The concept and name of “fugit” was developed by Mark Garman in “Semper Tempus Fugit,” *RISK*, May 1989.
if the option should be exercised at a node, set the fugit at that node equal to its period \(t\)

if the option should not be exercised at a node, set the fugit to the risk-neutral expected fugit over the next period

(3) the number calculated in this fashion at the beginning of the first period \((i=0)\) is the current fugit

In other terms, let \(f\) be the fugit appropriately subscripted to indicate its nodal location. Let \(\xi\) be a binary variable also subscripted to indicate its associated node and set equal to 0 if the option is not to be exercised at that node and set equal to 1 if the option is to be exercised at that node. For example, if \(n=2\), work backwards as follows:

\[
\begin{align*}
  f_{uu} &= f_{ud} = f_{dd} = 2 \\
  f_u &= \xi_u 1 + (1-\xi_u)[pf_{uu} + (1-p)f_{ud}] \\
  f_d &= \xi_d 1 + (1-\xi_d)[pf_{ud} + (1-p)f_{dd}] \\
  f &= \xi 0 + (1-\xi)[pf_u + (1-p)f_d]
\end{align*}
\]

As a final touch, to annualize the fugit, multiply the resulting value for \(f\) by \(t/n\).

**Sixth**, the sensitivity of the current option value to the underlying asset price is known as the option's "delta." In binomial context, it is convenient to approximate the delta using finite difference methods. In particular the binomial delta equals:

\[
\Delta = \frac{(C_u - C_d)}{[(u-d)\delta S]}
\]

which will be recognized as the number of shares in the current replicating portfolio.

The sensitivity of the current delta to the underlying asset price is called the option's "gamma." This can also be approximated with finite differences as:

\[
\Gamma = \frac{(\Delta_u - \Delta_d)}{[(u-d)\delta S]}
\]

where \(\Delta_u = \frac{(C_{uu} - C_{ud})}{[u(u-d)\delta S]}\) and \(\Delta_d = \frac{(C_{ud} - C_{dd})}{[d(u-d)\delta S]}\)

The sensitivity of the current option value to a one day reduction in time-to-expiration is called the option's

\[\tag{7}\]

Ideally, delta and gamma should be measured by comparing different option values, given different possible underlying asset prices at the current time. The calculations given here suffer from the difficulty that the option values are assessed at a nearby but future times. This creates a bias in the calculations which, of course, can be made as small as desired by increasing \(n\), the number of binomial steps for a fixed time period. However, a more efficient approach to handle this problem is to start the tree exactly two steps before the present time. As a result, for the present time three option values will be determined in the tree: \(C_{uu}, C_{ud}\), and \(C_{dd}\) where \(C_{ud}\) is now interpreted as the current value of the option. Now calculate:

\[
\begin{align*}
  \Delta &= \frac{(C_{uu} - C_{dd})}{[(uu-dd)\delta S]} \\
  \Delta^+ &= \frac{(C_{uu} - C_{ud})}{[u(u-d)\delta S]} \quad \text{and} \quad \Delta^- = \frac{(C_{ud} - C_{dd})}{[d(u-d)\delta S]} \\
  \Gamma &= \frac{(\Delta^+ - \Delta^-)}{[(u-d)\delta S]} \\
  \Theta &= \frac{(C_{ud}-C)}{4t/n}
\end{align*}
\]
"theta." As calculated here, it will mean the dollar change in the option value $C$ created by reducing the time-to-expiration $t$ by one calendar day, other things equal. This also can be approximated using forward finite differences as:

$$
\theta = (C_{ud} - C) - (2t/n)
$$

To interpret this sensitivity as the change in the value of the option after a one calendar day reduction in the time-to-expiration, scale it by dividing by 365.

The sensitivity of the percentage change of the option value to the percentage change in the underlying asset price is called the option's "omega." This can be approximated by:

$$
\Omega = \Delta (S/C)
$$

Seventh, the sensitivity of the current option value to the interest rate is known as the option's "rho". As calculated here, it will measure the dollar change of the option value $C$ caused by a .01 change in one plus the discrete annualized rate of interest $r$, other things equal (e.g. interest rate changes from 10% to 11%). To be consistent with the single-period binomial interest return $r$, we must have:

$$
r = r^{1/n}
$$

To calculate rho, following a two-period example, roll backwards through the binomial tree as follows:

$$
C_u = \max\{[\phi u S - \phi K], [pC_{uu} + (1-p)C_{ud}] / r\}
$$

$$
C_d = \max\{[\phi d S - \phi K], [pC_{ud} + (1-p)C_{dd}] / r\}
$$

$$
C = \max\{[\phi S - \phi K], [pC_u + (1-p)C_d] / r\}
$$

At each step, pause to calculate rho:

$$
\frac{\partial C_u}{\partial r} = 0 \quad \text{if } C_u = \phi u S - \phi K \text{ (exercise early)}, \text{ otherwise }
$$

$$
\left[\frac{\partial (p/r) / \partial r}{\partial C_u} + \frac{\partial (1-p) / \partial r}{\partial C_{ud}}\right] C_{ud}
$$

$$
\frac{\partial C_d}{\partial r} = 0 \quad \text{if } C_d = \phi d S - \phi K \text{ (exercise early)}, \text{ otherwise }
$$

$$
\left[\frac{\partial (p/r) / \partial r}{\partial C_d} + \frac{\partial (1-p) / \partial r}{\partial C_{dd}}\right] C_{dd}
$$

$$
\frac{\partial C}{\partial r} = 0 \quad \text{if } C = \phi S - \phi K \text{ (exercise early)}, \text{ otherwise }
$$

$$
\left[p \left(\frac{\partial C_u}{\partial r} + (1-p) \frac{\partial C_d}{\partial r}\right)\right] / r + \left[p \frac{\partial (p/r)}{\partial r} C_u + \frac{\partial (1-p)/r}{\partial r} C_d\right]
$$

In this last expression, the derivatives $\partial C_u/\partial r$ and $\partial C_d/\partial r$ are taken from the calculations in the previous step. The derivatives $\partial (p/r)/\partial r$ and $\partial ((1-p)/r)/\partial r$ which appear in both of these steps, can be calculated.

---

8 This method for calculating rho (as well as vega) was suggested by Eric Reiner. As an alternative technique, one can use a direct finite difference approximation of these derivatives. For example, first calculate the option value $C$ for a given value of $r$. Then calculate the option value $C'$ using the interest return $r' = r + .0001$, other things equal. The derivative $\partial C/\partial r$ is then approximated by $(C' - C) / .0001$. By comparison, the method used in this essay is somewhat more accurate and mathematically more elegant since it is part of the same approach of rolling backwards through the binomial tree that is used to calculate option values. Moreover, this method has the advantage of providing intermediate measures of rho throughout the entire binomial tree, should these be of interest.
explicitly using the chain rule, remembering that \( p = ((r/\delta) \cdot d)/ (u-d) \):

\[
\frac{\partial (p/r)}{\partial \sigma} = \left[ \frac{\partial (p/r)}{\partial r} \times [\partial r/\partial \sigma] \right] \times \left( \frac{t/n}{\partial u/\partial \sigma} \right) \times r^{1-(n/\gamma)}
\]

\[
\frac{\partial ((1-p)/r)}{\partial \sigma} = \left[ \frac{\partial ((1-p)/r)}{\partial r} \times [\partial r/\partial \sigma] \right] \times \left( \frac{t/n}{\partial u/\partial \sigma} \right) \times r^{1-(n/\gamma)}
\]

Scaling for the magnitude of the sensitivity, rho equals .01 \( \times \partial C/\partial \sigma \).

**Eighth**, the sensitivity of the current option value to the volatility of the underlying asset is known as the option’s “vega.” As calculated here, it will measure the dollar change in the option value \( C \) caused by a .01 change in \( \sigma \), other things equal (e.g. the volatility changes from .20 to .21). Again our method will require rolling backwards through the binomial tree using the following recursive rules:

\[
\frac{\partial C(j;n)}{\partial \sigma} = \begin{cases} 
0 & \text{if } C(j;n) = 0 \text{ (do not exercise at expiration)} \\
\phi(2) - n/u d^n S v(t/n) & \text{if } C(j;n) = u/\sigma d^n S \text{ (exercise at expiration)} 
\end{cases}
\]

\[
\frac{\partial C(j;n-i)}{\partial \sigma} = \phi(2) + i - n/u d^{n-i} S v(t/n) \quad \text{if } C(j;n-i) = \phi/\sigma d^{n-i} S - \phi K, \text{ otherwise}
\]

\[
\left[ \frac{\partial (p/r)}{\partial \sigma} \right] C(j+1:n-i+1) - C(j;n-i+1) \right] + \left( p/r \right) \left[ \partial C(j+1:n-i+1) / \partial \sigma \right] + \left( (1-p)/r \right) \left[ \partial C(j+1:n-i+1) / \partial \sigma \right]
\]

The derivative \( \partial (p/r)/\partial \sigma \) which appears in this last expression, can be calculated explicitly using the chain rule, remembering that \( p = ((r/\delta) \cdot d)/ (u-d) \):

\[
\frac{\partial (p/r)}{\partial \sigma} = \left[ \frac{\partial (p/r)}{\partial u} \times [\partial u/\partial \sigma] \right] = \left[ 1/ r (u^2-1)^2 \right] \times \left( \partial (r/\delta)(1+u^2)/2 \right) \times u \sqrt{t/n}
\]

Scaling for the magnitude of the sensitivity, vega equals .01 \( \times \partial C/\partial \sigma \).

**III. Risk-Neutral Discounting of the Expiration Date Payoff with an Explicit Solution**

As we have seen, if early exercise is either precluded (as in the case of European options) or never optimal, then equations (4), (5), and (9) can be combined into a single simple formula for the current value of the option:

\[
C = \left[ p^2 C_u + 2p(1-p)C_{wu} + (1-p)^2 C_d \right] / r^2
\]

As the tree size \( n \) is increased, it has been shown that defining \( u \) and \( d \) in terms of the volatility of \( \sigma \), risk-neutral lognormal distribution and the binomial \( r \) in terms of the annualized \( r \) as given above, produces binomial option values which converge to Black-Scholes values. With these transformations, as the tree size is increased, \( u \) approaches 1 from above and \( d \) approaches 1 from below. Thus, we have the limiting effect of smaller and smaller possible asset price movements over a single binomial interval as the elapsed

---

9 If the option is European, the procedure for calculating the current rho can be simplified to:

\[
\frac{\partial C}{\partial \sigma} = n \times \left[ \left[ \frac{\partial (p/r)}{\partial \sigma} \right] C_u + \left[ \frac{\partial ((1-p)/r)}{\partial \sigma} \right] C_d \right]
\]

time for each interval diminishes (but a larger and larger number and size of possible price changes over intervals of fixed elapsed time). In the limit, over each infinitesimal interval, the asset price moves infinitesimally up or down, never staying constant.\footnote{For an intuitive description of this limiting process, see Edward Seidenverg, "A Case of Confused Identity," Financial Analysts Journal, July-August 1988.}

With the limiting substitutions discussed above, as $n \to \infty$, using a central limit theorem it can be shown that

$$\log(S'/S)$$

approaches a normally distributed random variable with volatility $\sigma$. As a result, the Black-Scholes options formula can be derived as the continuous time limit of the binomial formula. We can also derive the Black-Scholes formula directly using risk-neutral probabilities. To do this, we assume:

1. the underlying asset follows a lognormal random walk, and

2. arbitrage arguments allow us to use a risk-neutral valuation approach -- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

In this case, we now revise our notation using $d$ as one plus the annualized dividend yield of the underlying asset and as before $r$ as one plus the annualized interest rate. In that case,

$$C = x^{-t}E[\max(0, \phi SR - \phi K)]$$

where $E$ is an expectation operator and $R = S'/S$ is a lognormally distributed random variable with mean $(r/d)$. Writing this in integral notation:

$$C = x^{-t} \int_{\log(K/S)}^{1} (\phi SR - \phi K) h(R) dR$$

where $h(R)$ is the lognormal density function. Using a change of variables, we can rewrite this as:

$$C = x^{-t} \int_{\log(K/S)}^{1} (\phi Se^u - \phi K) f(u) du$$

where $f(u)$ is the normal density function with mean $\mu t = [\log(r/d) - \frac{1}{2} \sigma^2]t$ and volatility $\sigma \sqrt{t}$.

Using another change of variables:

$$C = x^{-t} \int_{\log(K/S)}^{1} (\phi Se^{\mu t - \frac{1}{2} \sigma^2} - \phi K) n(z) dz$$

where $n(z)$ is the standard normal density function; that is,

$$n(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$$

Breaking this up into the difference between two integrals:
\[ C = \phi Sr^{-t} \int_{\log(K/r) - \mu \sigma \sqrt{t} \sigma \sqrt{t}}^{\phi} e^{\frac{\sigma^2 + \sigma^2}{2\sqrt{t}}} \phi \sigma \sqrt{t} n(z) \, dz - \phi K r^{-t} \int_{\log(K/r) - \mu \sigma \sqrt{t} \sigma \sqrt{t}}^{\phi} \phi \sigma \sqrt{t} n(z) \, dz \]

Substituting \( \log(r/d) - \frac{1}{2} \sigma^2 \) for \( \mu \) and factoring out \( e^{\frac{\sigma^2 + \sigma^2}{2\sqrt{t}}} \):

\[ C = \phi Sd^{-t} \int_{\log(K/r) - \sigma \sqrt{t} \sigma \sqrt{t}}^{\phi} e^{\frac{\sigma^2 + \sigma^2}{2\sqrt{t}}} \phi \sigma \sqrt{t} n(z) \, dz - \phi K r^{-t} \int_{\log(K/r) - \sigma \sqrt{t} \sigma \sqrt{t}}^{\phi} \phi \sigma \sqrt{t} n(z) \, dz \]

Using a little algebra:

\[ C = \phi Sd^{-t} N(\phi x) - \phi K r^{-t} N(\phi x - \phi \sigma \sqrt{t}) \quad \text{where} \quad x = \frac{\log(Sd^{-t}/Kr^{-t})}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t} \]

Having completed our formal argument, we now make some observations.

First, proportional changes in \( S \) and \( K \) result in the same proportional change in \( C \). That is:

\[ (\alpha C) = \phi (\alpha S) d^{-t} N(\phi x) - \phi (\alpha K) r^{-t} N(\phi x - \phi \sigma \sqrt{t}) \]

where \( x = \frac{\log((\alpha S) d^{-t}/(\alpha K) r^{-t})}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t} \)

This property is called homogeneity of degree one in the underlying asset price and striking price.

Second, the option sensitivities, delta, gamma and theta, are:

\[ \Delta = \frac{\partial C}{\partial S} = \phi d^{-t} N(\phi x) \]

\[ \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{d^{-t} n(x)}{S \sigma \sqrt{t}} \]

\[ \Theta = \frac{\partial C}{\partial t} = \phi S d^{-t} \log(d) N(\phi x) - \phi K r^{-t} \log(r) N(\phi x - \phi \sigma \sqrt{t}) - \frac{n(x) \sigma S d^{-t}}{2 \sqrt{t}} \]

To give theta the interpretation of the number of dollars per day that the option will change in value should the underlying asset price remain unchanged, as before, divide the above calculated number by 365.

IV. Reinterpret the Input Variables in a Binomial Tree or in the Black-Scholes Formula

The two most important examples of this valuation approach are the valuation of options on forward contracts and options on currency exchange rates.

Options on Futures. To see how to value options on forward contracts,\(^{12}\) we must first understand the

---

\(^{12}\) Futures differ from forward contracts in a number of ways. The daily revaluation feature of futures is probably the most significant difference for the purpose of valuation and hedging. Profits and losses from futures contracts are paid as they occur, whereas, for forwards, these are only paid in lump sum on the delivery date. Typically, for valuation and hedging purposes, this difference is ignored in practice, and futures are treated as if they were forward contracts.
basic arbitrage relation between a forward and its underlying asset. The payoff on the delivery date of a forward contract is:

\[ S^* - F \]

where \( F \) is the forward price set in the present and \( S^* \) is the price of the underlying asset on the delivery date. \( T \) will denote the time-to-delivery. In contrast to an option, a forward represents the obligation, not merely the option to deliver the underlying asset.

The following "arbitrage table" shows that a portfolio consisting of a purchased forward contract, shorting \( d^T \) units of the underlying asset and lending \( Fr^T \) dollars has exactly zero payoff on the delivery date. Therefore, to eliminate arbitrage opportunities, the current cost of this investment must be zero. This can only be true if the forward price equals its concurrent underlying asset price times the ratio of the interest return to the payout return, raised to the power of the time to the delivery date.

**Arbitrage Table: Forward Contracts**

<table>
<thead>
<tr>
<th>Present Date</th>
<th>Delivery Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Forward Contract</td>
<td>0</td>
</tr>
<tr>
<td>Short ( d^T ) units of Underlying Asset</td>
<td>( Sd^T )</td>
</tr>
<tr>
<td>Lend ( r^T ) units of Forward Price</td>
<td>( -Fr^T )</td>
</tr>
<tr>
<td>Total</td>
<td>( Sd^T - Fr^T )</td>
</tr>
</tbody>
</table>

To eliminate arbitrage opportunities:

\[ Sd^T - Fr^T = 0 \]

Therefore:

\[ F = S(r/d)^T \]

With this preliminary result, we are now ready to build a binomial model of options on forward contracts. To keep the notation clear, let:

- \( S \) = current underlying asset price
- \( r(\Delta t) \) = one plus the interest rate (over a binomial move)
- \( u \) = one plus the underlying asset rate of appreciation if increase (up)
- \( d \) = one plus the underlying asset rate of appreciation if decrease (down)
- \( d(\delta) \) = one plus the underlying asset payout rate (over a binomial move)
- \( F \) = current forward price (delivery date in \( m \) periods)
- \( u' \) = one plus forward price rate of appreciation if increase

---

\( ^{13} \) This table assumes there are no trading costs, costs of storing the underlying asset, penalties to short selling or other market imperfections, that the underlying asset payout return is known in advance, and that the underlying asset is held for investment and not consumption purposes. Dividends received between the present date and the delivery date are reinvested in the underlying asset which continues to pay out at the return \( d \).
$d' = \text{one plus forward price rate of appreciation if decrease}$

where to prevent simple arbitrage between the underlying and cash assets: $u > (r/\delta) > d$.

Examine what can happen over the first period in an $m$-period binomial tree, where a call on the forward matures in $n$ moves, and the underlying forward delivery date occurs after $m > n$ moves. During the first period, the forward price, a cash investment of $B$ dollars, a portfolio containing $\Delta$ forward contracts and the cash investment, and the price of a call on the forward price each change as follows:

- Forward
- Cash

\[ (u'F - F)\Delta + rB \]

\[ (d'F - F)\Delta + rB \]

\[ C_u \]

\[ C_d \]

The argument here contains the important amendment that the current value of a portfolio containing the forward contract and cash is simply the value of the cash, since the forward contract is currently worth zero. In parallel with our earlier binomial model, we try to replicate the value of the option by holding a portfolio of the underlying forward and cash:

\[ (u'F - F)\Delta + rB = C_u \]

\[ (d'F - F)\Delta + rB = C_d \]

Solving these equations for $\Delta$ and $B$:

\[ \Delta = (C_u - C_d)/(F(u' - d')) \]

\[ B = [p'C_u + (1-p')C_d]/r \]

where: $p' = (1 - d')/(u' - d')$
To prevent arbitrage, the current value $C$ of the option must equal the current cost of its replicating portfolio. Since the present value of a forward contract struck at the current forward price is zero:

$$C = B = \left[ p'C_v - (1-p')C_{v'} \right]/r$$

Compared to our earlier results for options on underlying assets, we have the same formula, except we replace:

$$S \rightarrow F, \quad \delta \rightarrow r, \quad u \rightarrow u' \text{ and } d \rightarrow d'$$

Similar modifications can be made to the Black-Scholes formula to transform it from valuing options on assets to options on forwards. Here is the argument. To prevent arbitrage between the forward contract and its underlying asset, recall that:

$$F = S(r/d)^T$$

Since this must hold at all times, in particular on the option expiration date:

$$F' = S'(r/d)^{T-q}$$

In addition, the payoff from an option on the forward with time-to-expiration $t$ is:

$$\max[0, F' - K]$$

Rewriting this:

$$\max[0, F' - K] = \max[0, S'(r/d)^{T-q} - K] = (r/d)^{T-q}\max[0, S' - K(d/r)^{T-q}]$$

Therefore, the option on the forward should have the same payoff as $(r/d)^{T-q}$ call options on the underlying asset with time-to-expiration $t$ and striking price $K(d/r)^{T-q}$. Since these positions have the same payoff, to prevent arbitrage they must have the same current value.

The Black-Scholes formula for a call on an asset with time-to-expiration $t$ and with payoff $\max[0, S' - K]$ is:

---

14 We can derive the relation of $p$ to $p'$: to prevent arbitrage between the forward and its underlying asset:

$$F = S(r/\delta)^T$$

$$u'F = uS(r/\delta)^{T-1} \quad \text{and} \quad d'F = dS(r/\delta)^{T-1}$$

so $u' = u/(r/\delta)$ and $d' = d/(r/\delta)$

therefore: $p' = [(r/\delta) \cdot d']/(u-d') = p$
\[ C = Sd^{-r}N(x) - K \cdot \frac{\log(Sd^{-r}/Kr^{-\frac{r}{2}})}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t} \]

Thus, under the conditions stated, we can value an option on a forward by using the Black-Scholes formula for \((r/d)^{r/2}\) call options where we substitute \(K(d/r)^{r/2}\) for \(K\) in the formula, \(F(d/r)^{r/2}\) for \(S\) in the formula, and multiply the resulting call value by \((r/d)^{r/2}\):

\[
C = [ (x/d)^{r/2} \times ( [F(d/r)^{r/2}] d^{-x/2}N(x) - [K(d/r)^{r/2}] x^{-\frac{r}{2}}N(x-\sigma \sqrt{t}) ) \]
\[
\text{where } x = \frac{\log([F(d/r)^{r/2}] d^{-x/2})}{[K(d/r)^{r/2}] x^{-\frac{r}{2}}} + \frac{1}{2} \sigma \sqrt{t}
\]

Simplifying:

\[
C = x^{-\frac{r}{2}} [[FN(x) - KN(x-\sigma \sqrt{t})]] \quad \text{where } x = \frac{\log(F/K)}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t}
\]

Note that this is the same result as taking the Black-Scholes formula for the underlying asset and substituting \(F\) for \(S\) and setting the payout return \(d\) equal to the interest return \(r\).\(^{15}\)

**Options on Currency Exchange Rates.** Similar transformations to the binomial model and Black-Scholes formula can be made to handle options on exchange rates. Here is our notation:

- \(X\) = current currency exchange rate ($/£)
- \(F\) = current forward currency exchange rate ($/£)
- \(u\) = one plus exchange-rate % appreciation if increase (up)
- \(d\) = one plus exchange-rate % appreciation if decrease (down)
- \(r(f)\) = one plus foreign interest rate (over a binomial move)
- \(r(d)\) = one plus domestic interest rate (over a binomial move)

where to prevent simple arbitrage between domestic and foreign bonds: \(u > (r/d) > d\).

It will help our intuition to derive the no arbitrage relation between currency exchange forward contracts and their underlying exchange rate.\(^{16}\) The following arbitrage table shows that this relation is the same as the relation between a forward price and its underlying asset, but where the exchange rate \(X\) replaces the underlying asset price \(S\), and the foreign interest return \(r\) replaces the payout return \(d\).

---

\(^{15}\) Why is it unnecessary to replace the volatility of the underlying asset return with the volatility of the change in the forward price? Because they are the same. To see this, recalling our binomial argument:

\[
u^* = \frac{u}{r/\delta} \quad = \log u^* = \log u \cdot \log(r/\delta)
\]
\[
d^* = \frac{d}{r/\delta} \quad = \log d^* = \log d \cdot \log(r/\delta)
\]

Since the logarithms of the two returns differ only by a constant, the variances of their logarithms must be the same.

\(^{16}\) The payoff from a currency exchange rate forward is \(X^* - F\). For example, if \(F = 2\), then the buyer agrees to pay $2 to receive £1 on the delivery date; but if then \(X^* = 2.50\), the buyer can subsequently exchange the £1 for $2.50 making a profit of $.50.
Arbitrage Table: Currency Forward Contracts

<table>
<thead>
<tr>
<th>Present Date</th>
<th>Delivery Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Currency Forward</td>
<td>0</td>
</tr>
<tr>
<td>Borrow $r_i^T$ units of Foreign Currency</td>
<td>$Xr_i^T$</td>
</tr>
<tr>
<td>Lend $r^T$ units of Domestic Currency</td>
<td>-$Fr^T$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>$Xr_i^T - Fr^T$</td>
</tr>
</tbody>
</table>

To eliminate arbitrage opportunities: $Xr_i^T - Fr^T = 0$

Therefore: $F = X(t/t_i)^T$

As we shall see, a very similar transformation is required to value options on currency exchange rates. Again, using a binomial model, during the first period, the exchange rate, a domestic cash investment of $B$ (dollars), a foreign cash investment of $B_f$ (pounds), a portfolio containing the two, and the price of a call on the currency exchange rate can change as follows:

\[
\begin{align*}
X & \quad \text{Exchange-Rate} \\
\text{Domestic Cash} & \quad \text{Foreign Cash} \\
\end{align*}
\]

\[
\begin{align*}
X + B & \quad \text{Foreign and Domestic Cash} \\
\end{align*}
\]

\[
\begin{align*}
C & \quad \text{Call on Exchange Rate} \\
\end{align*}
\]
Again, the replication argument implies:

\[ uXrB_i + rB = C_u \quad \text{and} \quad dXrB_i + rB = C_d \]

\[ B_i = (C_u - C_d)/(r_i(u_d-d)) = \Delta \quad \text{and} \quad B = (dC_u - uC_d)/[(u_d-r)] \]

\[ C = [pC_u + (1-p)C_d]/r \quad \text{with} \quad p = (r/r_i - d)/(u_d-r) \]

Therefore, the option pricing formula is identical to the formula for an option on an asset, but with \( r \) replacing dividend return \( \delta \), the current exchange rate \( X \) replacing the underlying asset price \( S \), and the volatility of the exchange-rate replacing the volatility of the underlying asset. It can be shown that these same transformations carry over to the Black-Scholes formula.

V. Risk-Neutral Discounting of the Expiration Date Payoff with Numerical Integration

Recall the step in our risk-neutral discounting argument of the expiration date payoff:

\[ C = r^{-t} \int_{\log(K/r)}^{\infty} \phi(s) \cdot \sigma \sqrt{t} dz \]

where \( n(z) \) is the standard normal density function with mean \( \mu t = [\log(r/d) - \frac{1}{2} \sigma^2]t \) and volatility \( \sigma \sqrt{t} \).

In that argument, we were able to go further and "simplify" this expression by writing it in terms of the weighted sum of two standard normal distribution functions. However, for many types of options, such simplification is not possible, and we must apply numerical integration techniques directly to the above expression. All such techniques rely on discretizing the continuous density function by sampling from it at suitable intervals. That is, we approximate the option value by:

\[ r^{-t} \sum_{j=0}^{n} p_j \max[0, \phi(S e^{(u_d-d)\sqrt{t}z_j} - K)] \]

where the \( p_j \) is some suitably discretized approximation to the density \( n(z) \). We will describe here one such discretization using a binomial density.\(^\text{17}\) The standard normal density can be approximated by a \((+1,-1)\) symmetric arithmetic binomial random walk:

\[ \text{17 For easy to follow descriptions of other approaches to numerical integration, see William Press, Saul Teukolsky, William Vettering, and Brian Flannery, Numerical Recipes, 2nd Edition (Cambridge University Press 1992).} \]
Given \( n \) steps, for each terminal outcome \( z_j \), the probability \( p_j \) of that outcome is:

\[
p_j = \frac{1}{2^n} \frac{n!}{j!(n-j)!}
\]

where

\[
z_j = \frac{j - (n-j)}{\sqrt{n}}
\]

for \( j = 0, 1, 2, 3, \ldots, n \).

\( z_j \) is standardized by dividing by \( \sqrt{n} \) to make the variance of \( z_j \) equal to one.

For example, for 3 and 4 step trees, the binomial standard normal approximation table is:

For \( n = 3 \):

\[
\begin{array}{ccc}
  j & p_j & z_j \\
  \hline
  3 & .125 & 1.73205 \\
  2 & .375 & .57735 \\
  1 & .375 & .57735 \\
  0 & .125 & -1.73205 \\
\end{array}
\]

For \( n = 4 \):

\[
\begin{array}{ccc}
  j & p_j & z_j \\
  \hline
  4 & .0625 & 2 \\
  3 & .25 & 1 \\
  2 & .375 & 0 \\
  1 & .25 & -1 \\
  0 & .0625 & -2 \\
\end{array}
\]

As \( n \to \infty \), the table comes closer and closer to approximating the standard normal density, and the option value of approaches its continuous density value.

VI. Monte-Carlo Simulation

Once we have derived the binomial tree for the movements in the underlying asset price, rather than work backwards recursively from the current date to the expiration date, European options can also be valued by working forward successively along single paths through the tree to the expiration date. Here is how this works. At the current date, record the current underlying asset price and select an up move \( u \) or down move \( d \) at random according to their associated risk-neutral probabilities \( p \) and \( 1-p \). Record the resulting underlying asset price at the end of the first binomial period \( uS \) or \( dS \), whichever is selected. Given that an up move was selected, during the second period again select an up or a down move randomly according the same risk-neutral probabilities. Continue in this manner, tracing a path to the end of the tree. Record the ending price of the underlying asset long that path (say \( S_1 \), where the subscript 1 indicates the first monte-carlo path). Calculate the realized payoff of the option \( \max[0, \phi S_1 - \phi K] \). Follow a second randomly selected path from the beginning to the end of the tree in the same manner, and record its associated
option payoff \( \max[0, \phi S_2 - \phi K] \). Do this \( m \) times, thereby accumulating \( m \) realized payoffs. The approximation to the current value of the option is then:

\[
r^t(\Sigma [0, \phi S_i - \phi K])/m
\]

Option deltas can be approximated by running a second monte-carlo simulation but with a slightly larger current underlying asset price, other things equal. The finite difference approximation to the delta is then the ratio of the option value generated from the second simulation minus the option value from the original simulation divided by the difference between the two current underlying asset prices. It is important that "other things equal" includes using exactly the same randomly selected paths through the binomial tree in both simulations (usually accomplished by setting the seed value for the random number generator equal for both simulations). Rho (vega) can be approximated similarly except by varying the interest rate (volatility) and holding other things equal in the two simulations.

The following table shows how the accuracy of the monte-carlo simulation depends on the number of paths. For example, using 1000 paths the expected absolute deviation of the resulting option value from the true value is about 59 cents. Note that increasing the number of paths by a factor of 10 only (approximately) doubles the accuracy.

<table>
<thead>
<tr>
<th>monte-carlo paths</th>
<th>mean value</th>
<th>absolute deviation</th>
<th>time in seconds per option</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$9.772</td>
<td>$1.87</td>
<td>.03</td>
</tr>
<tr>
<td>100</td>
<td>9.757</td>
<td>1.06</td>
<td>.30</td>
</tr>
<tr>
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<td>9.757</td>
<td>.59</td>
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</tr>
<tr>
<td>10000</td>
<td>9.757</td>
<td>.32</td>
<td>27.63</td>
</tr>
</tbody>
</table>

Option valued is a European at-the-money call expiring in 1 year. The current underlying asset price is $100 with an annualized volatility of 20% and an annualized dividend yield of 5%. The annualized interest rate is 10% and the tree size is 100 steps.

For some comparable benchmarks, the Black-Scholes formula executes at .00025 seconds per option, valuation using a binominal backwards recursive procedure takes .025 seconds per option, and valuation using expiration date payoff numerical integration takes .0077 seconds per option.

Of our six methods for solving for the value of options, monte-carlo simulation is the least efficient and should only be used when the other methods fail -- or when you have don't want to analyze the option problem but don't mind waiting (perhaps a long time) for your computer.
Appendix: Numerical Example Using Binomial Trees

Index (S) = 100  
Yrs (t) = 1.00  
IntRate = .10  
Vol (σ) = .20  
DivYld = .15  
TreeSize (n) = 5

\[ u = e^{σ\sqrt{t/n}} = 1.09356 \quad d = e^{-σ\sqrt{t/n}} = .914440 \]

\[ r = (1 + \text{IntRate})^{1/n} = 1.01924 \quad δ = (1 + \text{DivYld})^{1/n} = 1.02835 \]

\[ p = \frac{(r/δ) - d}{u - d} = .42824 \]
\[ (1-p) = \frac{u - (r/δ)}{u - d} = .57176 \]

Underlying Asset Price Binomial Tree

\[
\begin{align*}
100 & \quad [109.36] \quad [119.59] \quad [130.78] \quad 156.39 \\
91.44 & \quad [100] \quad [109.36] \quad [109.36] \\
83.62 & \quad [91.44] \quad [91.44] \\
76.47 & \quad [83.62] \quad [83.62] \\
69.92 & \quad [76.47] \quad [76.47] \\
63.94 & \quad \quad \quad [69.92] \\
\end{align*}
\]

Standard Option Binomial Tree (Values)

Amer/Euro = Amer  
Call/Put = Call  
Strike (K) = 100  
YrsToExp (t) = 1.00

\[
\begin{align*}
5.18 & \quad [11.21] \quad [19.59] \quad 56.39 \\
2.62 & \quad [5.31] \quad [10.44] \quad [3.93] \\
2.62 & \quad [1.65] \quad [1.65] \\
0.69 & \quad [0.00] \quad [0.00] \\
0.00 & \quad [0.00] \quad [0.00] \\
0.00 & \quad \quad \quad [0.00] \\
\end{align*}
\]

boundary condition: \[ C_n = \max[S_n - K, 0] \]

recursive rule: \[ C_i = \max\{ S_i - K, [pC_u + (1-p)C_d]/r \} \]

exercise indicated by \[ \]

20
### 1.00 BSE: STANDARD EUROPEAN OPTIONS (LOGNORMAL)

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<tr>
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### 5.00 BIN: STANDARD OPTIONS (BINOMIAL)

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### 10.00 BIN: STANDARD OPTIONS (BINOMIAL)

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<tr>
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\[
\begin{align*}
up &= e^{v(t/n)} \cdot 0.5 = 1.07852 \\
int &= r \cdot (t/n) = 1.01371 \\
down &= e^{-v(t/n)} \cdot 0.5 = 0.92719 \\
div &= d \cdot (t/n) = 1.00699
\end{align*}
\]

\[
S(j; i) = S(0; 0) \cdot (up^i) \cdot (down^{n-i}) \text{ for } j = 0, 1, 2, \ldots, i
\]

\[
F(j; i) = S(j; i) \cdot (int/div)^{(n-i)}
\]

\[
p = \frac{[(int/div) - down] + [up - down]}{0.51807}
\]

\[
(1-p) = \frac{[up - (int/div)] + [up - down]}{0.46841}
\]

boundary: \[P(j; n) = \max(0, K - S(j; n))\]

recursive: \[P(j; i) = \max[K - S(j; i), [pP(j+1; i+1) + (1-p)P(j; j+1)]/int]\]
Vega: Standard European Call (Lognormal)
Strike=100, IntRate=1, Vol=2, DivYld=0.05

Rho: Standard European Call (Lognormal)
Strike=100, IntRate=1, Vol=2, DivYld=0.05

Omega: Standard European Call (Lognormal)
Strike=100, IntRate=1, Vol=2, DivYld=0.05

Binomial Minus Lognormal Standard European Call Values
TreeSize (Index=Strike=100, TrsToExp=1, IntRate=1, Vol=2, Yld=0.05)
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Packages

by
Mark Rubinstein
December 9, 1991

The payoff of a standard European call can be described mathematically as:

\[ \max[0, S' - K] \]

where \( K \) is the option striking price and \( S' \) is the value of the underlying asset at the expiration date of the option. The "max" notation symbolizes that the payoff will be one of the quantities in the brackets, whichever is larger. The value of the call is simply the present value of this payoff, which we will denote as \( C \). Black and Scholes provide one way to calculate this present value. There are several other alternative valuation models, but they all have in common the same objective which is to calculate the present value of the payoff.

The valuation formula for a standard European put can be deduced from the fact that the payoff from a put equals the payoff from a call plus a short position in one share of the underlying asset and a loan of the present value of the striking price:

\[ \max[0, K - S'] = \max[0, S' - K] - S' + K \]

Therefore, the value of an otherwise identical put \( P = C - Sd^t + Kr^t \), where \( S \) is the current price of the underlying asset, \( t \) is the time-to-expiration of the option, \( r \) is one plus the riskless rate of interest, and \( d \) is one plus the payout rate of the underlying asset (the foreign interest rate in a currency context).

This is our first and simplest example of a package: an option which is equivalent to a portfolio containing only standard European calls and possibly cash and the underlying asset itself. For a put, this portfolio contains one otherwise identical call, \( d^t \) units of the underlying asset, and a loan of \( Kr^t \) dollars. Whenever an option can be so decomposed, it is particularly straightforward to value once we have a technique for valuing a standard European call. This is the first "trick" to value exotic options: try to find some way to convert the payoff into the payoff from a portfolio of standard calls (possibly including the underlying asset and cash). Since it is comparatively easy to value the constituent standard calls, it then becomes easy to value the exotic option whose value simply equals the value of the corresponding portfolio of standard calls. The goal of this essay is to describe a variety of package options and show how they can be decomposed into standard call options.

Collars. Our next example is a collar which has payoff:

\[ \min[ \max(S', K_i), K_2 ] \quad (\text{where } 0 < K_i < K_2) \]

Collars have the same payoff as an investment in cash and a standard call except that the unlimited upside potential of a call is replaced by a maximum level \( K_2 \). The buyer of a collar has in effect sold off any payoffs above this level.

---

\(^1\) Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates.
Collars can be easily constructed by observing that the payoff can be rewritten as:

\[ K_1 + \max[0, S^* - K_1] - \max[0, S^* - K_2] \]

which is the payoff from lending the present value of \( K_1 \), buying a standard call with striking price \( K_1 \) and simultaneously selling a standard call with striking price \( K_2 \). Sale of the second call cancels any additional profits which would have been earned from the first call were the underlying asset price to end up greater than \( K_2 \).

**Break forwards.** A break forward has a payoff pattern similar to a call with its terms modified to have zero initial cost. At expiration, we can distinguish among three possible outcomes \((F < K)\):

- \( S^* < F \)
- \( F \leq S^* \leq K \)
- \( K < S^* \)

\[ \begin{align*}
S^* - K & = F - K < 0 \\
S^* - K & \leq 0 \\
S^* - K & > 0
\end{align*} \]

where \( F \) is the current forward price on the underlying asset with a time-to-delivery equal to the time-to-expiration of the option.

Since \( K > F \), it is possible to choose \( K \) such that the value of this contract at inception is zero. Unlike a standard call, a break forward can require a cash outflow at expiration but limited to at most \( F - K \). This makes it possible for the payoff pattern to be otherwise similar to that of a standard call while at the same time requiring no initial cost.

The payoff from a break forward is then equivalent to:

\[ (F - K) + \max[0, S^* - F] \quad \text{(where } F < K) \]

A break forward is a package since it can be decomposed into two separate securities (with the same underlying asset and time to maturity):

1. lend the present value of \( F - K \) (actually borrowing since \( F - K < 0 \))
2. buy a call option with striking price equal to the forward price \( F \)

**Range forwards.** Similarly, a range forward is a "zero-cost collar". At expiration, we can distinguish among four possible outcomes \((K_1 < F < K_2)\):

- \( S^* < K_1 \)
- \( K_1 \leq S^* \leq F \)
- \( F \leq S^* \leq K_2 \)
- \( K_2 < S^* \)

\[ \begin{align*}
S^* - F + K_1 - S^* & = K_1 - F < 0 \\
S^* - F & \leq 0 \\
S^* - F & \geq 0 \\
S^* - F - S^* + K_2 & = K_2 - F > 0
\end{align*} \]

Since \( K_1 < F \), it is possible to choose \( K_1 \) and \( K_2 \) such that the value of this contract at inception is zero.

---

2 This contract is also known as a "Boston option," a "forward with optional exit," and a "cancelable forward."

3 This contract is also known as a "flexible forward," a "cylinder option," a "option fence," a "mini-max," and a "forward band."

2 This contract is also known as a "Boston option," a "forward with optional exit," and a "cancelable forward."

3 This contract is also known as a "flexible forward," a "cylinder option," a "option fence," a "mini-max," and a "forward band."
Unlike a collar, a range forward can require a cash outflow at expiration limited at most to \( K_1 - F \). This makes it possible for the payoff pattern to be otherwise similar to that of a collar while at the same time requiring no initial cost.

The payoff from a range forward is then equivalent to:

\[
(S^* - F) + \max[0, K_1 - S^*] - \max[0, S^* - K_2] \quad \text{(where } K_1 < F < K_2)\]

A range forward is a package since its payoff can be decomposed into three separate securities, one of which is a forward contract which is itself a package containing the underlying asset and a loan (with the same underlying asset and time to maturity):

1. buying a forward contract at forward price \( F \)
2. buying a put option at striking price \( K_1 < F \)
3. selling a call option at striking price \( K_2 > F \)

**Portfolio insurance.** The payoff from portfolio insurance is:

\[
\max[K, S + \beta(S^* - S)] \quad \text{(where } 0 < \alpha > 1 \text{ and } \beta > 0)\]

In the current jargon, \( \alpha \) is referred to as the "upside capture" and \( \beta \) as the "upside gain." In the special case where \( \beta = 1 \), the payoff simplifies to \( \max[K, \alpha S^*] \). \( K \) can be interpreted as the floor. If \( \alpha S^* \) ends up greater than \( K \), then the payoff is some fraction \( \alpha \) of the ending dollar value of the underlying asset. Alternatively, in the special case where \( \alpha = 1 \), the payoff simplifies to \( \max[K, S + \beta(S^* - S)] \). The upside payoff changes to a fraction \( \beta \) of the increase in the dollar value of the underlying asset above its initial level \( S \).

Again, portfolio insurance is a package since its payoff can be rewritten in terms of the payoff a cash investment and \( \alpha_\beta \) standard call options:

\[
K + \alpha_\beta \max\{0, S^* - [K - S(1-\beta)]/\alpha_\beta\}
\]

The present value of this option equals \( Kr_t \) (the present value of the striking price) plus the present value of \( \alpha_\beta \) standard call options on the same underlying asset with the same time-to-expiration and striking price \( [K - S(1-\beta)]/\alpha_\beta \).

**Chooser options.** A chooser option is purchased in the present, but immediately after predetermined elapsed time \( t \) in the future allows the purchaser to choose whether the option is to be a European standard put or a call with a common striking price \( K \) and remaining time to expiration \( T-t \). The payoff from a chooser can be written as:

---


5 The following derivation of the value of chooser options was originally published in Mark Rubinstein, "Options for the Undecided," *Risk*, April 1991.
max[C^*(K, T-t), P^*(K, T-t); t]

Such an option is very easy to value. Using the put-call parity relation (which for European options holds at all points during the lives of the options), we can restate the payoff as:

\[
\max[C^*(C^* - S^*d^{r_q} + Kr^{r_q}); t] = C^*(K,T) + \max[0, Kr^{r_q} - S^*d^{r_q}; t]
\]

This means that the payoff from a chooser will be the same as the payoff from:

1. buying a call with underlying asset price \( S \), striking price \( K \) and time-to-expiration \( T \)
2. buying a put with underlying asset price \( Sd^{r_q} \), striking price \( Kr^{r_q} \) and time-to-expiration \( t \)

**Piecewise-Linear Payoff Options.** We can use these valuation techniques to value (and create) an arbitrary piecewise-linear payoff pattern. Imagine a graph of the pattern where the \( \alpha \) times the ending value of the underlying asset is plotted along the horizontal axis and the payoff from the option is plotted along the vertical axis. Let \( K_a \) be the vertical intercept of the payoff pattern along the vertical axis, \( K_1, K_2, K_3, K_4, \ldots \) be the kink points of the payoff pattern along the horizontal axis, and \( \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \) be the slopes of the payoff pattern between \( 0 \) and \( K_1 \), \( K_1 \) and \( K_2 \), \( K_2 \) and \( K_3 \), \( K_3 \) and \( K_4, \ldots \).

![Payoff Diagram](image)

This payoff can be converted into a portfolio of standard calls since the payoff can be shown to be equal to:

\[
K_0 + \alpha \lambda_0 S^* + \alpha(\lambda_1 - \lambda_0)\max[0, S^* - K_1/\alpha] + \alpha(\lambda_2 - \lambda_1)\max[0, S^* - K_2/\alpha] + \ldots
\]

In other words, the exotic payoff is equal to the payoff from investing \( K_0 \) in cash, buying \( \alpha \lambda_0 \) units of the underlying asset, buying \( \alpha(\lambda_1 - \lambda_0) \) standard call options with striking price \( K_1/\alpha \), buying \( \alpha(\lambda_2 - \lambda_1) \) standard call options with striking price \( K_2/\alpha \), etc.

The logic behind the formula goes like this. The payoff pattern can be viewed as raised everywhere above the horizontal axis to \( K_0 \). Replicating this portion requires cash. Next there is the additional amount over and above this because the payoff rises at the rate \( \alpha \lambda_0 \) above this cash amount as the underlying asset price rises. Replicating this portion requires an investment in \( \alpha \lambda_0 \) units of the underlying asset. But this investment in the underlying asset is capped out when \( \alpha S^* = K_1 \). Replicating this cap requires selling \( \alpha \lambda_0 \)

---

standard calls with striking price \( K_1 / \alpha \). To capture the portion of the payoff between \( \alpha S^* = K_1 \) and \( \alpha S^* = K_2 \), buy \( \alpha \lambda_1 \) standard calls with striking price \( K_1 / \alpha \). But since this call also has a capped return at \( \alpha S^* = K_2 \), it is necessary also to sell \( \alpha \lambda_1 \) standard calls with a striking price of \( K_2 / \alpha \), etc.

For example, in the case of a collar, set

\[
K_0 = K_1 < K_2, \quad \lambda_0 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = \lambda_3 = \lambda_4 = \cdots = 0, \quad \alpha = 1
\]

Or, in the case of a covered call (buy the underlying asset and write a call), set:

\[
K_0 = 0, \quad \lambda_0 = 1, \quad \lambda_1 = \lambda_2 = \lambda_3 = \cdots = 0, \quad \alpha = 1
\]
15.00 PLP: PIECEWISE LINEAR EUROPEAN OPTIONS (LOGNORMAL)

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| .85 | 2.45 | 2.92 | 3.29 | 3.49 | 3.57 | 3.59 | 3.58 | 3.56 | 3.54 |
| .100 | 6.07 | 4.72 | 4.15 | 3.93 | 3.88 | 3.94 | 4.06 | 4.23 | 4.41 |
| 1.15 | 2.74 | 3.97 | 4.83 | 5.50 | 6.08 | 6.59 | 7.06 | 7.49 | 7.90 |
| .85 | 2.92 | 4.10 | 4.31 | 4.41 | 4.52 | 4.65 | 4.81 | 4.97 | 5.15 |
| .200 | 5.20 | 4.07 | 4.40 | 4.89 | 5.39 | 5.87 | 6.33 | 6.76 | 7.16 |
| 1.15 | 3.42 | 5.68 | 6.80 | 7.67 | 8.43 | 9.09 | 9.70 | 10.25 | 10.75 |
| .85 | 3.51 | 4.59 | 4.89 | 5.24 | 5.61 | 5.99 | 6.36 | 6.72 | 7.07 |
| .300 | 4.51 | 4.67 | 5.72 | 6.64 | 7.45 | 8.15 | 8.79 | 9.36 | 9.88 |
| 1.15 | 4.25 | 7.03 | 8.55 | 9.76 | 10.77 | 11.64 | 12.41 | 13.08 | 13.69 |

15.02 PLP: PIECEWISE LINEAR EUROPEAN OPTIONS (LOGNORMAL)

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<td>4Strike</td>
<td>.00</td>
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| Value |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Vol | 1Strik | .05 | .29 | .52 | .76 | 1.00 | 1.24 | 1.48 | 1.71 | 1.95 |
| .20 | 29.86 | 29.19 | 28.54 | 27.90 | 27.27 | 26.66 | 26.07 | 25.48 | 24.91 |
| .100 | 19.90 | 19.46 | 19.02 | 18.60 | 18.18 | 17.77 | 17.38 | 16.99 | 16.61 |
| 40 | 9.95 | 9.73 | 9.51 | 9.30 | 9.09 | 8.89 | 8.69 | 8.49 | 8.30 |
| .20 | 29.86 | 29.19 | 28.54 | 27.90 | 27.27 | 26.66 | 26.06 | 25.47 | 24.90 |
| .200 | 19.90 | 19.46 | 19.02 | 18.60 | 18.18 | 17.77 | 17.37 | 16.98 | 16.60 |
| 40 | 9.95 | 9.73 | 9.51 | 9.30 | 9.09 | 8.89 | 8.69 | 8.49 | 8.30 |
| .30 | 29.86 | 29.19 | 28.53 | 27.88 | 27.23 | 26.57 | 25.91 | 25.26 | 24.61 |
| .300 | 19.90 | 19.46 | 19.02 | 18.58 | 18.14 | 17.68 | 17.23 | 16.77 | 16.31 |
| 40 | 9.95 | 9.73 | 9.51 | 9.29 | 9.05 | 8.81 | 8.56 | 8.31 | 8.06 |

15.01 PLP: PIECEWISE LINEAR EUROPEAN OPTIONS (LOGNORMAL)

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<td>4Strike</td>
<td>.00</td>
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</table>

| Value |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Vol | Captur | .05 | .29 | .52 | .76 | 1.00 | 1.24 | 1.48 | 1.71 | 1.95 |
| .90 | 99.52 | 97.38 | 95.56 | 93.93 | 92.43 | 91.00 | 89.64 | 88.32 | 87.05 |
| .100 | 100.53 | 100.11 | 99.23 | 98.23 | 97.18 | 96.10 | 95.00 | 93.91 | 92.81 |
| 1.10 | 109.73 | 108.51 | 107.37 | 106.24 | 105.11 | 103.97 | 102.83 | 101.69 | 100.56 |
| .90 | 99.54 | 98.38 | 97.50 | 96.60 | 95.65 | 94.66 | 93.66 | 92.64 | 91.60 |
| .200 | 101.42 | 102.17 | 101.94 | 101.39 | 100.69 | 99.88 | 99.01 | 98.09 | 97.14 |
| 1.10 | 109.76 | 109.36 | 108.89 | 108.26 | 107.50 | 106.66 | 105.76 | 104.82 | 103.85 |
| .90 | 99.70 | 99.93 | 98.87 | 99.54 | 99.04 | 98.42 | 97.71 | 96.94 | 96.12 |
| .300 | 102.31 | 104.25 | 104.68 | 104.62 | 104.30 | 103.81 | 103.20 | 102.51 | 101.75 |
| 1.10 | 109.96 | 110.92 | 111.21 | 111.11 | 110.77 | 110.26 | 109.64 | 108.94 | 108.16 |
Forward-Start Options

by
Mark Rubinstein
December 9, 1990

How much would you be willing to pay now for the following opportunity related to a prespecified underlying asset: after a known elapsed time \( t \) in the future (the "grant date"), you will receive at no extra cost a call option with time to expiration \( \tau \), and with a striking price set so that the call will be at-the-money at the time the option is granted? An example of such a forward-start option can be found in corporate incentive stock option arrangements, where employees look forward to receiving options, which are at-the-money on the day of grant.

If we assume only that

1. **homogeneity**: the call option value, when it is granted, will be homogeneous of degree one in the underlying asset price and the striking price

2. **state variable**: all uncertainty in valuing the option after time \( t \) is resolved once the underlying asset price after time \( t \) is known

3. **date-invariance**: the variables determining the value of the option are not date-dependent

4. **payout**: the underlying asset through the grant date has a known constant payout rate \( d \),

such an opportunity is surprisingly easy to value and to hedge.\(^2\)

Let:

\[ S = \text{current value of underlying asset,} \]

\[ S_t = \text{(uncertain) value of underlying asset after time} \ t, \]

\[ C(X, Y, \tau) = \text{value of a call with concurrent underlying asset price} \ X, \text{striking price} \ Y, \text{and time to expiration} \ \tau. \]

From the homogeneity assumption, the value of the forward-starting at-the-money call on the grant date can be written as:

\[ C(S_t, S_t, \tau) = S_t C(1, 1, \tau) \]

From the date-invariance assumption, no date subscript is required for the function \( C \). From the state variable assumption, \( C(1, 1, \tau) \) is known in advance since it is none other than the current value of an at-

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\(^1\) Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates. A version of this essay, under the title "Pay Now, Choose Later," has appeared in RISK, February 1991. Derivations of the formula for valuing forward-start options have recently appeared in two unpublished working papers: George Blazenko, Phelim Boyle and Ken Newport, "Valuation of Tandem Options," and Indi Rajasingham, "On Differential Option Strategies," University of California at Berkeley, December 1998. The derivation presented here is both more general and much simplified.

\(^2\) These assumptions hold for many approaches to option pricing including the Black-Scholes option pricing formula and its generalization to binomial price movements as well as for the jump-diffusion formula developed by Robert Merton.
the-money option with underlying asset price equal to 1 and with time to expiration \( \tau \).

If we can somehow arrange to make an investment now which will for sure produce exactly the cash flow \( S, C(1, 1, \tau) \) after time \( t \), then the current cost of this investment must be the value of the forward-start option. Interpreting \( C(1, 1, \tau) \) as a number of shares, to replicate now the value of the option after time \( t \) we need to hold \( C(1, 1, \tau) \) shares of underlying asset. Using the payout assumption, correcting for the loss of dividends over time \( t \),

\[
Sd^tC(1, 1, \tau)
\]

is then the current value of the forward-start option. Again, using the homogeneity assumption, we can rewrite this as

\[
d^tC(S, S, \tau).
\]

In other words, the value a forward-start option is simply the current value of \( d^t \) calls which are currently at-the-money, with time to expiration \( \tau \).

Ignoring dividends, replicating such an option up to the grant date is quite simple: from the current time to the grant date, we simply need to hold \( C(1, 1, \tau) \) shares of stock. Since \( C(1, 1, \tau) \) is a constant, a buy-and-hold (in contrast to a dynamic) strategy is required. Such a strategy is, of course, trivially self-financing.

We can quickly generalize these results in a number of ways. Without further complicating the above proof, the terms of the forward-start option can easily be generalized to permit the granting of options which are "proportionally in- or out-of-the-money." That is, we can write the contract so that on the grant date the call is worth \( C(S, \alpha S, \tau) \), where \( \alpha \) is a prespecified positive constant. Then, the current value of the option will be \( d^tC(S, \alpha S, \tau) \).

In some cases, we may be uncertain about the grant date perhaps because an employer is unwilling to make a definite commitment. In that case, if there are to be no payouts prior to the grant date, our results continue to hold. Since the seller can perfectly hedge his sale by continuing to hold \( C(1, 1, \tau) \) shares until whenever the grant date occurs, the seller can afford to sell the forward-start option for the current cost of this portfolio, namely \( SC(1, 1, \tau) \). However, if payouts are positive, then the longer the grant date is postponed, the lower the current value of the option since the holder of a forward-start option gets no benefit of payouts prior to the grant date.

Finally, if receipt of the option requires that we continue to be employed, if the probability of continued employment is independent of the underlying asset price, and if we can assume risk-neutrality toward the risk of being fired, then the above formula for the value of a forward-start option needs to be adjusted downward by multiplying it by one minus the probability of being fired.
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Compound Options

by
Mark Rubinstein
July 20, 1991

A compound option is like a standard option except that its underlying asset is itself a standard option. In this article, my objective is to value a variety of compound options in a Black-Scholes environment; that is:

(1) where the underlying asset can be assumed to follow a lognormal random walk, and

(2) where arbitrage arguments allow us to use a risk-neutral valuation approach -- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

The options literature contains a formula for the value of a call on a call; this paper generalizes this result to all four possible combinations: call on a call, put on a call, call on a put, and put on a put.\(^2\)

The payoff for an option on a option is:

$$\max[0, \phi PV_t[\max(0, \eta S^* - \eta k|T|)] - \phi K]$$

where the compound option has striking price \( K \) and time to expiration \( t \) and the underlying option has striking price \( k \) and time-to-expiration \( T > t \). \( S^* \) is the value of the asset underlying the underlying option after time \( t \) and \( PV_t[\cdot] \) means the present value after time \( t \) of the quantity in brackets. \( \eta \) is a binary variable set to 1 when the underlying option is a call and to -1 when the underlying option is a put. \( \phi \) is a binary variable set to 1 for a call on a call and a call on a put and set to -1 for a put on a call and a put on a put.

In a Black-Scholes environment, the formula for the present value after time \( t \) of the underlying option (c) is:

$$c(S_t, t) = \eta S_t d^{(T-t)}N(\eta z_t) - \eta k r^{(T-t)}N(\eta \sigma \sqrt{T-t})$$

where \( z_t = [\log(S_t d^{(T-t)}/k r^{(T-t)}) + \sigma \sqrt{T-t}] / \frac{1}{2} \sigma \sqrt{T-t} \)

and \( S_t \) is the underlying asset price after time \( t \), \( d \) is one plus the payout rate of the underlying asset, \( r \) is one plus the riskless interest rate, and \( N(a) \) is the area under a normal distribution function from \(-\infty\) to \( a \).

Of course, from the viewpoint of the current time, \( S_t \) is a random variable. Using the risk-neutrality principle the current value of the compound option (C) is then:

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\(^1\) Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates. A version of this essay, under the title, "Double Trouble," has appeared in RISK, December 1991-January 1992. The author thanks Eric Reiner for his careful reading of an earlier draft of this essay.

\[ C = r^t \mathbb{E}[\max(0, \phi c(S_t, t) - \phi K)] \]

where \( \mathbb{E} \) is an expectation operator. Writing this in integral form:

\[ C = r^t \int \max(0, \phi c(Se^u, t) - \phi K)f(u)du \]

with the integral taken over the region from \(-\infty\) to \(\infty\), \(u = \log(S_t/S)\),

\[ f(u) = \frac{1}{\sqrt{2\pi}} e^{-(u - \mu^2)/2\sigma^2} \]

with \(v = (u - \mu)/\sigma\sqrt{t},\ \mu = \log(r/d) - \frac{1}{2}\sigma^2\)

and \(S_t\) is the current price of the underlying asset. Since \(S_t\) is lognormal, \(u\) is normal so that \(f(u)\) is the normal density function.

To evaluate this integral, it is convenient to break the problem down into three components corresponding to the three payoff variables, \(S_t\), \(k\) and \(K\):

\[ [1] = \phi S_r^t \int e^{v N(\xi_1)}f(u)du = \phi \eta Sd^T N_1(\phi \eta x, \eta y; \phi \rho) \]

\[ [2] = \phi k r^t \int N(z_1 - \sigma /T-t)f(u)du = \phi k r^t N_2(\phi \eta x - \phi \eta \sigma /T, \eta y - \eta \sigma /T; \phi \rho) \]

\[ [3] = \phi K r^t \int f(u)du = \phi K r^t N(\phi \eta x - \phi \eta \sigma /T) \]

with each integral taken over the region \(\log(X/S)\) to \(\phi \infty\),

\[ x = [\log(Sd^T/Xr^t) + \sigma /\sqrt{T}] + \frac{1}{2}\sigma /\sqrt{T} \]

\[ y = [\log(Sd^T/kr^t) + \sigma /\sqrt{T}] + \frac{1}{2}\sigma /\sqrt{T} \]

\[ \rho = \sqrt{t/T} \]

and \(X\) solves:

\[ \eta Xd^{(T-t)} N(\eta z) - \eta kr^{(T-t)} N(\eta z - \eta \sigma /\sqrt{T-t}) - K = 0 \]

where \(z = [\log(Xd^{(T-t)}/kr^{(T-t)}) + \sigma /\sqrt{(T-t)}] + \frac{1}{2}\sigma /\sqrt{(T-t)}\).

Here, \(N_1(a, b; \rho)\) is the area under a standard bivariate normal distribution function covering the region from \(-\infty\) to \(a\) and \(-\infty\) to \(b\), where the two random variables have correlation \(\rho\).

The current value of the compound option is then \(C = [1] - [2] - [3]\).

\(X\) has a simple interpretation. It is just the critical underlying asset price \(S_t\) at which the value of the underlying standard call \(c(S_t, t)\) equals the derivative option striking price \(k\). \(X\) can easily be found numerically using a Newton-Raphson search procedure.

One interesting complication surrounding compound options is that the derivative of the option value with respect to time (the "theta" of the option) must take into account the passage of time as it affects the remaining time-to-expiration of the compound option \(t\) and the remaining time-to-expiration of the
underlying option (T); so that the compound call theta equals \(-[\frac{\partial C}{\partial t} + \frac{\partial C}{\partial T}]\).
## 21.00 CMP: COMPOUND EUROPEAN OPTIONS (LOGNORMAL)

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Chooser Options

by
Mark Rubinstein
February 6, 1991

Have you ever purchased a call but shortly thereafter wished you had purchased a put instead? If so, you might find "chooser" options interesting.

A standard chooser option is purchased in the present, but immediately after a predetermined elapsed time \( t \) in the future allows the purchaser to choose whether the option is to be a European standard put or a call with a common striking price \( K \) and remaining time to expiration \( T-t \). The payoff from a standard chooser can be written as:

\[
\max[ C'(K, T-t), P'(K, T-t); t] 
\]

Like a straddle, this is an option with a double face; but it will be cheaper than a straddle since the buyer must irrevocably choose, prior to the expiration date, whether the option is to have the payoffs of a call or a put.

Choosers are not creations proceeding from my own heat-oppres's brain; rather at least three OTC choosers have been marketed since July 1990 -- all by Bankers Trust in London, one on oil, one on the DAX index, and one on the BCI index.\(^2\)

A simple trick leads easily to the value of a standard chooser. Using the put-call parity relation (which for European options holds at all points during the lives of the options), we can restate the payoff as:

\[
\max[ C', (C' - S'd^{(T-t)} + Kr^{(T-t)}); t] = C'(K,T) + \max[ 0, Kr^{(T-t)} - S'd^{(T-t)}; t] 
\]

where \( S' \) is the uncertain value after elapsed time \( t \) of the underlying asset, \( d \) is one plus the payout rate on the underlying asset (or foreign interest rate in a foreign currency context), and \( r \) is one plus the rate of interest.

This means that the payoff from a chooser will be the same as the payoff from:\(^3\)

1. buying a call with underlying asset price \( S \), striking price \( K \) and time-to-expiration \( T \)
2. buying a put with underlying asset price \( Sd^{(T-t)} \), striking price \( Kr^{(T-t)} \) and time-to-expiration \( t \)

---

\(^1\) Mark Rubinstein is a professor of finance at the University of California and a principal of Leland O'Brien Rubinstein Associates. A version of this essay, under the title "Options for the Undecided," has appeared in RISK, April 1991.

\(^2\) Unlike the European choosers treated in this article, all the Bankers Trust choosers were American in the sense that the buyer could choose between an American call or an American put. For all these choosers, the call and put had the same striking price and time-to-expiration.

\(^3\) The payoff from a standard chooser can alternatively be decomposed into:

1. buying a put with underlying asset price \( S \), striking price \( K \) and time-to-expiration \( T \)
2. buying a call with underlying asset price \( Sd^{(T-t)} \), striking price \( Kr^{(T-t)} \) and time-to-expiration \( t \)
This decomposition rule provides a way to value standard choosers, whatever standard option pricing formula you wish to use. For example, in the case of Black-Scholes, the value of a standard chooser is:

\[
Sd^TN(x) - K\tau^TN(x - \sigma/\sqrt{T}) - Sd^TN(-y) + K\tau^TN(-y + \sigma/\sqrt{T})
\]

where \( x = [\log(Sd^T/\tau) + \sigma/\sqrt{T}] + \frac{1}{2}\sigma/\sqrt{T} \)

\( y = [\log(Sd^T/\tau) + \sigma/\sqrt{T}] + \frac{1}{2}\sigma/\sqrt{T} \)

and \( \sigma \) is the volatility of the underlying asset and \( N(a) \) is the area under a normal distribution function from \( -\infty \) to \( a \).

Consider five "Black-Scholes" standard chooser options, all with current underlying asset price \( S = 100 \), striking price \( K = 100 \), time-to-expiration \( T = 1 \) year, domestic interest rate \( r = .10 \), payout rate or foreign currency interest rate \( d = .05 \), and underlying asset volatility \( \sigma = .3 \). The options only differ by the time to the choice date. Using our formula, we can easily calculate the following chooser option values:

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In the two degenerate cases, \( t = 0 \) and \( t = 1 \), the chooser value is the same as the value of the call and the straddle, respectively. These extreme cases place a minimum and maximum on the value of the chooser.

A "complex" chooser option is similar to a standard chooser except that the either the call/put striking prices, call/put time-to-expirations, or both are not identical. The payoff from a complex chooser is written as:

\[ \max[C'(K_1, T_1), P'(K_2, T_2); t] \]

implying the chosen call (put) has striking price \( K_1 (K_2) \) and time-to-expiration \( T_1 - t (T_2 - t) \) on the choice date.

This substantially complicates the solution and prevents the complex chooser from being interpreted as a package of standard options. The "Black-Scholes" valuation formula for this option is:

\[
Sd^TN_2(x, y_1, \rho_1) - K_1\tau^TN_2(x - \sigma/\sqrt{T}; y_1, \sigma/\sqrt{T}; \rho_1) - Sd^TN_2(-x - \sigma/\sqrt{T}; -y_2, \rho_2) + K_2\tau^TN_2(-x + \sigma/\sqrt{T}; -y_2 + \sigma/\sqrt{T}; \rho_2)
\]

where \( x = [\log(Sd^T/\tau) + \sigma/\sqrt{T}] + \frac{1}{2}\sigma/\sqrt{T} \)

---

\[ ^4 \text{The formula for the valuation of complex chooser options is quite similar to the formulas for valuing compound options.} \]

Observe that the first two terms of the complex chooser formula are the same as the first two terms of the formula for a call on a call, and the third and fourth terms of the complex chooser formula are the same as the first two terms of the formula for a call on a put—except that the critical underlying asset price \( X \) is set to the level at which the value of standard call \( C'(K_1, T_1) \) will equal the value of a standard put \( P'(K_2, T_2) \), after elapsed time \( t \). \( X \) can be easily evaluated numerically using a Newton-Raphson search procedure.
\[ y_1 = \log(Se^{r_1T_1}/K_r^{r_1}) + \sigma \sqrt{T_1} + \frac{1}{2} \sigma \sqrt{T_1} \quad \text{and} \quad y_2 = \log(Se^{r_2T_2}/K_r^{r_2}) + \sigma \sqrt{T_2} + \frac{1}{2} \sigma \sqrt{T_2} \]

\[ \rho_1 = \sqrt{t/T_1} \quad \text{and} \quad \rho_2 = \sqrt{t/T_2} \]

and \( X \) solves: \[ Xd^{(r_1-t)}N(z_1) - K_r^{r_1-t}N(z_1 - \sigma \sqrt{(T_1-t)}) + Xd^{(r_2-t)}N(-z_2) - K_r^{r_2-t}N(-z_2 + \sigma \sqrt{(T_2-t)}) = 0 \]

where \( z_1 = [\log(Xd^{(r_1-t)}/K_r^{r_1-t}) + \sigma \sqrt{(T_1-t)}] + \frac{1}{2} \sigma \sqrt{(T_1-t)} \)

\[ z_2 = [\log(Xd^{(r_2-t)}/K_r^{r_2-t}) + \sigma \sqrt{(T_2-t)}] + \frac{1}{2} \sigma \sqrt{(T_2-t)} \]

Here \( N_x(a, b; \rho) \) is the area under a standard bivariate normal distribution function covering the portion from \(-\infty\) to \(a\) and \(-\infty\) to \(b\), where the two random variables have correlation \( \rho \).

Of course, it is a trivial matter to add the partially "American" feature that the buyer may choose to precommit to a call or a put at any time up to \(T\). All our results continue to hold in this more general case because it would never pay the buyer to precommit any sooner than required.
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Standard Chooser Minus Straddle Lognormal European Values

CStrike=PStrike=100, CYrs=PYrs=1, IntRate=.1, Vol=.2, DivYld=.05
Barrier Options

by
Mark Rubinstein and Eric Reiner
July 15, 1991

The payoff of a standard European option only depends on the price of the underlying asset on the expiration date. In particular, given the final price of the underlying asset, the payoff will be the same regardless of the path taken by the underlying asset during the life of the option to reach that final price. Whether the underlying asset price reaches a given price by first moving down and then up, or up and then down, matters not to the buyer or seller of the option. It is as if does not matter whether you travel from Paris to London by air or by chunnel, as long as you arrives in London at the appointed time.

The terminology used to describe this feature is "path-independence". More generally, the payoff from an option may depend on some aspect of the price path. For example, the payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during the life of the option, and the payoff of an "Asian" option depends on the average price. In this article, we will examine a simpler type of path-dependent option where the payoff depends not only on the final price of the underlying asset but also on whether or not the underlying asset has reached some other "barrier" price during the life of the option.

In this essay, our objective is to value a variety of these options in a Black-Scholes environment; that is,

1. where the underlying asset return can be assumed to follow a lognormal random walk, and

2. where arbitrage arguments allow us to use a risk-neutral valuation approach -- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

These options are in a sense intermediate between standard European and American options. They are like American options since their value depends on how the underlying asset price behaves through time. But they are simpler to value than American options since the critical boundary of the underlying asset price is determined in advance and specified in the contract. As a result, unlike American options, it will be possible to state "closed form" valuation solutions.

To do this we will need the density of the natural logarithm of the risk-neutral underlying asset return, \( u \):

\[
    f(u) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-u^2 / 2
}
\]

with \( u = (\mu t) / \sigma \sqrt{t}, \mu = \log(r/d) - 1/2 \sigma^2 \).

This is just a normal density function. \( r \) is one plus the rate of interest, \( d \) is one plus the payout rate of

---

1 Mark Rubinstein is professor of finance at the University of California at Berkeley, and he and Eric Reiner are a principal and Vice President, respectively, of Leland O'Brien Rubinstein Associates. A version of this essay, under the title "Breaking Down the Barriers," has appeared in RISK, September 1991.

2 To our knowledge, the only published solution for the options covered in this article has been for the down-and-out option (without adjustment for payouts). See, John Cox and Mark Rubinstein, Options Markets, page 410, Prentice-Hall, 1985.
the underlying asset, $\sigma$ is the volatility of the underlying asset, and $t$ is the time-to-expiration of the option.\(^3\)

We also need another density. Given that the underlying asset price first starts at $S$ above the barrier $H$, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up below the barrier at expiration is:

$$g(u) = e^{u\alpha/\sigma^2} \left(1/(\sigma \sqrt{2\pi t})\right)e^{-\frac{1}{2\sigma^2}u^2}$$

with $v = (u - 2\eta\alpha - \eta \mu t)/\sigma\sqrt{t}$, $\alpha = \log(H/S)$

This is a normal density premultiplied by $e^{u\alpha/\sigma^2}$. Here $\eta = 1$. Alternatively, given that the underlying asset price first starts below the barrier, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up at expiration below the barrier is the same expression but where $\eta = -1$.

1. **"In" Barrier Options**

Our first example is a down-and-in call. Although you pay for this option up front, you do not receive the call until the underlying asset price reaches a prespecified level termed the barrier or knock-in boundary, $H$.\(^4\) If, after elapsed time $\tau \leq t$, the underlying asset price hits the barrier, you then receive a standard European call with striking price $K$ and time-to-expiration $t - \tau$. On the other hand, if through elapsed time $t$, the barrier is never hit, then instead you receive a rebate $R$ at that time (at expiration). Expressed concisely, the payoff from this option is:

$$\begin{align*}
\text{down-and-in call:} & \quad \max[0, S^* - K] \quad \text{if for some } \tau \leq t, \ S(\tau) \leq H \\
(S > H) & \quad R \ (\text{at expiry}) \quad \text{if for all } \tau \leq t, \ S(\tau) > H
\end{align*}$$

where $S(\tau)$ is the price of the underlying asset after elapsed time $\tau$, and $S^* = S(t)$ is the price of the underlying asset at expiration. To be interesting, the initial price $S$ of the underlying asset must be greater than the barrier.

\(^3\) Since Black and Scholes are only interested in the price of the underlying asset at expiration, they can allow $r$, $d$, and $\sigma$ to be known functions of time. However, since the options discussed in this article depend in complex ways on the time paths of these variables, to keep matters simple, we assume here that these variables are constant through time.

\(^4\) In this case, the standard call is received conditional on the behavior of a random variable (underlying asset price). See, Mark Rubinstein, "Pay Now, Choose Later," RISK, February 1991, for an analysis of a standard call that is received unconditionally but at some prespecified future date.
The graphs below help to visualize what can happen:

**Call payoff \((K>H)\)**

\[
\text{prob}[S' > K: S(\tau) \leq H] + \text{prob}[S' > H: S(\tau) \leq H]
\]

**Call payoff \((K<H)\)**

\[
\text{prob}[H \geq S' > K]
\]

**Rebate payoff**

\[
\text{prob}[S' > H: S(\tau) \leq H]
\]

The lines in the first two graphs show paths where the barrier is crossed and the call is received. The line in the third graph shows a path which leads to a rebate payoff at expiry. The statements below state the probability that each outcome will occur. It is necessary to distinguish between two cases: one where \(K>H\) and one where \(K<H\). In the first case, to receive a positive payoff from the call, the underlying asset price must end up above the striking price while having first touched the barrier. On the other hand, the rebate is received only if the underlying price ends up above the barrier without ever having hit the barrier prior to expiration. For this first case, there are thus three types of outcomes:

\[
\begin{align*}
S' &> K \text{ conditional on } S(\tau) \leq H \text{ for some } \tau \leq t \implies \text{payoff} = S' - K \\
S' &\leq K \text{ conditional on } S(\tau) \leq H \text{ for some } \tau \leq t \implies \text{payoff} = 0 \\
S(\tau) &> H \text{ for all } \tau \leq t \implies \text{payoff} = R
\end{align*}
\]

In the second case, to receive a positive payoff from the call, the underlying asset price must end up above the barrier while having first touched the barrier, or end up below the barrier but above the striking price. Again, the rebate is only received if the underlying price ends up above the barrier without ever having hit the barrier prior to expiration. Here there are four types of outcomes:

\[
\begin{align*}
S' &> H \text{ conditional on } S(\tau) \leq H \text{ for some } \tau \leq t \implies \text{payoff} = S' - K \\
S' &\leq H \text{ and } S' > K \\
S' &\leq K \\
S(\tau) &> H \text{ for all } \tau \leq t
\end{align*}
\]

The second and third outcomes are simplified by the fact that since the underlying asset price starts out above the barrier, if the underlying asset price then finishes below the barrier, it must performe have breached the barrier at some time.

Consider first the \(K>H\) case. The value of the option is the the sum of two terms, the first (call payoff) corresponding to \(\text{prob}[S' > K: S(\tau) \leq H]\) and the second (rebate) corresponding to \(\text{prob}[S' > H: S(\tau) \leq H]\):

\[
\begin{align*}
[3] &= r^\tau \int \phi(Se^{\sigma u} - K)g(u)du = \phi Sd'(H/S)^{2\sigma}N(\eta y) - \phi Kr'(H/S)^{2\sigma}N(\eta y - \eta \sigma \sqrt{t}) \\
[5] &= Rr^\tau \int [f(u) - g(u)]du = Rr'[N(\eta y, - \eta \sigma \sqrt{t}) - (H/S)^{2\sigma}N(\eta y, - \eta \sigma \sqrt{t})]
\end{align*}
\]
where the first integral is taken over the region \( \log(K/S) \) to \( \eta^\infty \), the second integral is taken from \( \log(H/S) \) to \( \eta^\infty \),

\[
\begin{align*}
x &= [\log(S/H) + \sigma / t] + \lambda \sigma / t \\
y &= [\log(H^2 / SK) + \sigma / t] + \lambda \sigma / t \\
y_1 &= [\log(H/S) + \sigma / t] + \lambda \sigma / t \\
\lambda &= 1 + (\mu / \sigma^2)
\end{align*}
\]

\( N(\cdot) \) is the standard normal distribution function

and the binary variables, \( \eta \) and \( \phi \), are currently both set equal to 1.

The current value of the down-and-in call can then be expressed as

\[
C_{d\downarrow(K-H)} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \{ \eta = 1, \phi = 1 \}
\]

If, instead \( K < H \), we will need terms corresponding to \( \text{prob}[H \geq S > K] \) and \( \text{prob}[S' > H : S(t) \leq H] \), as well as the rebate term. Since

\[
\text{prob}[H \geq S > K] = \text{prob}[S' > K] - \text{prob}[S' > H]
\]

we have the three corresponding integrals:

\[
\begin{align*}
[1] &= r \int \phi(Se^u - K)f(u)du = \phi Sd^+N(\phi x) - \phi Kr^+N(\phi x - \phi \sigma / t) \\
[2] &= r \int \phi(Se^u - K)f(u)du = \phi Sd^-N(\phi x_1) - \phi Kr^-N(\phi x_1 - \phi \sigma / t) \\
[4] &= r \int \phi(Se^u - K)g(u)du = \phi Sd^+(H/S)^{2\lambda}N(\eta y_1) - \phi Kr^+(H/S)^{2\lambda}N(\eta y_1 - \eta \sigma / t)
\end{align*}
\]

where the first integral is taken over the region \( \log(K/S) \) to \( \phi^\infty \), the second integral is taken over the region \( \log(H/S) \) to \( \phi^\infty \), the third integral is taken over the region \( \log(H/S) \) to \( \eta^\infty \),

\[
x = [\log(S/K) + \sigma / t] + \lambda \sigma / t
\]

and the binary variables, \( \eta \) and \( \phi \), are currently both set equal to 1.

Using this we can write the current value of the down-and-in call as:

\[
C_{d\downarrow(K-H)} = [1] - [2] + [4] + [5] \{ \eta = 1, \phi = 1 \}
\]

Our next barrier option is an up-and-in call. This option is identical to a down-and-in call except that the underlying asset price starts out below instead of above the barrier. Expressed concisely, the payoff from this option is:
**up-and-in call:**

\[(S < H)\]

max[0, \( S^* - K \)] if for some \( \tau \leq t \), \( S(\tau) \geq H \)

R (at expiry) if for all \( \tau \leq t \), \( S(\tau) < H \)

The graphs below help to visualize what can happen:

**call payoff \((K>H)\)**

\[\text{prob}[S^* > K] + \text{prob}[H > S^* > K; S(\tau) \geq H] - \text{prob}[S^* < H; S(\tau) \geq H]\]

**call payoff \((K<H)\)**

\[\text{prob}[S^* \geq H] + \text{prob}[H > S^* > K; S(\tau) \geq H] - \text{prob}[S^* < H; S(\tau) \geq H]\]

**rebate payoff**

\[\text{prob}[S^* < H]\]

For the \( K>H \) case, the new quantities are \( \text{prob}[S^* < H] \) and \( \text{prob}[S^* < H; S(\tau) \geq H] \). The density corresponding to the former is of course \( f(u) \) and the density corresponding to the latter is identical to \( g(u) \), but with \( \eta = -1 \). Therefore,

\[C_{ui(K>H)} = [1] + [5] \quad \{\eta = -1, \phi = 1\}\]

For the \( K<H \) case, we first restate

\[\text{prob}[H > S^* > K; S(\tau) \geq H] = \text{prob}[S^* < H; S(\tau) \geq H] - \text{prob}[S^* \leq K; S(\tau) \geq H]\]

Then, we can write immediately that

\[C_{ui(K<)} = [2] - [3] + [4] + [5] \quad \{\eta = -1, \phi = 1\}\]

For our next options, down-and-in puts and up-and-in puts, we simply provide graphs from which the stated results can easily be inferred:
**down-and-in put:** 
\( (S > H) \)

\[
\begin{align*}
S(\tau) & \quad S(\tau) & \quad S(\tau) \\
S & \quad S & \quad S \\
K & \quad K & \quad K \\
H & \quad H & \quad H
\end{align*}
\]

put payoff \((K>H)\)  
\( \text{prob}[S^* \leq H] \)  
\( + \text{prob}[H < S^* < K; S(\tau) \leq H] \)

**put payoff \((K<H)\)**  
\( \text{prob}[S^* < K] \)

**rebate payoff**  
\( \text{prob}[S^* > H; S(\tau) \leq H] \)

\[
P_{d,i(K>H)} = [2] - [3] + [4] + [5] \quad \{\eta=1, \phi=-1\}
\]

\[
P_{d,i(K<H)} = [1] + [5] \quad \{\eta=1, \phi=-1\}
\]

**up-and-in put:**  
\( (S < H) \)

\[
\begin{align*}
S(\tau) & \quad S(\tau) & \quad S(\tau) \\
K & \quad K & \quad K \\
H & \quad H & \quad H \\
S & \quad S & \quad S
\end{align*}
\]

put payoff \((K>H)\)  
\( \text{prob}[H \leq S^* < K] \)  
\( + \text{prob}[S^* < H; S(\tau) \geq H] \)

put payoff \((K<H)\)  
\( \text{prob}[S^* < K; S(\tau) \geq H] \)

rebate payoff  
\( \text{prob}[S^* < H] \)

\[
P_{u,i(K>H)} = [1] - [2] + [4] + [5] \quad \{\eta=-1, \phi=-1\}
\]

\[
P_{u,i(K<H)} = [3] + [5] \quad \{\eta=-1, \phi=-1\}
\]
II. "Out" Barrier Options

Corresponding to each of these four "in" barrier options are four "out" options. For example, in a down-and-out call, a standard call comes into existence when the down-and-out is issued, but the standard call is extinguished prior to expiration if the underlying asset price ever drops below the knock-out boundary, \( H \). In that case the buyer of the option may be paid a fixed rebate, \( R \). Otherwise, if the underlying asset price never drops below \( H \), the down-and-out call will have the same payoff as a standard call. Expressed concisely, the payoff from this option is:

\[
\text{down-and-out call: } \begin{cases} 
\max(0, S - K) & \text{if for all } \tau \leq t, \ S(\tau) > H \\
R \text{ (at hit)} & \text{if for some } \tau \leq t, \ S(\tau) \leq H
\end{cases}
\]

Here is one possible use of a down-and-out call. Suppose you are holding a covered call but will be forced to liquidate the underlying asset if its price falls sharply. If you sold a down-and-out call in place of a standard call, you could arrange to have the call liquidated automatically at the same time.

If the rebate \( R = 0 \), the following parity relationship makes it easy to write down the values of down-and-out calls:

\[
\text{payoff from standard option} = \text{payoff from down-and-out option} + \text{payoff from down-and-in option}
\]

To see this, suppose you own otherwise identical down-and-out and down-and-in options with no rebates. If the common barrier is never hit, then you receive the payoff from a standard option; if the common barrier is hit, as the down-and-out option is extinguished, the down-and-in option delivers you a standard option identical to the one you lost when the down-and-out option was cancelled. Thus, even in this case, you end up receiving the payoff from a standard option.

The only difficulty comes from the rebate. For "in" options, it is not possible to receive the rebate prior to expiration, since one continues to remain in doubt about whether or not the barrier will never be hit. However, for an "out" option, it is possible as well as customary for the rebate to be paid the moment the barrier is hit. This complicates the risk-neutral valuation problem since the rebate may now be received at a random rather than prespecified time. Thus, we need an additional density of the first passage time \( (\tau) \) for the underlying asset price to hit the barrier:

\[
h(\tau) = (\eta \alpha / \sigma \tau \sqrt{2\pi \tau}) e^{-\frac{\eta^2}{2\tau}}
\]

with \( \eta = (\eta \alpha + \eta \mu \tau) / \sigma \sqrt{\tau} \)

Here, \( \eta = 1 \) if the barrier is being approached from above and \( \eta = -1 \) if the barrier is being approached from below. The present value of the rebate is then the expected rebate discounted by the interest rate raised to the power of the first passage time:

\[
[6] = R \int_0^\tau h(\tau) d\tau = R(\{(H/S)^{\eta z}N(\eta z) + (H/S)^{\eta b}N(\eta z - 2\eta b \sigma \sqrt{\tau})\}]
\]

where the region of integration is from 0 to \( t \), and

\[
z = [\log(H/S) \div \sigma \sqrt{t}] + b \sigma \sqrt{t}
\]

---

\(^5\) This is not actually a new density since it can be derived by differentiating the integral of \( g(u) \) with respect to \( t \).
\[ a = \mu / \sigma^2, \quad b = \left[ \sqrt{\mu^2 + 2\log(\eta)\sigma^2} \right] / \sigma^2 \]

Using these relationships, we can now write down the valuation solutions for the down-and-out call and the three remaining "out" options:

\[
\begin{align*}
C_{do(K>H)} &= [1] - [3] + [6] \quad \{ \eta=1, \, \phi=1 \} \\
C_{do(K<H)} &= [2] - [4] + [6] \quad \{ \eta=1, \, \phi=1 \}
\end{align*}
\]

**up-and-out call:** \( (S < H) \)

\[
\begin{align*}
\text{max}[0, S' - K] & \quad \text{if for all } \tau \leq t, \quad S(\tau) < H \\
R \text{ (at hit)} & \quad \text{if for some } \tau \leq t, \quad S(\tau) \geq H
\end{align*}
\]

\[
C_{uo(K>H)} = [6] \quad \{ \eta=-1, \, \phi=1 \}
\]

\[
\]

**down-and-out put:** \( (S > H) \)

\[
\begin{align*}
\text{max}[0, K - S'] & \quad \text{if for all } \tau \leq t, \quad S(\tau) > H \\
R \text{ (at hit)} & \quad \text{if for some } \tau \leq t, \quad S(\tau) \leq H
\end{align*}
\]

\[
\]

\[
P_{do(K<H)} = [6] \quad \{ \eta=1, \, \phi=-1 \}
\]

**up-and-out put:** \( (S < H) \)

\[
\begin{align*}
\text{max}[0, K - S'] & \quad \text{if for all } \tau \leq t, \quad S(\tau) < H \\
R \text{ (at hit)} & \quad \text{if for some } \tau \leq t, \quad S(\tau) \geq H
\end{align*}
\]

\[
P_{uo(K>H)} = [2] - [4] + [6] \quad \{ \eta=-1, \, \phi=-1 \}
\]

\[
P_{uo(K<H)} = [1] - [3] + [6] \quad \{ \eta=-1, \, \phi=-1 \}
\]

At first, it may be surprising that the rebate provides the only contribution to the value of an up-and-out call when the striking price is greater than the barrier. But it is easy to see why. Since \( S < H < K \), in order for the underlying asset price to end up above the striking price it must first breach the barrier, but in this event, the call is extinguished. Similarly, a down-and-out put will also only be valued for the rebate when the striking price is less than the barrier.
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Strike=100, IntRate=1, Vol=2, DivYld=5, Barrier=95, Rebate=2

Delta: Down-and-Out, Up-and-Out European Call (Lognormal)
Strike=100, IntRate=1, Vol=2, DivYld=5, Barrier=95, Rebate=2

Gamma: Down-and-Out, Up-and-Out European Call (Lognormal)
Strike=100, IntRate=1, Vol=2, DivYld=5, Barrier=95, Rebate=2
Down-and-out Minus Standard Call Lognormal European Values

Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=90, Rebate=2
Down-and-out Minus Standard Call Lognormal European Gammas
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=90, Rebate=2
Up-and-out Call Lognormal European Values

Strike=100, IntRate=.1, Vol=.2, Yld=.05, Barrier=110, Rebate=10
Up-and-out Call Minus Vertical Spread Lognormal European Values

Strike=100, IntRate=.1, Vol=.2, Yld=.05, Barrier=110, Rebate=10
Binary Options

by

Mark Rubinstein and Eric Reiner

July 31, 1991

In contrast to the smooth payoff patterns of standard options, binary options tend to have discontinuous payoffs, switched completely one way or the other depending on whether the underlying asset price satisfies some condition. This article considers a wide variety of binaries, first those with path-independent payoffs and then the more complex barrier binaries with path-dependent payoffs.

In this article, our objective is to value a variety of these options in a Black-Scholes environment; that is,

1) where the underlying asset return can be assumed to follow a lognormal random walk, and

2) where arbitrage arguments allow us to use a risk-neutral valuation approach -- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

Under these conditions, the (Black-Scholes) formula for a standard option is

\[ C = \phi S_d N(\phi x) - \phi K_r N(\phi x - \phi \sigma \sqrt{t}) \text{ where } x = \left[ \log \left( \frac{S_0}{K} \right) + \frac{d(\sigma^2 t)}{2} \right] \sqrt{t} \]

S is the present value of the underlying asset, K is the striking price, r is one plus the rate of interest, d is one plus the payout rate of the underlying asset, t is the time-to-expiration of the option, \( \sigma \) is the volatility of the underlying asset, \( N(\cdot) \) is the standard normal distribution function, and the binary variable \( \phi \) is set equal to 1 for a call and -1 for a put.

I. Path-Independent Binary Options

Cash-or-nothing calls and puts. The simplest binary call (put) pays off nothing if the underlying asset price (S) finishes below (above) the striking price (K), or pays out a predetermined constant amount (X) if the underlying asset finishes above (below) the striking price:

\[ 0 \text{ if } \phi S^t \leq \phi K, \quad \text{or } X \text{ if } \phi S^t > \phi K \]

with \( \phi = 1 \) for a call, and \( \phi = -1 \) for a put.

In a Black-Scholes environment, such an option is easy to value. Recall that the Black-Scholes formula can

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1 Mark Rubinstein is professor of finance at the University of California at Berkeley, and he and Eric Reiner are a principal and Vice President, respectively, of Leland O'Brien Rubinstein Associates. A version of this essay, under the title "Unscrambling the Binary Code," has appeared in RISK, October 1991.

2 See the article by Mike Hudson, "The Value in Going Out," in RISK, March 1991, p. 31, for a brief taxonomy of some of these contracts (there referred to as digital options).

3 Since the values of standard European options depend only on the price of the underlying asset at expiration, it is possible to let r, d, and \( \sigma \) be known functions of time. However, since the options discussed in this article depend in complex ways on the paths taken by the underlying asset through time, to keep matters simple, we assume here that these variables are constant.
be decomposed into the difference between two terms: the unprotected present value of the underlying asset price conditional upon exercising the option -- \( \text{Sd}^t \mathbb{N}(\phi x) \) -- and the present value of the striking price -- \( \text{K} \) -- times the risk-neutral probability of exercising the option -- \( \mathbb{N}(\phi x - \phi \sigma \sqrt{t}) \). If the predetermined payoff of the cash-or-nothing call (\( X \)) were equal to the striking price (\( K \)), then a cash-or-nothing call would be like a standard written call except that, although the writer receives the striking price, he is under no obligation to deliver the underlying asset. Such a "partial call" would be then be worth \( \text{Kr}^t \mathbb{N}(x - \sigma \sqrt{t}) \) to the writer. More generally, allowing \( X \neq K \), the value of a cash-or-nothing call would be \( \text{Xr}^t \mathbb{N}(\phi x - \phi \sigma \sqrt{t}) \) with \( \phi = 1 \). Similarly, the value of a cash-or-nothing put is the same, but with \( \phi = -1 \).

**Asset-or-nothing calls and puts.**\(^4\) A somewhat more complex binary option has the following payoffs:

\[
0 \quad \text{if } \phi S' \leq \phi K \quad \text{or} \quad S' \quad \text{if } \phi S' > \phi K
\]

These options are similar to cash-or-nothing options, except that when they pay off, the amount is not predetermined but rather is equal to the underlying asset price at expiration. To value these, we look instead to the first term of the Black-Scholes formula. This gives us exactly what we need: the unprotected present value of the underlying asset price conditional upon exercising the option. So the value of an asset-or-nothing call is simply \( \text{Sd}^t \mathbb{N}(\phi x) \) with \( \phi = 1 \), and the value of an asset-or-nothing put is the same, but with \( \phi = -1 \).

**Gap options.** Our next options are just slightly more complex than standard options:

\[
0 \quad \text{if } \phi S' \leq \phi K \quad \text{or} \quad \phi S' - \phi x \quad \text{if } \phi S' > \phi K
\]

These options highlight the dual role played by the striking price in a standard option: \( K \) not only determines whether the option finishes in- or out-of-the-money but also the size of the resulting payoff (\( X \)). There is no logical necessity for these two functions to be vested in a single number. The "gap" is defined as \( X - K \). Positive gap calls will clearly sell for less than standard calls while positive gap puts will be worth more than their standard counterparts. The value of a gap option may be derived by subtracting a cash-or-nothing option from an asset-or-nothing option:

\[
C = \phi Sd^t \mathbb{N}(\phi x) - \phi Xr^t \mathbb{N}(\phi x - \phi \sigma \sqrt{t}) \quad \text{where} \quad x = \left[ \log \left( \frac{\text{Sd}^t}{\text{K}^t} \right) + \sigma \sqrt{t} \right] + \frac{1}{2} \sigma \sqrt{t}
\]

almost identical to the Black-Scholes formula for a standard option. In a special case, if \( X \) is chosen so that \( C = 0 \), the option is a "pay-later option," since it can be interpreted as if the buyer pays for the option at expiration, and only if at that time \( \phi S' > \phi K \).\(^5\)

**Supershares.** In an article published in 1976, Nils Hakansson proposed a financial intermediary which would hold an underlying portfolio and issue claims called "supershares" against this portfolio to investors.\(^6\) A

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\(^4\) These options are discussed in John Cox and Mark Rubinstein, *Options Markets*, page 460, Prentice-Hall, 1985.


supershare is a security, which on its expiration date entitles its owner to a given dollar value proportion of the assets of the underlying portfolio, provided the value of those assets on that date lies between a lower value $K_i$ and an upper value $K_h$. Otherwise, the supershare expires worthless. That is, the payoff is:

$$\begin{align*}
0 & \quad \text{if } S^* < K_i \\
S^*/K_i & \quad \text{if } K_i \leq S^* < K_h \\
0 & \quad \text{if } K_h \leq S^*
\end{align*}$$

It is easy to see that such an option has the same payoff as $(1/K_i)$ bullish vertical spreads of asset-or-nothing calls, where the purchased asset-or-nothing call has striking price $K_i$ and the written asset-or-nothing call has striking price $K_h$. That is,

$$C = (Sd^+/K_i)[N(x_i) - N(x_h)]$$

$$x_i = \left[\log(Sd^+/K_i) + \sigma \sqrt{t}\right] + \frac{\nu \sigma \sqrt{t}}{2}, \quad x_h = \left[\log(Sd^+/K_h) + \sigma \sqrt{t}\right] + \frac{\nu \sigma \sqrt{t}}{2}$$

These securities can be shown to be building blocks for constructing other more common securities such as purchased standard calls and puts.

**Portfolios of binary options.** Another way to construct arbitrary piecewise linear payoff patterns is to make more direct use of cash-or-nothing and asset-or-nothing calls. For example, consider the following payoff pattern:

![Payoff Pattern Diagram]

where $\lambda_1 = (y_2-y_1)/(x_2-x_1)$, $\lambda_2 = (y_2-y_3)/(x_2-x_3)$, $\lambda_3 = (y_4-y_5)/(x_3-x_4)$, and $\lambda_4 = (y_5-y_6)/(x_4-x_5)$ are the slopes of diagonal line segments moving from left to right.

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7 The associated essay in this series, "Packages", describes a rule for using a portfolio of standard call options, and possibly cash and the underlying asset itself, to construct generalized piecewise linear payoff patterns. However, since these "basis" securities all have continuous payoffs, all portfolios constructed from them must also have continuous payoff patterns. Using cash-or-nothing and asset-or-nothing calls permits discontinuous payoffs as well (as illustrated in the example). For a given set of breakpoints, this approach also permits greater control over the slope of the payoff pattern between breakpoints than can be provided with supershares.
Let \( C(K) \) be the current value of a cash-or-nothing call paying off $1 if and only if \( S_t > K \), and let \( A(K) \) be the current value of an asset-or-nothing call paying off \( S_t^\gamma \) if and only if \( S_t > K \). The above payoff pattern can be replicated by the following portfolio:

\[
(y_1 - \lambda_1 x_1)[C(x_1) - C(x_2)] + \lambda_1[A(x_1) - A(x_2)]
\]

\[
(y_2 - \lambda_2 x_2)[C(x_2) - C(x_3)] + \lambda_2[A(x_2) - A(x_3)]
\]

\[
(y_3 - \lambda_3 x_3)[C(x_3) - C(x_4)] + \lambda_3[A(x_3) - A(x_4)]
\]

\[
(y_4 - \lambda_4 x_4)[C(x_4) - C(x_5)] + \lambda_4[A(x_4) - A(x_5)]
\]

II. Binary Barrier Options

Just as we can value path-independent binary options by selecting pieces of the Black-Scholes formula for standard options, we can also construct solutions for binary barrier options from pieces of the formulas for barrier options.\(^8\)

First, we will need the density of the natural logarithm, \( u \), of the underlying risk-neutral asset return

\[
f(u) = (1/\sigma \sqrt{2\pi t})e^{u^2/2}
\]

where \( v = (u - \mu t)/\sigma \sqrt{t} \), \( \mu = \log(r/d) - \frac{1}{2}\sigma^2 \)

Second, we need the density, \( g(u) \), of the natural logarithm of the underlying asset return, given that the asset price first starts at \( S \) above the barrier \( H \), crosses the barrier at least once, but ends up greater than \( H \) at expiration:

\[
g(u) = e^{2u\mu/\sigma^2}(1/\sigma \sqrt{2\pi t})e^{-u^2/2}
\]

where \( v = (u - 2\eta \alpha - \eta \mu t)/\sigma \sqrt{t} \), \( \alpha = \log(H/S) \)

This is a normal density premultiplied by \( e^{2u\mu/\sigma^2} \). Here \( \eta = 1 \). Alternately, given that the underlying asset price first starts below the barrier, the density of the logarithmic return when the underlying asset price breaches the barrier but ends up below the barrier at expiration is the same expression but with \( \eta = -1 \).

Finally, we need an additional density for the first time \( \tau \) that the underlying asset price crosses the barrier:

\[
h(\tau) = (-\eta \alpha/\sigma \tau \sqrt{2\pi \tau})e^{-\nu^2/2}
\]

where \( \nu = (-\eta \alpha + \eta \mu \tau)/\sigma \sqrt{\tau} \)

\( h(\tau) \) is often called the first passage time density.

Because we plan to define 28 types of path-dependent binary options with 44 different formulas, it is convenient to provide some notation for the desired pieces:

\(^8\) The notation used here is consistent with the notation used in our earlier article, "Breaking Down the Barriers", RISK, September, 1991.
\[ \begin{align*}
[1A] &= \text{S}r^i \int e^{u_f(u)du} = \text{Sd}^i N(\phi x) \\
[1C] &= \text{Xr}^i \int f(u)du = \text{Xr}^i N(\phi x - \phi \sigma /t) \\
[2A] &= \text{Sr}^i \int e^{u_f(u)du} = \text{Sd}^i N(\phi x_i) \\
[2C] &= \text{Xr}^i \int f(u)du = \text{Xr}^i N(\phi x_i - \phi \sigma /t) \\
[3A] &= \text{Sr}^i \int e^{x g(u)du} = \text{Sd}^i (H/S)^{2\lambda N(\eta y)} \\
[3C] &= \text{Xr}^i \int g(u)du = \text{Xr}^i (H/S)^{2\lambda N(\eta y - \eta \sigma /t)} \\
[4A] &= \text{Sr}^i \int e^{x g(u)du} = \text{Sd}^i (H/S)^{2\lambda N(\eta y_i)} \\
[4C] &= \text{Xr}^i \int g(u)du = \text{Xr}^i (H/S)^{2\lambda N(\eta y_i - \eta \sigma /t)} \\
[5] &= \text{X} \int r^i h(\tau)d\tau = X[(H/S)^{r^i+b}N(\eta z) + (H/S)^{r^i+b}N(\eta z - 2 \eta b \sigma /t)]
\end{align*} \]

where integrals [1A] and [1C] are taken over the region \( \log(K/S) \) to \( \phi \infty \), the integrals [2A] and [2C] are taken over the region \( \log(H/S) \) to \( \phi \infty \), the integrals [3A] and [3C] are taken over the region \( \log(K/S) \) to \( \eta \infty \), the integrals [4A] and [4C] are taken over the region \( \log(H/S) \) to \( \eta \infty \), and integral [5] is taken over the region 0 to \( t \), and

\[
\begin{align*}
x &= [\log(S/K) - \sigma /t] + \lambda \sigma /t \\
x_i &= [\log(S/H) + \sigma /t] + \lambda \sigma /t \\
y &= [\log(H^2/SK) + \sigma /t] + \lambda \sigma /t \\
y_i &= [\log(H/S) + \sigma /t] + \lambda \sigma /t \\
z &= [\log(H/S) + \sigma /t] + b \sigma /t \\
\lambda &= 1 + (\mu /\sigma^2) \\
a &= \mu /\sigma^2, \quad b = [1(\mu^2 + 2(\log r)\sigma^2)] /\sigma^2
\end{align*}
\]

and the binary variables, \( \eta \) and \( \phi \), each take values of either 1 or -1.
We now catalogue our valuation results. Options 1-4 all have payoffs that are received the moment the barrier is breached:

(1) **down-and-in cash-(at hit)-or-nothing**: payoff: $X$ (at hit) if for some $\tau \leq t$, $S(\tau) \leq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) > H$
\hspace{1cm} value: [6] \{ $\eta = 1$ \}

(2) **up-and-in cash-(at hit)-or-nothing**: payoff: $X$ (at hit) if for some $\tau \leq t$, $S(\tau) \geq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) < H$
\hspace{1cm} value: [6] \{ $\eta = -1$ \}

The payoff from an "in" cash-(at hit)-or-nothing option is identical to the rebate portion of a down-and-out or up-and-out option where the rebate equals $X$.

(3) **down-and-in asset-(at hit)-or-nothing**: payoff: $S(\tau)$ (at hit) if for some $\tau \leq t$, $S(\tau) \leq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) > H$
\hspace{1cm} value: [6] \{ $\eta = 1, X = H$ \}

(4) **up-and-in asset-(at hit)-or-nothing**: payoff: $S(\tau)$ (at hit) if for some $\tau \leq t$, $S(\tau) \geq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) < H$
\hspace{1cm} value: [6] \{ $\eta = -1, X = H$ \}

The valuation of "in" asset-(at hit)-or-nothing options may at first seem to be difficult because the buyer appears to receive a random amount at a random time. However, in the continuous price path world of Black-Scholes, the value of the underlying asset ($S(\tau)$) at the moment the barrier is hit must be equal to the barrier itself ($H$).

All the remaining options have payoffs received only at expiration. Options 5-8 have positive payoffs requiring only that the barrier ($H$) be breached at some time before expiration:

(5) **down-and-in-cash-(at expiry)-or-nothing**: payoff: $X$ (at expiry) if for some $\tau \leq t$, $S(\tau) \leq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) > H$
\hspace{1cm} value: [2C] + [4C] \{ $\eta = 1, \phi = -1$ \}

(6) **up-and-in-cash-(at expiry)-or-nothing**: payoff: $X$ (at expiry) if for some $\tau \leq t$, $S(\tau) \geq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) < H$
\hspace{1cm} value: [2C] + [4C] \{ $\eta = -1, \phi = 1$ \}

(7) **down-and-in-asset-(at expiry)-or-nothing**: payoff: $S(t)$ (at expiry) if for some $\tau \leq t$, $S(\tau) \leq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) > H$
\hspace{1cm} value: [2A] + [4A] \{ $\eta = 1, \phi = -1$ \}

(8) **up-and-in-asset-(at expiry)-or-nothing**: payoff: $S(t)$ (at expiry) if for some $\tau \leq t$, $S(\tau) \geq H$
\hspace{1cm} 0 if for all $\tau \leq t$, $S(\tau) < H$
\hspace{1cm} value: [2A] + [4A] \{ $\eta = -1, \phi = 1$ \}

The payoff from an "in" asset-(at expiry)-or-nothing is identical to the payoff from a down-and-in or up-and-in
call for which both the striking price and rebate are zero.

Options 9-12 have positive payoffs requiring only that the barrier (H) not be breached before expiration; they are the "out" complements to options 5-8.

(9) **down-and-out cash-or-nothing:** 
\[
(S > H) \quad \text{payoff: } X \text{ (at expiry) if for all } \tau \leq t, \; S(\tau) > H \\
\text{0 if for some } \tau \leq t, \; S(\tau) \leq H \\
\text{value: } [2C] \cdot [4C] \{ \eta = 1, \phi = 1 \}
\]

(10) **up-and-out cash-or-nothing:** 
\[
(S < H) \quad \text{payoff: } X \text{ (at expiry) if for all } \tau \leq t, \; S(\tau) < H \\
\text{0 if for some } \tau \leq t, \; S(\tau) \geq H \\
\text{value: } [2C] \cdot [4C] \{ \eta = -1, \phi = -1 \}
\]

(11) **down-and-out asset-or-nothing:** 
\[
(S > H) \quad \text{payoff: } S(t) \text{ (at expiry) if for all } \tau \leq t, \; S(\tau) > H \\
\text{0 if for some } \tau \leq t, \; S(\tau) \leq H \\
\text{value: } [2A] \cdot [4A] \{ \eta = 1, \phi = 1 \}
\]

(12) **up-and-out asset-or-nothing:** 
\[
(S < H) \quad \text{payoff: } S(t) \text{ (at expiry) if for all } \tau \leq t, \; S(\tau) < H \\
\text{0 if for some } \tau \leq t, \; S(\tau) \geq H \\
\text{value: } [2A] \cdot [4A] \{ \eta = -1, \phi = -1 \}
\]

The payoff of a down-and-out cash-or-nothing option is equal to \(-X/H\) times the cash portion of the payoff of a down-and-out call with striking price \(H\), and the payoff of an up-and-out cash-or-nothing is equal to \(X/H\) times the cash portion of an up-and-out put with striking price \(H\). Similarly, the payoff from an "out" asset-or-nothing is identical to the payoff from a down-and-out or up-and-out call with both the striking price and rebate equal to zero.

These results may be obtained from formulas 5-8 by noting that a portfolio consisting of an "out" option and its complementary "in" will always pay off at expiry, regardless of whether or not the barrier is crossed. Consequently, we can derive parity relations equating the sum of the present values of the two options to the present value of delivery at expiration of cash or the underlying asset. For example, the value of a down-and-out cash-or-nothing (9) is merely the difference between the present value of \(X\) delivered unconditionally at expiry \((X^\delta)\) and our result for option 5.

Options 13-16 have positive payoffs requiring not only that the barrier (H) be breached, but also that the underlying asset finish above a given level (K).

(13) **down-and-in cash-or-nothing:** 
\[
(S > H) \quad \text{payoff: } X \text{ (at expiry) if } S(t) > K \text{ and for some } \tau \leq t, \; S(\tau) \leq H \\
\text{0 if } S(t) < K \text{ or for all } \tau \leq t, \; S(\tau) > H \\
\text{value (K>H): } [3C] \{ \eta = 1 \} \\
\text{value (K<H): } [1C] \cdot [2C] + [4C] \{ \eta = 1, \phi = 1 \}
\]

(14) **up-and-in cash-or-nothing:** 
\[
(S < H) \quad \text{payoff: } X \text{ (at expiry) if } S(t) > K \text{ and for some } \tau \leq t, \; S(\tau) > H \\
\text{0 if } S(t) < K \text{ or for all } \tau \leq t, \; S(\tau) \leq H \\
\text{value (K>H): } [1C] \{ \phi = 1 \} \\
\text{value (K<H): } [2C] \cdot [3C] + [4C] \{ \eta = -1, \phi = -1 \}
\]

Suppose \(X\) were equal to the striking price of a down-and-in or up-and-in call. Then the striking price
portion of the payoff of these calls has the same value as an otherwise identical "in" cash-or-nothing call.

(15) **down-and-in asset-or-nothing call:**
\[
\begin{align*}
\text{payoff: } & S(t) \text{ (at expiry) if } S(t) > K \text{ and for some } \tau \leq t, \ S(\tau) \leq H \\
& 0 \text{ if } S(t) < K \text{ or for all } \tau \geq t, \ S(\tau) > H \\
\text{value } & (K>H): \quad [3A] \quad \{ \eta = 1 \} \\
\text{value } & (K<H): \quad [1A] - [2A] + [4A] \quad \{ \eta = 1, \phi = 1 \}
\end{align*}
\]

(16) **up-and-in asset-or-nothing call:**
\[
\begin{align*}
\text{payoff: } & S(t) \text{ (at expiry) if } S(t) > K \text{ and for some } \tau \leq t, \ S(\tau) \geq H \\
& 0 \text{ if } S(t) < K \text{ or for all } \tau \leq t, \ S(\tau) < H \\
\text{value } & (K>H): \quad [1A] \quad \{ \phi = 1 \} \\
\text{value } & (K<H): \quad [2A] - [3A] + [4A] \quad \{ \eta = -1, \phi = 1 \}
\end{align*}
\]

An "in" asset-or-nothing call has the same value as the asset portion of the payoff of an otherwise identical down-and-in or up-and-in call.

Options 17-20 have positive payoffs requiring not only that the barrier (H) be breached but also that the underlying asset finish **below** a given level (K).

(17) **down-and-in cash-or-nothing put:**
\[
\begin{align*}
\text{payoff: } & X \text{ (at expiry) if } S(t) < K \text{ and for some } \tau \leq t, \ S(\tau) \leq H \\
& 0 \text{ if } S(t) < K \text{ or for all } \tau \leq t, \ S(\tau) > H \\
\text{value } & (K>H): \quad [2C] - [3C] + [4C] \quad \{ \eta = 1, \phi = -1 \} \\
\text{value } & (K<H): \quad [1C] \quad \{ \phi = -1 \}
\end{align*}
\]

(18) **up-and-in cash-or-nothing put:**
\[
\begin{align*}
\text{payoff: } & X \text{ (at expiry) if } S(t) < K \text{ and for some } \tau \leq t, \ S(\tau) \geq H \\
& 0 \text{ if } S(t) < K \text{ or for all } \tau \leq t, \ S(\tau) < H \\
\text{value } & (K>H): \quad [1C] - [2C] + [4C] \quad \{ \eta = -1, \phi = -1 \} \\
\text{value } & (K<H): \quad [3C] \quad \{ \eta = -1 \}
\end{align*}
\]

Suppose X were equal to the striking price of a down-and-in or up-and-in put. Then the striking price portion of the payoff of these puts has the same value as an otherwise identical "in" cash-or-nothing put.

(19) **down-and-in asset-or-nothing put:**
\[
\begin{align*}
\text{payoff: } & S(t) \text{ (at expiry) if } S(t) < K \text{ and for some } \tau \leq t, \ S(\tau) \leq H \\
& 0 \text{ if } S(t) < K \text{ or for all } \tau \leq t, \ S(\tau) > H \\
\text{value } & (K>H): \quad [2A] - [3A] + [4A] \quad \{ \eta = 1, \phi = -1 \} \\
\text{value } & (K<H): \quad [1A] \quad \{ \phi = -1 \}
\end{align*}
\]

(20) **up-and-in asset-or-nothing put:**
\[
\begin{align*}
\text{payoff: } & S(t) \text{ (at expiry) if } S(t) < K \text{ and for some } \tau \leq t, \ S(\tau) \geq H \\
& 0 \text{ if } S(t) < K \text{ or for all } \tau \leq t, \ S(\tau) < H \\
\text{value } & (K>H): \quad [1A] - [2A] + [3A] \quad \{ \eta = -1, \phi = -1 \} \\
\text{value } & (K<H): \quad [3A] \quad \{ \eta = -1 \}
\end{align*}
\]

An "in" asset-or-nothing put has the same value as the asset portion of the payoff of an otherwise identical down-and-in or up-and-in put.

Options 21-24 have positive payoffs requiring not only that the barrier (H) **not** be breached but also that the underlying asset finish **above** a given level (K).
Suppose \( X \) were equal to the striking price of a down-and-out or up-and-out call. Then the striking price portion of the payoff of these calls has the same value as an otherwise identical "out" cash-or-nothing call. It may seem surprising that an up-and-out cash-or-nothing call should be worth nothing when the striking price is greater than the barrier. But it is easy to see why. Since \( S < H < K \), in order for the underlying asset to end up above \( K \), it must first breach the barrier \( H \), but in this event, the call is extinguished.

An "out" asset-or-nothing call has the same value as the asset portion of the payoff of an otherwise identical down-and-out or up-and-out call.

Options 25-28 have positive payoffs requiring not only that the barrier (\( H \)) not be breached but also that the underlying asset finish below a given level (\( K \)).

Suppose \( X \) were equal to the striking price of a down-and-out or up-and-out put. Then the striking price portion of the payoff of these puts has the same value as an otherwise identical "out" cash-or-nothing put.
(27) down-and-out asset-or-nothing put: 
\[ (S > H) \]
\[
\text{payoff: } S(t) \text{ (at expiry) if } S(t) < K \text{ and for all } \tau \leq t, \ S(\tau) > H \\
0 \text{ if } S(t) < K \text{ or for some } \tau \leq t, \ S(\tau) \leq H \\
\text{value (K>H): } [1A] - [2A] + [3A] - [4A] \ \{\eta = 1, \ \phi = -1\} \\
\text{value (K<H): } 0
\]

(28) up-and-out asset-or-nothing put: 
\[ (S < H) \]
\[
\text{payoff: } S(t) \text{ (at expiry) if } S(t) < K \text{ and for all } \tau \leq t, \ S(\tau) < H \\
0 \text{ if } S(t) < K \text{ or for some } \tau \leq t, \ S(\tau) \geq H \\
\text{value (K>H): } [2A] - [4A] \ \{\eta = -1, \ \phi = -1\} \\
\text{value (K<H): } [1A] - [3A] \ \{\eta = -1, \ \phi = -1\}
\]

An "out" asset-or-nothing put has the same value as the asset portion of the payoff of an otherwise identical down-and-out or up-and-out call.

Just as there are "parity" relations between our results 5-8 and 9-12, so too, can we find equalities which link sums of the values of options 12-28 with the values of less complex securities. For example, a portfolio consisting of an "in" option from the list 13-20 and its "out" complement from 21-28 is equivalent to one of the path-independent binaries we discussed earlier. Alternatively, an "in" call from 13-16 together with the corresponding put from 17-20 comprise a position identical to one of the barrier-only "in" options from 5-8. Similar results link positions constructed from the "out" calls and puts 21-24 and 25-28 with barrier-only "out" options from 9-12.

We may also combine options 13-28 in ways which yield more familiar securities. Each of the barrier options examined in our previous article may be built from an asset-or-nothing option from this list and a corresponding cash-or-nothing with \(X\) set equal to \(K\). For example, a down-and-in call may be synthesized from a long asset-or-nothing down-and-in call and a short cash-or-nothing down-and-in call with the same striking price and with \(X = K\).

Finally, we can construct barrier- and strike-dependent gap options and supershares analogous to those presented in section I for for path-dependent conditions. However, perhaps ...
### 91.00 DIG: ALL-OR-NOTHING EUROPEAN OPTIONS (LOGNORMAL)

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<tr>
<th>Call/P Strike</th>
<th>Cash/Aset = Cash</th>
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<td>50.94</td>
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<td>51.23</td>
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<td>105</td>
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### 92.00 GAP: GAP EUROPEAN OPTIONS (LOGNORMAL)

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### 93.00 SSE: SUPERSHARES (LOGNORMAL)

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### 65.00 BIB: BINARY BARRIER OPTIONS (LOGNORMAL)

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#### 65.00 BIB: BINARY BARRIER OPTIONS (LOGNORMAL)

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#### 65.00 BIB: BINARY BARRIER OPTIONS (LOGNORMAL)

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Lookback Options

by
Mark Rubinstein
March 3, 1991

The payoff of a standard European option only depends on the price of the underlying asset on the expiration date. In particular, given the final price of the underlying asset, the payoff will be the same regardless of the path taken by the underlying asset during the life of the option to reach that final price. Whether the underlying asset price reaches a given price by first moving down and then up, or up and then down, matters not to the buyer or seller of the option. It is as if did not matter whether you travelled from Paris to London by air or by chunnel, as long as you arrived in London at the appointed time.

The terminology used to describe this feature is "path-independence". More generally, the payoff from an option may depend on some aspect of the price path. For example, the payoff of an "Asian" option depends on the average price experienced over time of the underlying asset, and the payoff a barrier option depends on whether or not the underlying asset has previously reached some other barrier price during its life. In this article, we will examine still another type of path-dependent option -- lookback options -- where the payoff depends not only possibly on the price at expiration but also on the minimum or maximum price experienced by the underlying asset during at least some portion of the life of the option.

In this essay, our objective is to value a variety of these options in a Black-Scholes environment; that is,

(1) where the underlying asset return can be assumed to follow a lognormal random walk, and

(2) where arbitrage arguments allow us to use a risk-neutral valuation approach -- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

Lookback options have payoffs:

\[
\text{call: } \max[0, S_n - \min(S_0, S_1, \ldots, S_n)] \\
\text{put: } \max[0, \max(S_0, S_1, \ldots, S_n) - S_n]
\]

where \(S_0, S_1, \ldots, S_n\) is the sequence of sequentially observed prices of the underlying asset observed over the life of the option, with \(S_0\) being the price at inception and \(S_n\) being the price at the expiration time. The Black-Scholes environment valuation formula for a lookback call is (where prices are observed continuously):

\[
C = Sd^\lambda - Mr^\lambda N[(b + \mu t)/\sigma \sqrt{t}] + Mr^\lambda e^{(\lambda - 1)/\lambda} N[(-b - \mu t)/\sigma \sqrt{t}] - Sd^\lambda N[(b - \mu t - \sigma^2 t)/\sigma \sqrt{t}]
\]

---

1 Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates.

2 The original paper on these options is by Barry Goldman, Howard Sosin and Mary Ann Gatto. "Path Dependent Options: Buy at the Low, Sell at the High," Journal of Finance, December 1979. The results of this paper were generalized to include a non-zero payout rate on the underlying asset (or a non-zero foreign interest rate in a currency option context) by Mark Garman in "Recollection in Tranquility," Risk Magazine, March 1989.
where $b = \log(S/M)$, $\mu = \log(r/d) - \frac{1}{2}\sigma^2$, and $\lambda = \frac{1}{2}\sigma^2/\log(r/d)$

All variables are interpreted as before with the addition that $M_S$ is the current minimum price that the underlying asset has experienced so far during the life of the option. To value a lookback put, use the same formula, but reverse the signs of the arguments of $N$, reverse the sign of the resulting formula value, and reinterpret $M_S$ as the current maximum price the underlying asset has experienced so far during the life of the option.

The valuation of a lookback option is simplified by the following trick: in a certain sense a lookback is not really an option since it will always pay to exercise a lookback option at expiration. To see this for a lookback call, observe that

$$\max[0, S_n - \min(S_0, S_1, \ldots, S_n)] = S_n - \min(S_0, S_1, \ldots, S_n)$$

Unlike standard options, lookback puts can have positive deltas since the best that could happen to the buyer of a lookback put would be for the underlying asset price first to establish at new high and then fall precipitously to end up quite low by expiration.
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<thead>
<tr>
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"Asian" Options

by

Mark Rubinstein

March 3, 1991

The payoff of a standard European option only depends on the price of the underlying asset on the expiration date. In particular, given the final price of the underlying asset, the payoff will be the same regardless of the path taken by the underlying asset during the life of the option to reach that final price. Whether the underlying asset price reaches a given price by first moving down and then up, or up and then down, matters not to the buyer or seller of the option. It is as if did not matter whether you travelled from Paris to London by air or by chunnel, as long as you arrived in London at the appointed time.

The terminology used to describe this feature is "path-independence". More generally, the payoff from an option may depend on some aspect of the price path. For example, the payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during the life of the option, and the payoff a barrier option depends on whether or not the underlying asset has previously reached some other barrier price during its life. In this article, we will examine still another type of path-dependent option -- "Asian" options -- where the payoff depends not only possibly on the price at expiration of the underlying asset but also on the average price experienced by the underlying asset during at least some portion of the life of the option.

In this essay, our objective is to value a variety of these options in a Black-Scholes environment\(^2\); that is,

1. Where the underlying asset return can be assumed to follow a lognormal random walk,

2. Where arbitrage arguments allow us to use a risk-neutral valuation approach --- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

We will consider two types of averages: geometric and arithmetic. While almost all traded Asian options are based on arithmetic averages, geometric average options are of interest because (unlike the arithmetic case) they have closed-form solutions and because these solutions can be used to approximate the values of arithmetic average options.

Average price calls and puts. The payoff of an arithmetic average price option is:

---

\(^1\) Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates.

\(^2\) While there are now several papers available dealing with average price options, the first paper to my knowledge to provide a reasonably efficient solution algorithm is by A.G.Z. Kemna and A.C.F. Vorst, "Options on Average Asset Values," Faculty of Economics working paper, Erasmus University, Rotterdam, 1987. This paper uses a control variate Monte-Carlo numerical technique to value average price options first applied to problems in financial economics by Phelim Boyle in "Options: A Monte-Carlo Approach," Journal of Financial Economics, May 1977. For a more recent paper and another approach to valuation than that considered here, see Edmond Levy, "The Valuation of Average Rate Currency Options," Midland Montagu, London, April 1990. My paper extends the results of these papers to permit an averaging period which can begin some time after the current time (therefore only covering a subset of the life of the option), or an averaging period which has begun before the current date (to allow the option to be valued midway through its life), and a discrete (non-continuous) underlying asset price sampling frequency.
\[ \max\{0, \phi[\Sigma S_i/(n+1)] - \phi K\} \]

where \( S_i \), for \( i = 0, 1, 2, ..., n \), is a sequence of sequentially observed prices of the underlying asset observed over a prespecified time period which may be coincident with, be a subset of, or overlap the life of the options, with observations sampled at prespecified time intervals. Thus, \( \Sigma S_i/(n+1) \) is the arithmetic average price of the underlying asset observed over the prespecified observation interval and observation frequency. \( \phi \) is a binary variable set to 1 for a call and -1 for a put.

Average price options are sometimes used to reduce incentives for manipulation of the underlying price at expiration. They are also useful for investors who are obligated to buy the same type of asset frequently over a known horizon. For example, options on the average price can be used to cap the total costs of these purchases. Moreover, these options will be cheaper than the purchase of a series of standard options, each maturing at successive periods over the horizon. Capping each transaction in this manner would cost the present value of \( \Sigma \min(S_i, K) \) while purchasing \( n \) average price options would cost the present value of \( \min(\Sigma S_i, nK) \) -- substantially less, reminiscent of the fact that an option on a portfolio will be less costly than a portfolio of options.

To this point, although it may be possible, no one has published a "closed-form" formula for valuing these options. Instead, researchers have resorted to numerical methods. We will employ the Monte-Carlo control variate method first applied to problems in financial economics by Phelim Boyle.

The logic behind the control variate approach is as follows. Say you are trying to estimate \( A^* \) by Monte-Carlo simulation. Say you have a prior reasonable estimate \( B^* \). Let \( A \) be a single value for \( A^* \) from one Monte-Carlo path (assumed to be an unbiased estimate of \( A^* \)) and let \( B \) be a single value for \( B^* \) along the same Monte-Carlo path (assumed to be an unbiased estimate of \( B^* \)). Unbiasedness means that \( A^* = E(A) \) and \( B^* = E(B) \). If \( A \) and \( B \) are sufficiently correlated, then a good way to estimate \( A^* \) is to use the equation:

\[ A^* = B^* + (A - B) \]

where \( A^* \) is our estimate of \( A^* \). Over an infinite number of Monte-Carlo paths, we will have:

\[ A^* = B^* + E(A - B) = B^* + E(A) - E(B) = E(A) = A^* \]

Of course, with an infinite number of Monte-Carlo paths, the control variate approach will not be an improvement (but it won't hurt either). But over a finite number of paths, it can be a considerable improvement. To see this, note that

\[ \text{var}(A^*) = \text{var}(A) + \text{var}(B) - 2 \text{cov}(A,B) \]

Our method will be an improvement if this is less than \( \text{var}(A) \) -- which we would get without the control variate approach. Now \( \text{var}(A^*) \) will be less than \( \text{var}(A) \) whenever

\[ \text{cov}(A,B) > \frac{1}{2} \text{var}(B) \]

which will be true whenever

\[ \rho(A,B) > \frac{1}{2} \frac{\text{std}(B)}{\text{std}(A)}. \]

In other words if the correlation between \( A \) and \( B \) is sufficiently high, we will have an improvement. Intuitively, whenever the errors in \( A \) (from \( A^* \)) and \( B \) (from \( B^* \)) are highly enough correlated, subtracting...
B from A, tends to leave the true difference between B* and A*.

A useful control variate (B) is the value of an otherwise identical option on the geometric average of the underlying asset prices observed over the same interval and sampled at the same frequency \((t_1, S_1)^{1/(n+1)}\). In contrast, to the arithmetic case, a Black-Scholes environment closed-form solution is easy to derive since the geometric average will be lognormally distributed if each of the observed underlying asset prices is lognormally distributed. Such a property is not available for the arithmetic case because a sum of lognormally distributed random variables is not lognormally distributed.

It is useful to distinguish among three time intervals:

- \(t_0\) = the time before averaging begins
- \(t_1\) = the time since averaging has begun
- \(t_2\) = the time remaining for averaging

There are three cases to consider:

- \(t_0 = t_1 = 0\): averaging starts at the current date
- \(t_0 > 0, t_1 = 0\): averaging begins after the current date
- \(t_0 = 0, t_1 > 0\): averaging begins before the current date

In every case the time-to-expiration will be \(t = t_0 + t_2\) (since it is assumed that the option always expires simultaneously with the end of the averaging period). Let \(h\) be the observation frequency. The Black-Scholes environment formula for the value of a call on the geometric average price is:

\[
C = \text{Xr}^e^{M+\Sigma^2}N(x) - Kr^e^{-\Sigma}N(x - \Sigma) \quad \text{where} \quad x = \left[\log(X/K) + M\right] + \Sigma
\]

where

\[M = [t_0 + t_2(t_2 + h) - 2(t_1 + t_2 + h)] \times \left[\log(r/d) - \frac{1}{2}\sigma^2\right]\]

\[\Sigma^2 = [t_0 + t_2(t_2 + h)(2t_2 + h) \times 6(t_1 + t_2 + h)^2] \cdot \sigma^2\]

Here, \(A\) is the current geometric average of the underlying asset price so far during the averaging period. If the averaging period is yet to start, then \(A = 1\). In the special case where sampling is continuous (\(h = 0\)) and the averaging period and the remaining life of the option coincide \((t_0 = t_1 = 0\) so that \(t = t_2\)), this formula simplifies to:

\[
C = \text{Sd}^w r^e^{-\sigma^2/12}N[x] - Kr^e^{-\sigma^2/3}\left[\log(Sd^w/Kr^w) + \sigma^2/(t/3)\right]
\]

where \(x = [\log(Sd^w/Kr^w)] + \sigma^2/(t/3)\) + \(1/4\sigma^2/(t/3)\)

The value of an average price call will always be less than the value of an otherwise identical standard call because an average price option behaves as if it has a shortened time to expiration.

One interesting complication surrounding average price (and average strike) options is that the derivative of the option value with respect to time (the “theta” of the option) must take into account the change in the averaging period as well as the remaining time-to-expiration of the option.

---

3 This formulation also defines \(A\) to exclude the current underlying asset price if the averaging period has already begun.
Average strike puts and calls. The payoff of an arithmetic average strike option is:

$$\max\{0, \phi S_n - \phi[\Sigma S_i/(n+1)]\}$$

Average strike options can be used by a seller to place a floor on the selling price of a sequence of sales of the same type of asset over a known horizon. The cost of an average strike option will be cheaper than trying to achieve the same floor by buying a sequence of standard options. Certain dividend reinvestment plans and employee stock purchase plans can be interpreted as average strike options. The benefits from employee pension plans can often be fruitfully analyzed as an average strike option. The contributions to such a plan are often a proportion of salary so that the total contribution depends on the entire salary history of the employee. On the other hand, the distribution at retirement is often a fixed percentage of the terminal salary level. To account for this type of payoff one need only modify the payoff function for an average strike call to:

$$\max\{0, \alpha S_n - [\Sigma \beta_i S_i/(n+1)]\}$$

where $\alpha$ is the percentage of the terminal salary received at retirement and the $\beta_i$ are the proportions of annual salary contributed to the pension plan brought forward to terminal value at the interest rate.

Again, no closed-form solution has yet been published to value this option; and again the Monte-Carlo control variate approach will be used. The control variate will be the value of an otherwise identical geometric average strike option. In this case we are interested in the joint distribution function of the value of the underlying asset at expiration $S_n$ and the geometric average underlying asset price $(\Pi, S)_{1/(n+1)}$. This might seem to be a more difficult problem than valuing an average price option where we are concerned with just one random variable. However, thanks to a valuation trick invented by William Margrabe, the valuation problem can be converted into one involving a single random variable -- the ratio of $S_n$ to $(\Pi, S)_{1/(n+1)}$. Since this ratio is itself a single lognormally distributed random variable, the same solution technique use to value average price options can be used to value average strike options.

We reproduce here only the special case for the geometric average strike call where the sampling is continuous ($h=0$) and the averaging period and the remaining life of the call coincide ($t_0 = t_1 = 0$ so that $t = t_2$). 

$$C = Sd^{-1}\{N[x] - (d/r)w e^{\sigma^2/12}N[x - \sigma^{2/3}(t/3)]\}$$

where $x = [\log(r/d)^{1/2} + 14\sigma^2] + \sigma^{2/3}(t/3)$

---

4 Eric Reiner has brought to my attention what is probably the first paper containing valuation results for average strike options - an unpublished working paper by Phelim Boyle and David Emanuel titled "Mean Dependent Options," August 1985.


6 I want to thank Eric Reiner for pointing out an error in my original derivation of this formula.
### 41.00 GAR: GEOMETRIC AVERAGE PRICE EUROPEAN OPTIONS

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### 43.00 AAR: ARITHMETIC AVERAGE PRICE EUROPEAN OPTIONS

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### 51.00 GAK: GEOMETRIC AVERAGE STRIKE EUROPEAN OPTIONS

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### 53.00 AAK: ARITHMETIC AVERAGE STRIKE EUROPEAN OPTIONS

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Exchange Options

by

Mark Rubinstein¹
June 7, 1991

Options to exchange one asset for another ("exchange options") arise quite naturally in a number of significant financial arrangements including bond futures contracts, investment performance incentive fees, dual-currency option bonds, options whose striking price is an average of the experienced underlying asset price during the life of the option, margin accounts, mortgage standby commitments, stock tender offers, etc.²

To represent the payoff at expiration from an exchange option, let:

\[
S_1^* = \text{current (final) price of underlying asset 1} \\
S_2^* = \text{current (final) price of underlying asset 2} \\
C^* = \text{current (final) price of exchange option} \\
C^* = \max(0, S_2^* - S_1^*)
\]

Exchange options can either be interpreted as a call on asset 2 with a striking price equal to the future value of asset 1, or a put on asset 1 with a striking price equal to the future value of asset 2.

From this, it is easy to see that a European option on the worse performing or better performing of two underlying assets can also be valued in terms of an exchange option since:

\[
\begin{align*}
\min(S_1^*, S_2^*) &= S_2^* - \max(0, S_2^* - S_1^*) \\
\max(S_1^*, S_2^*) &= S_1^* + \max(0, S_2^* - S_1^*)
\end{align*}
\]

For example, the present value of a European option delivering the worse performing of two underlying assets can be decomposed into the present value of the second asset minus the present value of an option to exchange the first asset for the second asset.

Margarbe's "Black-Scholes" formula for an exchange option (including payouts) can be written as:

\[
C = S_2 d_2 N(x) - S_1 d_1 N(x - \Sigma \sqrt{t}) \text{ where } x = \left[ \log(S_2 d_2^* / S_1 d_1^*) + \Sigma \sqrt{t} \right] + \frac{1}{2} \Sigma \sqrt{t} \text{ and } \Sigma^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2
\]

Here, \(S_1\) and \(S_2\) are the current underlying asset prices, \(t\) is the time-to-expiration, \(d_1\) and \(d_2\) are the underlying asset respective payout rates, \(\sigma_1\) and \(\sigma_2\) are their respective volatilities, \(\rho\) is the

¹ Mark Rubinstein is a professor of finance at the University of California at Berkeley and a principal of Leland O'Brien Rubinstein Associates. A version of this essay, under the title "One for Another," has appeared in RISK, July-August 1991.

² Several of these applications were developed in William Margrabe's original article "The Value of an Option to Exchange One Asset for Another," Journal of Finance, March 1978.
correlation coefficient of the natural logarithms of one plus their rates of return, \( t \) is the time-to-expiration, and \( N(a) \) is the area under a normal distribution function from \(-\infty\) to \( a \).

This article provides a previously unpublished binomial derivation of this formula. As in the case of standard calls and puts, the binomial approach clarifies the intuitive economic intuition behind Margrabe’s formula and can be used to handle American exchange options as well.\(^3\)

At first, it might appear a binomial approach will not work since after one period there are four potential outcomes: assets 1 and 2 both up, assets 1 and 2 both down, asset 1 up and 2 down, asset 1 down and 2 up. We know that in the limit of such a process, as the time interval between binomial moves goes to zero, we want the distribution of the two random variables to be jointly lognormal. As a result, the ratio of the two variables will be univariate lognormal. This suggests the first trick which is to model this ratio as univariate binomial. Therefore, restate the payoff as:

\[
C^* = S_1^* \max[0, (S_2^*/S_1^*) - 1]
\]

In a two-period setting, assume the relative price of the two assets moves binomially each period as follows:

\[
\begin{align*}
S_2/S_1 & \quad \begin{cases} 
\text{uu}(S_2/S_1) \\
\text{ud}(S_2/S_1)
\end{cases} \\
\text{d}(S_2/S_1) & \quad \begin{cases} 
\text{dd}(S_2/S_1)
\end{cases}
\end{align*}
\]

where \( u \) and \( d \) are the possible one plus the rates of growth of the relative prices of the two assets over each binomial period after payouts.

Of course, the relative price paths \( \text{ud} \) and \( \text{du} \) will end up at the same point, but our subsequent argument will be clearer if we temporarily keep these two paths distinct.

In the last period, let us examine what we would do if the relative price had moved up in the first period. Since \( S_{u2}/S_{u1} = \text{uu}(S_2/S_1) \) and \( S_{u2}/S_{d1} = \text{ud}(S_2/S_1) \):

\[
\begin{align*}
C_{uu} &= S_{u1} \max[0, \text{uu}(S_2/S_1) - 1] \\
C_{ud} &= S_{d1} \max[0, \text{ud}(S_2/S_1) - 1]
\end{align*}
\]

It will be convenient to define:

\[
\begin{align*}
X_{uu} &= \max[0, \text{uu}(S_2/S_1) - 1] \quad (1) \\
X_{ud} &= \max[0, \text{ud}(S_2/S_1) - 1] \quad (2)
\end{align*}
\]

\(^3\) See the associated essay in this series, "Standard Options," for a development of the binomial method for analyzing standard options.
so that $C_{uu} = S_{uu1}X_{uu}$ and $C_{ud} = S_{ud1}X_{ud}$.

This appears more complicated than the binomial argument for a standard option because the underlying asset prices, $S_{uu1}$ and $S_{ud1}$, are themselves random variables. For example, $S_{uu1}$ stands for the price of underlying asset 1 conditional on the path of the relative prices being up the first period and up the second period. But there are an infinite number of pairs of absolute levels of $S_{uu1}$ and $S_{ud1}$ that could result in these relative moves. For example, suppose $S_1 = S_2 = 100$ and $u = 1.2$. One way for path $uu$ to evolve would be for the two assets to follow

$$(100, 100) \rightarrow (120, 100) \rightarrow (144, 100).$$

But another possibility would be

$$(100, 100) \rightarrow (240, 200) \rightarrow (72, 50).$$

Furthermore, since $S_{uu1}$ and $S_{ud1}$ are random variables, then so too are $C_{uu}$ and $C_{ud}$. This might worry us - but, let's wait and see if this will turn out to create a problem.

Proceeding in a manner similar to the standard binomial argument, let $\Delta_{u1}$ and $\Delta_{u2}$ be the numbers of units of underlying assets one and two held in the replicating portfolio during the second period if the underlying price relative moved up during the first period. Then,

$$C_{uu} = S_{uu1}\delta_1\Delta_{u1} + S_{uu2}\delta_2\Delta_{u2} = S_{uu1}\delta_1\Delta_{u1} + S_{uu1}\delta_2 u(S_2/S_1)\Delta_{u2}$$

$$C_{ud} = S_{ud1}\delta_1\Delta_{u1} + S_{ud2}\delta_2\Delta_{u2} = S_{ud1}\delta_1\Delta_{u1} + S_{ud1}\delta_2 u(S_2/S_1)\Delta_{u2}$$

where $\delta_1$ ($\delta_2$) is one plus the payout rate of asset one (two) over each binomial period. Again, this seems to be more complex than the binomial procedure for standard options because the deltas ($\Delta_{u1}$ and $\Delta_{u2}$) each seem to be random variables since they depend on the absolute levels of the prices of the two underlying assets. However, this is not the case. Indeed, in the next step we will see that the deltas only depend on the relative prices of the underlying assets. To see this, combine the above two sets of two equations as follows:

$$C_{uu} = S_{uu1}\{\delta_1\Delta_{u1} + uu\delta_2(S_2/S_1)\Delta_{u2}\} = S_{uu1}X_{uu}$$

$$C_{ud} = S_{ud1}\{\delta_1\Delta_{u1} + ud\delta_2(S_2/S_1)\Delta_{u2}\} = S_{ud1}X_{ud}$$

This gets the absolute levels of the two asset prices to cancel out, so that:

$$\delta_1\Delta_{u1} + uu\delta_2(S_2/S_1)\Delta_{u2} = X_{uu}$$

$$\delta_1\Delta_{u1} + ud\delta_2(S_2/S_1)\Delta_{u2} = X_{ud}$$

Solving these simultaneous equations for $\Delta_{u1}$ and $\Delta_{u2}$:

$$\Delta_{u1} = (uX_{ud} - dX_{uu})/\delta_1(u-d)$$

$$\Delta_{u2} = (X_{uu} - X_{ud})/[u(u-d)\delta_2(S_2/S_1)]$$

This proves that the deltas only depend on relative underlying asset prices, not the absolute levels of these prices.

Considering the potential of early exercise at the end of period one and to prevent riskless arbitrage we
must also have:

\[ C_u = \max\{[S_{u2} - S_u], [S_{u1} \Delta_{u1} + S_{u2} \Delta_{u2}]\} \]

Substituting for \( \Delta_{u1} \) and \( \Delta_{u2} \), a little algebra shows that:

\[ C_u = S_{u1} \max\{[u(S_2/S_1) - 1], [pX_{u1} + (1-p)X_{u2}]/\delta_1\} \quad \text{where} \quad p = ((\delta_1/\delta_2) - d)/(u - d) \]

Similarly, when instead the underlying asset falls during the first period:

\[ C_d = S_{d1} \max\{[d(S_2/S_1) - 1], [pX_{d1} + (1-p)X_{d2}]/\delta_1\} \]

\[ X_{u1} = \max[0, du(S_2/S_1) - 1] \quad (3) \]

\[ X_{d1} = \max[0, dd(S_2/S_1) - 1] \quad (4) \]

Thus, the decision to exercise the option depends only on relative not absolute prices.

To value the option at the beginning of the first period, as before define:

\[ X_u = \max\{[u(S_2/S_1) - 1], [pX_{u1} + (1-p)X_{u2}]/\delta_1\} \quad (5) \]

\[ X_d = \max\{[d(S_2/S_1) - 1], [pX_{d1} + (1-p)X_{d2}]/\delta_1\} \quad (6) \]

so that \( C_u = S_{u1}X_u \) and \( C_d = S_{d1}X_d \). A similar analysis shows that

\[ C = S_1 \max\{[(S_2/S_1) - 1], [pX_u + (1-p)X_d]/\delta_1\} \quad (7) \]

This proves that one can value an American exchange option binomially by simply working backwards on the binomial tree of relative prices, using only the current price \( S_1 \) at the last step. In particular, with two periods, work backwards solving equations (1-4), then equations (5-6), and finally equation (7).

The binomial argument for options to exchange one asset for another is formally equivalent to the binomial argument for standard calls except that:

relative prices replace underlying asset prices

\( \delta_1 \) replaces the interest rate

\( \delta_2 \) replaces the payout rate of the underlying asset

1 replaces the striking price

\( \Sigma \) (volatility of \((S_2^-/S_1^-)\)/\(\sqrt{t}\)) replaces the the volatility of the underlying asset

\( C/S_1 \) replaces the resulting current call value

Since the binomial formula for a standard European call can lead to the Black-Scholes formula in the limit of continuous time, to derive the Black-Scholes environment formula for an exchange option, we need only make these same substitutions in the Black-Scholes formula for a standard call option.
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Currency-Translated Options

by Eric Reiner
January 15, 1992

In their 1973 paper, Fischer Black and Myron Scholes showed that options on equities could be valued exactly through an ingenious dynamic hedging argument: an option writer could, in theory, exactly offset his exposure to the underlying asset by continuously buying and selling shares of stock. Since the overall delta and, consequently, the net risk of his portfolio are always zero, the value of his holdings must appreciate at the riskless interest rate. This observation leads naturally to the risk-neutral pricing principle enunciated by John Cox and Stephen Ross: derivative assets may be valued as if the expected rate of return of the underlying asset were equal to the riskless rate. In particular, the value of a European option is simply the discounted present value of its payoff under the risk-neutral probability distribution for the stock price at expiration. Assuming that the instantaneous standard deviation of an asset's return is proportional to its price so that the risk-neutral density is lognormal, Black and Scholes found the value of a European call with payoff

\[ C^* = \max[0, S^* - K] \]

to be:

\[ C = Sd^tN(x) - Kr^tN(x - \sigma_s\sqrt{t}), \]

with \[ x = \log(Sd^t/Kr^t) + \sigma_s\sqrt{t} \]

In these expressions, \( S \) is the current price of the underlying asset, \( t \) is the time to expiration of the option, \( K \) is its striking price, \( r \) is one plus the annualized rate of interest, \( d \) is one plus the annualized proportional payout rate of the underlying asset, \( \sigma_s \) is the underlying asset annualized price volatility, and \( N(\cdot) \) is the standard normal distribution function. The value of an otherwise identical put may be obtained by negating the arguments of the normal functions and then negating the entire expression for \( C \). The option writer's replicating hedge portfolio consists of

\[ \Delta_s = d^tN(x) \]

shares of stock\(^2\) and a (short) cash position of

\[ B = -Kr^tN(x - \sigma_s\sqrt{t}). \]

In a foreign exchange context, Mark Garman and Steven Kohlhagen\(^4\) have shown that similar arguments may be applied to value options on currencies, but with the foreign interest rate replacing the dividend yield. Their formula for the value of a European call on a foreign currency with payoff

\(^1\) Eric Reiner is a senior vice-president of Leland O'Brien Rubinstein Associates. He thanks Mark Rubinstein for several helpful comments on a previous draft of this essay. A version of this essay has appeared under the title "Quanto Mechanics," in RISK 5 (March 1992), pp. 59-63.

\(^2\) In this essay, the values of random variables at the expiration of an option are indicated by a superscript \( \cdot \), and foreign-denominated values will be denoted by a prime \( \cdot' \).

\(^3\) To simplify this discussion, we assume that option writers hedge in the spot market and that they borrow and lend both domestic and foreign cash at the corresponding riskless rates.

\[ C' = \max[0, X' - K] \]
\[ C = Xr_tN(y) - KxN(y - \sigma_x\sqrt{t}), \]

with \( y = [\log(Xr_t/K) + \sigma_x\sqrt{t}] + 1/2\sigma_x\sqrt{t}. \)

Here, \( X \) is the spot exchange rate expressed as the price in the domestic currency of a unit of foreign currency, \( r_t \) is one plus the annualized foreign riskless rate, and \( \sigma_x \) is the annualized exchange rate volatility. The corresponding replicating portfolio is

units of foreign currency and
\[ B' = r_tN(y) \]
\[ B = -Kr_tN(y - \sigma_x\sqrt{t}) \]

in domestic cash. Note that these expressions are identical to those obtained by Black and Scholes in an equity setting, but with \( r_t \) replacing \( d \) and \( \sigma_x \) replacing \( \sigma_s \).

I. Menu of Currency-Translated Options

In a global equity market, it is possible to link foreign equity and currency exposures in a variety of interesting ways: investors may choose to combine their investments in foreign equities with differing degrees of protection against adverse moves in exchange rates, equity prices, or combinations of these. Four scenarios, in roughly increasing order of complexity, and the payoffs that match them, are:

1. An investor wants to participate in gains in a foreign equity, desires protection against losses in that equity, but is unconcerned about the translation risk arising from a potential drop in the exchange rate. Such an investor might desire the payoff of a foreign equity call struck in foreign currency:
\[ C_1^* = X'max[0, S' - K'] \]

where \( S' \) is the equity price in its own currency after time \( t \) and \( K' \) is a foreign currency amount. In this formula, \( X' \) appears in front of the maximum function, indicating that the final payoff must be converted into domestic currency.

2. An investor wishes to receive any positive returns from the foreign market, but wants to be certain that those returns are meaningful when translated back into his own currency. For him, it is the product of the foreign asset price and the exchange rate at expiration that is important, and he might be interested in a payoff like that of a foreign equity call struck in domestic currency:
\[ C_2^* = \max[0, S'X' - K], \]

where \( K \) is now a domestic currency amount and \( X' \) multiplies \( S' \) only, representing translation of the foreign equity value into domestic terms.\(^5\)

3. An investor wants, as in the first scenario, to capture upside returns on his foreign equity investment, but

\(^5\) Payoffs 1 and 2 are two of the four possible call payoffs combining a foreign or domestic equity and a foreign or domestic striking price. The remaining two are \( \max[0, S' - K] \) (domestic equity/domestic strike) and \( \max[0, S' - K'X'] \) (domestic equity/foreign strike). The first of these is just the payoff corresponding to the Black-Scholes formula. We will obtain a valuation formula for the second (in a slightly disguised form) in the course of valuing payoff 2.
now desires to hedge away all exchange risk by fixing in advance a rate at which the payoff will be converted into domestic currency, effectively linking a foreign equity option with a currency forward. This investor’s desired payout pattern is that of a fixed exchange rate foreign equity call:  

\[ C^*_6 = X_0 \max[0, S^* - K'] = \max[0, S^*X_0 - K], \]

where \( X_0 \) is the rate at which the translation will be made. The two equivalent forms of the payoff arise from the choice of expressing the striking price in foreign or domestic terms.

4. An investor desires foreign equity exposure regardless of whether the foreign equity market rises or falls, but wishes to place a floor on the exchange rate component of his investment. This strategy, which combines a currency option with an equity forward to create a variable quantity FX option called an equity-linked foreign exchange call\(^7\), is the complement to the contract in scenario 3, and has the payoff:

\[ C^*_4 = S^* \max[0, X' - K]. \]

Since the payoffs of each of these options resemble closely those of standard equity and currency calls, and since the products of the (risk-neutrally) lognormal variables which appear in these payoffs are also lognormally distributed, it is not surprising that closed-form valuation formulas quite similar to the Black-Scholes equation may be derived for each contract. What is remarkable, sometimes even counterintuitive, is the way that market parameters -- interest and dividend rates, volatilities, and the correlation between the equity and currency price changes -- appear in some of these formulas. The remainder of this essay shows how these valuation results may be obtained.\(^8\)

II. Valuation and Replicating Portfolios

**Foreign equity call struck in foreign currency.** The first case, that of a foreign equity call with striking price \( K' \) also in foreign currency is particularly simple. The necessary observation is that \( X' \) plays an almost trivial role in the payoff: whatever payment in the foreign currency results from exercise of the option is just converted at the spot exchange rate at expiration. To make use of this point, consider an option writer located in the country where \( S' \) is traded. She is indifferent (in a frictionless market) between this call -- paying off in your currency -- and a standard option on \( S' \) paying off in her own. Accordingly, the present

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\(^6\) These options are also known as "quanto's," and are traded both in the over-the-counter market and on the American Stock Exchange.

\(^7\) Bankers Trust has offered such contracts under the name "Elf-X". Analogous contracts were originally discussed by Alan Marcus and David Modest in their paper "The Valuation of a Random Number of Put Options: An Application to Agricultural Price Supports," *Journal of Financial and Quantitative Analysis* 21 (March 1986), pp. 73-86.

\(^8\) In each of our four cases, the components of the payoff must be chosen so that the units of payment are consistent. For example, to a U.S. investor with an underlying British equity exposure these payoffs are:

- \( C^*_1 \) (\$/share) = \( X' \max[0, S^* \max(\$P/\text{share}) - K'] \)
- \( C^*_2 \) (\$/share) = \( \max[0, S^* \max(\$P/\text{share}) - K'] \)
- \( C^*_3 \) (\$/share) = \( X_0 \max[0, S^* \max(\$P/\text{share}) - K'] \)
- \( C^*_4 \) (\$/share) = \( S^* \max(\$P/\text{share}) \max[0, X' \max(\$P/\text{share}) - K'] \)

3
value in foreign units of the call is just given by the Black-Scholes formula. To obtain the domestic currency value of the option we need only invoke the law of one price, which requires that identical contracts have equivalent prices in all markets, to find (by multiplying by the spot exchange rate):

\[ C_t = S_t X d^+ N(x_t) - K_t X r_t^+ N(x_t - \sigma_s \sqrt{t}), \]

with \( x_t = \left[ \log(S_t d^+/K_t r_t^+) + \sigma_s \sqrt{t} \right] + \frac{1}{2} \sigma_s^2 t, \)

where \( \sigma_s \) is the volatility of \( S' \). This argument also allows us to identify easily the replicating portfolio as:

\[ \Delta_x = d^+ N(x_t), \]

shares of stock and

\[ B' = -K_t X r_t^+ N(x_t - \sigma_s \sqrt{t}), \]

units of foreign cash, identical to the holdings of the foreign-based option writer.

**Foreign equity call struck in domestic currency.** A very similar line of reasoning may be used to value the second type of foreign equity call, which is struck in domestic currency. Again we may adopt the viewpoint of the foreign-based option writer. To her, the payoff looks like:

\[ C_2' = \max(0, S' - KX'), \]

where \( X' = 1/X \) is the exchange rate quoted at the price of a unit of (our) domestic currency in terms of her (our foreign) currency. But this is merely the payoff of an option to exchange one asset (K units of our currency) for another (a share of stock), and she may easily find\(^9\) that this contract is worth:

\[ C_2' = S' d^+ N(x_2) - KX r^+ N(x_2 - \sigma_{x'} \sqrt{t}), \]

with \( x_2 = \left[ \log(S' d^+/KX r^+) + \sigma_{x'} \sqrt{t} \right] + \frac{1}{2} \sigma_{x'}^2 t \)

and \( \sigma_{x'} = \sqrt{\sigma_x^2 + \sigma_s^2 - 2p_{x', s} \sigma_x \sigma_s} \).

Here \( \sigma_x = \sigma_s \), \( \sigma_{x'} \) is the relative volatility of \( S' \) and \( X' \), and \( p_{x', s} \) is the correlation between the rate of return of \( S' \) and that of \( X' \). Again, we may multiply by the exchange rate and replace \( X' \) everywhere by \( 1/X \) to obtain the domestic value of this option. Only one step requires care: the rate of return for \( X' \) is exactly the negative of that for \( S \), so that the correlation coefficient \( p_{x, x'} \) between \( S' \) and \( X \) is equal to \( -p_{x, x'} \). The final valuation formula is:

\[ C_2 = S' X d^+ N(x_2) - K r^+ N(x_2 - \sigma_{x'} \sqrt{t}), \]

with \( x_2 = \left[ \log(S' X d^+/K r^+) + \sigma_{x'} \sqrt{t} \right] + \frac{1}{2} \sigma_{x'}^2 t \)

and \( \sigma_{x'} = \sqrt{\sigma_y^2 + \sigma_x^2 + 2p_{x', s} \sigma_y \sigma_x} = \sigma_{x'y} \).

Note that the differences between the arguments of the normal distribution functions are notational only, and

are introduced purely for convenience. This contract may be replicated with a current position of

\[ \Delta_v = d' \hat{N}(x_2) \]

shares of stock and

\[ B' = -Kr'd \hat{N}(x_2 - \sigma_{sx} \sqrt{t}) \]

in domestic cash.

Before proceeding to the more complicated valuation results for options 3 and 4, it is useful to study the formulas we have obtained for the simpler cases above. In particular, the results for option 2 are quite revealing. \( S' \) and \( X \) always appear together as a product in these expressions, replacing \( S \) in the Black-Scholes equation just as the product volatility \( \sigma_{sx} \) replaces \( \sigma_s \). Further, the foreign interest rate shows up nowhere, while the currency-translated stock price is discounted by \( d' \) only. Similarly, there is no foreign cash position in the replicating portfolio. It is as if the product \( S'X \) were itself a domestically traded equity with volatility \( \sigma_{sx} \) and a proportional payout rate \( d-1 \) and we could apply the risk-neutral pricing approach to value derivative assets on \( S'X \) as an underlying stock.

That this is the case is a consequence of Mark Garman’s relativity principle: the laws of finance are the same in all frames of reference, so that assets are valued identically in all markets. To see these results more directly, we can use the following argument. Suppose that the foreign equity were also traded domestically. Then, to preclude arbitrage, its price could only be \( S'X \) and its volatility would be that of the product of \( S' \) and \( X \), namely \( \sigma_{sx} \). Further, the proportional payout rate \( d-1 \) would be identical in both markets. We could then apply the Black-Scholes hedging argument to derive the result for \( C_2 \) given above.

The foregoing analysis allows us to develop two parallel sets of simple rules for valuing payoffs like \( C_1^* - C_2^* \) in a Black-Scholes environment:

**A. Domestic Market Method**

1. Replace \( S' \) by \( (S'X)^*/X' \) everywhere in the payoff function to express it in terms of the hypothetical domestically traded asset \( S'X \).

2. Value the resulting payoff under the risk-adjusted joint probability distribution for \( (S'X)^* \) and \( X' \), where \( S'X \) and \( X \) have yield rates \( d-1 \) and \( r-1 \), volatilities \( \sigma_{sx} \) and \( \sigma_s \) and correlation \( \rho_{sx,x} = (\sigma_s \rho_{sx} + \sigma_s)/\sigma_{sx} \).\(^{10}\)

3. Discount the payoff at the domestic interest rate \( r-1 \) to obtain \( C \), the present value of the option.

4. Evaluate the delta \( \Delta_x \) and the foreign cash position \( B' \) by taking partial derivatives of \( C \) with respect to \( S'X \) and \( X \), respectively. Determine the domestic cash position from:

\[ B = C - \Delta_x S'X - B'X. \]

**B. Foreign Market Method**

1. Divide the payoff by \( X' \) and substitute \( X^* = 1/X' \) everywhere to express it in terms of the reciprocal exchange rate \( X' \).

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\(^{10}\) This peculiar form for the correlation coefficient shows up because \( S'X \) is a composite of the price processes for \( S' \) and \( X \). One can show that the covariance of the return rates of \( S'X \) and \( X \) is \( \sigma_s (\sigma_s \rho_{sx} + \sigma_s) \), from which the expression for \( \rho_{sx,x} \) follows directly.
2. Value the resulting payoff under the risk-neutral joint probability distribution for $S'$ and $X'$, where $S'$ and $X'$ have yield rates $d-1$ and $r-1$, volatilities $\sigma_s$ and $\sigma_x$, and correlation $\rho_{sX} = \rho_{sX'}$.

3. Discount the payoff at the foreign interest rate $r_f$ to obtain $C'$, the present value of the option, in foreign currency terms.

4. Evaluate the delta $\Delta_x'$ and the domestic cash position $B$ by taking partial derivatives of $C'$ with respect to $S'$ and $X'$, respectively. Determine the foreign cash position from:

$$B' = C' - \Delta_s S' - BX'.$$

5. Convert $C'$ from foreign to domestic units by multiplying by $X$ and replacing $X'$ everywhere by $1/X$.

It is not too difficult to show that these procedures lead to identical results, but in practice it is often easier to apply one rather than the other. For example, the first option discussed here is most conveniently valued -- as shown above -- by approach B. In contrast, valuation of the second payoff is readily accomplished by procedure A.

**Fixed exchange rate foreign equity call.** Our remaining two options are best handled by the "foreign market" technique. To begin, payoff 3 can be rewritten in reciprocal units as:

$$C_2' = X_2 X'\max[0, S' - K']$$

Next, we express this payoff in the form

$$X_2 X'e^{r'}\max[0, S'e^{r'} - K']$$

where $u$ and $v$ are the natural logarithms of one plus the returns of $S'$ and $X'$, respectively. Then integrate its product with the risk-neutralized joint density for $u$ and $v$:

$$f(u,v) = \frac{1}{\sigma_s \sigma_x \sqrt{2\pi}} \exp\left[-\frac{(u - \mu_u)^2}{2\sigma_x^2} - \frac{(v - \mu_v)^2}{2\sigma_x^2}\right]$$

with $w = \left\{(u - \mu_u)^2/\sigma_x^2 + 2\rho_{SX}(u - \mu_u)(v - \mu_v)/\sigma_x^2\sigma_x^2 + (v - \mu_v)^2/\sigma_x^2\right\}/(1 - \rho_{SX}^2)$

with $\mu_u = \log(r_f/d) - 1/2\sigma_u^2$ and $\mu_x = \log(r_f/r) - 1/2\sigma_x^2$.

Note that the sign preceding $2\rho_{sx}$ is positive for the reasons discussed earlier. The region of integration may be restricted to $\log(K'/S') \leq u < \infty$ and $-\infty < v < \infty$, and the quadrature with respect to $v$ is readily performed to leave only exponential functions in $u$. The final expression for the option value in foreign currency becomes:

$$C_2' = X_2 X'\max\left\{(S'/(rd/r)^{t'})\exp(-\rho_{sx}\sigma_s \sigma_x t)N(x_3) - K'r'N(x_5 - \sigma_v \sqrt{t})\right\}$$

with $x_3 = \left\{\log(S'd'/K'r') - \rho_{sx}\sigma_s \sigma_x t + \sigma_v \sqrt{t}\right\} + 1/2\sigma_v^2 t$,

corresponding to a domestic value of:

$$C_3 = X_3\{S'(rd/r)^{t'}\exp(-\rho_{sx}\sigma_s \sigma_x t)N(x_3) - K'r'N(x_5 - \sigma_v \sqrt{t})\}$$

The curious appearance of this formula is not all that is remarkable, however: the replicating portfolio
\[ \Delta_x = (X_0/X)(rd/t)^s \exp(-\rho s \sigma_x \sigma_s t)N(x_s) \]

units of equity, \( B = C_0 \) in domestic currency, and \( B' = -\Delta_x S' \) in foreign cash!

How can we interpret these results? It is best to begin by understanding the relationships between the components of the replicating portfolio. To a domestic option writer, a position of \( \Delta_x \) units of the foreign equity creates a proportional currency exposure which must be exactly offset by \( B' \). The net value of these holdings is zero, so that the entire call price remains in cash. Alternatively, a foreign writer of this contract carries exchange rate risk equal to the present value of the option, which she must replicate with a FX exposure equal to \( C_0 \). Consequently, she must borrow in her own currency the total amount necessary to set up her hedge position in equity.

It is more interesting to explain the "discount factors" which appear before each of the normal integrals in the formula for the option value and which are combined in the expression for \( x_{0'} \). The second term in the formula represents the exercise probability-weighted present value of having to pay out the striking price, \( K'X_{0'} \). That this value is discounted at the domestic interest rate is not surprising, since it is just a domestic cash amount. The term proportional to the equity price is more challenging to understand. Let us first assume that currency and stock price movements are uncorrelated. Then, to account for the fact that the exchange rate is fixed, we must adjust the payoff for \( S' \) by a factor proportional to the accumulated interest rate differential over the life of the option. This introduces the factor \( (r/r_0) \). But, if the exchange rate and underlying equity are correlated, this adjustment is not sufficient. For example, if the correlation is positive, then increases in \( S' \) which raise the value of the underlying equity option will be partially offset by increases in \( X \) which decrease the value of the FX protection. This effect is accounted for by the exponential factor, which just corrects for the correlation between the returns of \( S' \) and \( X \). The relevant volatility in \( x_s \) is just that of the underlying equity option, namely \( \sigma_s \).

**Equity-linked foreign exchange call.** It will be helpful in the analysis of our fourth payoff to make use of the following property: the symmetry arguments which make it possible to transform the Black-Scholes formula for the value of a call into that of a put by changing a few signs are equally applicable to each of the results derived above, as well as to the formula we shall derive shortly for \( C_4' \). To see this, consider what must be done to transform a call payoff into that of a put. First, the sign of the payoff must be reversed, and this passes through directly to the expression for the present value. Second, the limits of integration over the log-return of one of the underlying assets must be changed from \( +\infty \) to the logarithm of the ratio of the striking price to the present value of the asset (for the upper limit) and from that logarithm to \( -\infty \) for the lower limit. This has the effect of negating the arguments of the normal distribution functions.

We can now utilize this property to value an equity-linked FX call. To begin, express the payoff in "foreign terms" as:

\[ C_4' = S' \max[0, 1 - KX'] = KS' \max[0, (1/K) - X'] \]

In this form the underlying call reappears as a put, reflecting the fact that a call on one currency struck in a second is equivalent to a put on the second times a factor equal to the call striking price. Our analysis can now be simplified if we recognize that this payoff is in perfect analogy to that for a fixed exchange rate foreign equity put -- for which we already have valuation results. We need only interchange \( S' \) and \( X' \) (and their respective volatilities and payout rates) and replace \( X_0 \) by \( K \) and \( K' \) by \( 1/K \) to find a foreign market value of:

\[ C_4' = S'd'N(x_s) - KS'X'(rd/t)^s \exp(-\rho s \sigma_x \sigma_s t)N(x_s - \sigma_x' t) \]
with \( x_s = \{ \log(X_t^r/K^r) + \rho_{s,x} \sigma_s \sigma_x t + \sigma_s \sqrt{t} \} + \frac{1}{2} \sigma_x^2 t. \)

corresponding to a domestic value of:

\[ C_d = S'Xd'N(x_d) - KS'(r_d/r_t)^t \exp(-\rho_{s,x} \sigma_s \sigma_x t) N(x_d - \sigma_x \sqrt{t}). \]

Here again, the replicating positions are unusual. The net equity exposure from this contract is just proportional to its value, so that \( \Delta_s = C_d/(S'X). \) The required foreign currency holdings may be found by differentiation to equal:

\[ B' = (KS'/X)(r_d/r_t)^t \exp(-\rho_{s,x} \sigma_s \sigma_x t) N(x_d - \sigma_x \sqrt{t}), \]

which must be exactly offset by domestic cash, so that \( B = -B'X. \)

III. Summary

Each of the valuation formulas derived in this essay may be obtained by modifying one or more of the parameters in the Black-Scholes equation. Consequently, our results are concisely summarized by the following table:

<table>
<thead>
<tr>
<th>Option Type</th>
<th>Corresponding Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black-Scholes (domestic equity)</td>
<td>S, K, r, d, ( \sigma_s )</td>
</tr>
<tr>
<td>Garman-Kohlhagen (currency)</td>
<td>X, K, r, ( r_t ), ( \sigma_x )</td>
</tr>
<tr>
<td>Foreign equity/foreign strike</td>
<td>S'X, K'X, ( r_t ), d, ( \sigma_{s'} )</td>
</tr>
<tr>
<td>Foreign equity/domestic strike</td>
<td>S'X, K, r, d, ( \sigma_{s-x} )</td>
</tr>
<tr>
<td>Fixed exchange rate foreign equity</td>
<td>S'X_0, K'X_0, ( r ), ( (r_d/r_t) \exp(\rho_{s-x} \sigma_{s-x}) ), ( \sigma_{s'} )</td>
</tr>
<tr>
<td>Equity-linked foreign exchange</td>
<td>S'X, KS', ( (r_d/r_t) \exp(\rho_{s-x} \sigma_{s-x} \sigma_x) ), d, ( \sigma_x )</td>
</tr>
</tbody>
</table>
Value: Foreign Equity Call Struck in Domestic Currency

Index=Strike=100, IntRate=.1, Vol=XVol=.2, Yld=.05, ExchRate=2
Value: Fixed Exchange-Rate Foreign Equity Call (Lognormal)

Index=Strike=100, Yrs=1, DInt=FInt=.1, Vol=XVol=.2, Exch=FExch=.2
Value: Equity-Linked Foreign-Exchange Call (Lognormal)

Index = 100, XStrike = 1.2, Vol = 2, XVol = 15, Dint = 1, FInt = 0.75
European Rainbow Options

by
Mark Rubinstein
July 11, 1991

Two-color rainbow options are here defined as options on two risky underlying assets which cannot by some
trick of reasoning be valued as if they were options on one underlying asset. In this essay, my objective
is to value a variety of these options in a Black-Scholes environment; that is:

(1) where the two underlying assets can be assumed to follow a jointly lognormal random
walk, and

(2) where arbitrage arguments allow us to use a risk-neutral valuation approach -- discount
the expected payoff of the options at expiration by the riskless interest rate, where both
underlying asset prices are expected to appreciate at the same riskless rate less payouts.

It will turn out that it will be necessary to value these options using either bivariate normal distribution
functions which lead to closed-form solutions, or integrals over bivariate normal density functions which
cannot be simplified.

I. Options with Closed-Form Solutions

The options literature contains closed-form valuation solutions for a class of two-color rainbow options.
Armed with the result for an option delivering the best or two risky assets and cash, we can write down
immediately the solutions to several others. Before discussing a previously unpublished numerical valuation
technique for more complex options, it will be useful to reexamine the cases where a closed-form solution
is possible.

Options delivering the best of two risky assets and cash. These options have the payoff:

$$\max[S_1^*, S_2^*, K]$$

where $S_1^*$ and $S_2^*$ are the prices of the underlying risky assets at expiration, and $K$ is a fixed amount
of cash potentially received at expiration.

---

1 Mark Rubinstein is professor of finance at the University of California at Berkeley and a principal of Leland O'Brian Rubinstein
Associates. A version of this essay, under the title "Somewhere Over the Rainbow," has appeared in RISK, November 1991. The
author thanks Eric Reiner for his careful reading of an earlier draft of this article.

2 For examples of these, see my article about options on the product of two asset prices which appeared as "Two in One," RISK,
May 1991, and my article about American exchange options which appeared as "One for Another," RISK, July-August 1991. In both
of these cases, closed-form European valuation formulas can be derived in terms only of univariate normal distribution functions.

3 The original papers deriving the valuation formulas for these options are by Rene Stulz, "Options on the Minimum or the
working paper, Louisiana State University, August 1981. The equation (A.6) in Stulz, Appendix 2, for a delta of the option, can be
simplified by omitting the second and third terms since their sum can be shown to be equal to 0. An extension to more than two
assets is contained in Herb Johnson, "Options on the Maximum or the Minimum of Several Assets," Journal of Financial and
Quantitative Analysis, September 1987. My essay provides both a more transparent proof as well as a slight generalization that
allows for payouts, or a foreign interest rate in a currency context.
The most straightforward Black-Scholes valuation technique is to calculate the expected payoff from the option in a "risk-neutral" world in which \( S_1 \) and \( S_2 \) are expected to appreciate at the interest rate less payouts and discount this back to the present at the riskless interest rate. That is,

\[
C = r^t E\left[ \max(S_1^*, S_2^*, K) \right]
\]

where \( r \) is one plus the rate of interest, \( t \) is the time-to-expiration, and \( E \) is an expectation operator. Writing this as a double integral:

\[
C = r^t \int \int \max[S, e^x, S, e^y, K]f(x,y)dx dy
\]

with both integrals taken over the region \(-\infty \) to \( \infty \) and

\[
x = \log(S_1^*/S_1) \quad \text{and} \quad y = \log(S_2^*/S_2)
\]

\[
f(x,y) = \frac{1}{\sqrt{2\pi}(1-p^2)\sigma_1 \sigma_2} e^{-\frac{u}{2}}
\]

with \( u = \left[ \frac{(x-\mu_1)^2}{\sigma_1^2} - 2p \frac{(x-\mu_1)(y-\mu_2)}{\sigma_1 \sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right] / (1-p^2) \)

\[
\mu_1 = \log(r/d_1) - \frac{1}{2}\sigma_1^2 \quad \text{and} \quad \mu_2 = \log(r/d_2) - \frac{1}{2}\sigma_2^2
\]

Here \( S_1 \) and \( S_2 \) are the current values of the two underlying assets, \( d_1 \) and \( d_2 \) are one plus the payout rates of the two underlying assets, \( \sigma_1 \) and \( \sigma_2 \) are the volatilities of the underlying assets, and \( p \) is the correlation of the natural logarithms of one plus the rates of return of the two underlying assets. Since \( S_1^* \) and \( S_2^* \) are jointly lognormal, \( x \) and \( y \) will be jointly normal, so that \( f(x,y) \) is the bivariate normal density function.

To evaluate this integral, it is first convenient to break the problem down into three components corresponding to the circumstances under which the payoff is received from each security:

1. \([1]\) \( r^t \) times the expected value of \( S_1^* \) given that \( S_1^* > K \) and \( S_2^* < S_2 \)

2. \([2]\) \( r^t \) times the expected value of \( S_2^* \) given that \( S_2^* > K \) and \( S_2^* < S_2 \)

3. \([3]\) \( r^t \) times \( K \) times the probability that \( K > S_1^* \) and \( K > S_2^* \)

The option value will be the sum of these three components; that is, \( C = [1] + [2] + [3] \).

To derive our valuation formula, we will need four density functions:

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} \quad \text{with} \quad v_1 = \frac{x - \mu_1}{\sigma_1}\sqrt{t}
\]

\[
f(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{y^2}{2\sigma_2^2}} \quad \text{with} \quad v_2 = \frac{y - \mu_2}{\sigma_2}\sqrt{t}
\]

\[
f(x|y) = \frac{1}{\sqrt{2\pi}(1-p^2)\sigma_1 \sigma_2} e^{-\frac{w_1}{2}}
\]

with \( w_1 = \left[ (x - \mu_1) - p(\sigma_1/\sigma_2)(y - \mu_2) \right] / (1-p^2)\sigma_1 \sqrt{t} \)

\[
f(y|x) = \frac{1}{\sqrt{2\pi}(1-p^2)\sigma_2 \sigma_1} e^{-\frac{w_2}{2}}
\]

2
with \( w_2 = \frac{((y - \mu_2^t) - \rho(\sigma_2/\sigma_1)(x - \mu_1))^2}{(1 - \rho^2)} \sigma_2^2 t \)

The first two are just normal densities, and the last two are conditional normal densities.

Using these densities, the three components can be written as:

\[
[1] = S_1 r^t \left[ \int f(y|x)dy \right] e^{r(t)} \left[ S_1 d_1^t \{ N[y_1] - N[-x_1, y_1; \rho_1] \} \right]
\]

with the exterior integral taken over the region \( \log(K/S_1) \to \infty \) and the interior integral taken over the region \( -\infty \to x - \log(S_2/S_1) \);

\[
[2] = S_2 r^t \left[ \int f(x|y)dx \right] e^{r(t)} \left[ S_2 d_2^t \{ N[y_2] - N[-x_2, y_2; \rho_2] \} \right]
\]

with the exterior integral taken over the region \( \log(K/S_2) \to \infty \) and the interior integral taken over the region \( -\infty \to y - \log(S_1/S_2) \);

\[
[3] = Kr^t \left[ \int f(y|x)dy \right] f(x) \left[ S_2 d_2^t \{ N[x_1 + \sigma_1 \sqrt{t}, -x_2 + \sigma_2 \sqrt{t}; \rho \} \right]
\]

with the exterior integral taken over the region \( -\infty \to \log(K/S_1) \) and the interior integral taken over the region \( -\infty \to \log(K/S_2) \)

and

\[
\begin{align*}
x_1 &= \left[ \log(S_1 d_1^t/\sqrt{K t}) + \sigma_1 \sqrt{t} \right] + \frac{1}{2} \sigma_1 \sqrt{t} \\
y_1 &= \left[ \log(S_1 d_1^t/\sqrt{K t}) + \sigma_1 \sqrt{t} \right] + \frac{1}{2} \Sigma \sqrt{t} \\
\Sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \\
\rho_1 &= (\rho \sigma_2^2 - \sigma_1^2) / \Sigma \\
\rho_2 &= (\rho \sigma_2^2 - \sigma_1^2) / \Sigma
\end{align*}
\]

where \( N(\cdot) \) is the standard normal distribution function and \( N_2(a, b; c) \) is the area under a bivariate standard normal distribution function from \( -\infty \to a, -\infty \to b \) with correlation \( c \).

**Calls and puts on the maximum of two risky assets.** These options have payoff:

\[
\max[0, \phi \max(S_1^t, S_2^t) - \phi K]
\]

where \( \phi \) is a binary variable set equal to 1 for a call and -1 for a put.

A valuation formula for a call on the maximum can easily be derived from our formula for an option on the best of two risky assets and cash, since the payoff from the former equals the payoff from the latter plus borrowing the present value of the striking price:

\[
\max[0, \max(S_1^t, S_2^t) - K] = \max[S_1^t, S_2^t, K] - K
\]

Thus, the value of a call on the maximum is just \( C - Kr^t \).

A valuation formula for a put on the maximum can easily be derived from the formula for a call on the maximum since the put has the same payoff as the purchase of an otherwise identical call on the maximum,
the sale of an option on the better of two risky assets, and a loan of the present value of the striking price:

\[ \max[0, K - \max(S_1^*, S_2^*)] = \max[0, \max(S_1^*, S_2^*) - K] - \max(S_1^*, S_2^*) + K \]

Options on the better of two risky assets are a special case of these options, since if \( K=0 \), then the payoff reduces too:

\[ \max[0, \max(S_1^*, S_2^*) - K] = \max[0, \max(S_1^*, S_2^*)] = \max(S_1^*, S_2^*) \]

**Calls and puts on the minimum of two risky assets.** These options have payoff:

\[ \max[0, \phi \min(S_1^*, S_2^*) - \phi K] \]

A valuation formula for a call on the minimum can easily be derived from the formula for a call on the maximum since a call on the minimum has the same payoff as the purchase of a standard call on the first underlying asset with striking price \( K \), the purchase of standard call on the second underlying asset with striking price \( K \), and the sale of an otherwise identical call on the maximum:

\[ \max[0, \min(S_1^*, S_2^*) - K] = \max[0, S_1^* - K] + \max[0, S_2^* - K] - \max[0, \max(S_1^*, S_2^*) - K] \]

A valuation formula for a put on the minimum can easily be derived from the formula for the call on the minimum since the put has the same payoff as the purchase of an otherwise identical call on the minimum, the sale of an option on the worse of two risky assets, and a loan of the present value of the striking price:

\[ \max[0, K - \min(S_1^*, S_2^*)] = \max[0, \min(S_1^*, S_2^*) - K] - \min(S_1^*, S_2^*) + K \]

Options on the worse of two risky assets are a special case of these options, since if \( K=0 \), then the payoff reduces to:

\[ \max[0, \min(S_1^*, S_2^*) - K] = \max[0, \min(S_1^*, S_2^*)] = \min(S_1^*, S_2^*) \]

**II. Options without Closed-Form Solutions**

Superficially innocuous generalizations of the foregoing types of options immediately preclude the derivation of closed-form valuation formulas. For example,

- **spread options:** \( \max[0, \phi (S_2^* - S_1^*) - \phi K] \)
- **portfolio options:** \( \max[0, \phi (n_1S_1^* + n_2S_2^*) - \phi K] \)
- **dual-strike options:** \( \max[0, \phi_1 (S_1^* - K_1), \phi_2 (S_2^* - K_2)] \)

where \( n_1 \) and \( n_2 \) are the numbers of units of the two assets in a portfolio and the binary variables \( \phi_1 \) and \( \phi_2 \) are either 1 or -1.

Numerical techniques for valuing these options have been published by Phelim Boyle, Jeremy Evnine and
Stephen Gibbs. Here we provide a similar but alternative methodology which has certain advantages. Our problem is to find a convenient way to evaluate integrals of the form:

$$ C = \int \int g(x,y)f(x,y)dxdy $$

where $g(x,y)$ is the payoff function of the option and the integrals range from $-\infty$ to $\infty$. For example, for spread options, $g(x,y) = \max[0, \phi(S, e^r - S, e^r) - \phi K]$. Our approach will be to approximate the continuous bivariate normal density $f(x,y)$ with a discrete bivariate binomial density.

First suppose we want to approximate a standard normal univariate density using a univariate binomial density. The standard normal density can be approximated by a $(+1, -1)$ symmetric arithmetic binomial random walk:

```
0 1
-1
-2
```

Given $n$ steps, for each terminal outcome $z_j$, the probability $p_j$ of that outcome is:

$$ p_j = \frac{1}{2^n}[n!/(n-j)!] $$

where

$$ z_j = \frac{j - (n-j)}{\sqrt{n}} $$

for $j = 0, 1, 2, ..., n$

$z_j$ is standardized by dividing by $\sqrt{n}$ to make the variance of $z_j$ equal to one at the end of the tree.

For example, for 3 and 4 step trees, the binomial standard normal approximation table is:

---

4 See their article, "Numerical Evaluation of Multivariate Contingent Claims," Review of Financial Studies, Volume 2, Number 2, 1989. Both approaches approximate a bivariate normal distribution with a bivariate binomial distribution. In their case, the correlation effects are built into the probabilities permitting simple move sizes. However, it becomes possible for some probabilities to be negative if parameters are not chosen carefully and negative probabilities produce incorrect results. In the approach presented here the correlation effects are built into the move sizes with simple probabilities so that this is not a problem.

5 Other numerical integration techniques were tried including Gauss-Legendre and Gauss-Hermite quadrature. The method presented here provided the greatest computational efficiency. Hua He in his article, "Convergence from Discrete- to Continuous-Time Contingent Claims Prices," Review of Financial Studies, Volume 3, Number 4, 1990, has shown that a bivariate normal distribution can be approximated by a process in which the paired random variables can only move to three outcomes at each step. While theoretically very elegant, as the number of steps becomes larger, the convergence of this process to bivariate normal is extremely slow compared to the four-outcome process used here.
Of course, as \( n \to \infty \), the table comes closer and closer to approximating the standard normal density.

Now let us see the natural way to approximate a standard bivariate normal density function. If \( u \) and \( v \) are independent standard normal (binomial) random variables, then \( w \) and \( z \) defined by

\[
w = u \quad \text{and} \quad z = \rho u + \sqrt{1 - \rho^2} v
\]

will be standard bivariate normal (binomial) with correlation \( \rho \). To verify this, since \( u \) and \( v \) are standard variates, \( E(u) = E(v) = 0 \), \( \text{var}(u) = \text{var}(v) = 1 \), and since they are independent \( \text{cov}(u, v) = 0 \). Using this, it is easy to show that

\[
(1) \ E(w) = E(z) = 0 \\
(2) \ \text{var}(w) = \text{var}(z) = 1 \\
(3) \ \text{cov}(w, z) = \rho
\]

In addition \( (w, z) \) are normal (binomial) because they are weighted sums of normal (binomial) random variables. This result gives us an easy way to manufacture bivariate normal or bivariate binomial random variables out of univariate variables.\(^6\)

Given \( n \) steps, there are \((n+1)^2\) possible paired outcomes, \((w_{j,k}, z_{j,k})\), for \( j = 0, 1, 2, 3, ..., n \) and \( k = 0, 1, 2, 3, ..., n \). Each paired outcome \((w_{j,k}, z_{j,k})\) has probability \( p_{j,k} = p_j p_k \), where

\[
p_j = \frac{1}{2^j [n!/(n-j)!]} \quad p_k = \frac{1}{2^n [n!/(k!(n-k)!)]}
\]

\[
w_{j,k} = [j - (n-j)]/\sqrt{n} \quad z_{j,k} = \rho [j - (n-j)]/\sqrt{n} + \sqrt{1 - \rho^2} [k - (n-k)]/\sqrt{n}
\]

for \( j = 0, 1, 2, 3, ..., n \) and \( k = 0, 1, 2, 3, ..., n \)

For example, suppose \( \rho = .5 \) and \( n = 4 \), then:

\[
\begin{array}{c|c|c}
\hline
j & p_j & z_j \\
\hline
3 & .125 & 1.73205 \\
2 & .375 & .57735 \\
1 & .375 & .57735 \\
0 & .125 & -1.73205 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c}
\hline
j & p_j & z_j \\
\hline
4 & .0625 & 2 \\
3 & .25 & 1 \\
2 & .375 & 0 \\
1 & .25 & -1 \\
0 & .0625 & -2 \\
\hline
\end{array}
\]

\(^6\) In fact, these formulas are often used in Monte-Carlo simulation to generate bivariate random draws given a technique for drawing univariate random variables.
<table>
<thead>
<tr>
<th>j</th>
<th>k</th>
<th>p_{jk}</th>
<th>u_{jk}</th>
<th>v_{jk}</th>
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To approximate our non-standard variables \((x,y)\), we must magnify by the standard deviation and tack on the mean return:

\[ x_{jk} = \mu_x t + \sigma_x \sqrt{w_{jk}} \quad \text{and} \quad y_{jk} = \mu_y t + \sigma_y \sqrt{w_{jk}} \]

Now we are ready to put this together to value the option:

\[ C = r^T \sum \sum p_{jk} g(x_{jk}, y_{jk}) \]

Finally, a word about computer efficiency. At \( n = 100 \), the number of terms in the double summation is \( 101^2 = 10,201 \), and the numerical procedure can be expected to produce at least as much accuracy as a 100 step univariate binomial tree for a standard option. For European options we are only concerned with the ending nodes and do not need to work backwards through the tree. Working backwards in the binomial approach for standard American options, we traverse \((101 \times 102) \times 2 = 5151\) nodes. So evaluating a two-color rainbow option by our approach should take about twice the computer time than evaluating a standard American option by working backwards through a 100 step tree. Given \( n \), each correlation \( p \) has a unique standard bivariate binomial tree. As a result, it is possible to calculate the standard bivariate binomial tree once and for all, store it in a file, and recall it whenever rainbow options involve underlying assets with the corresponding correlation.
### 71.00 RB2: TWO-COLOR RAINBOW EUROPEAN OPTIONS

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### 71.00 RB2: TWO-COLOR RAINBOW EUROPEAN OPTIONS

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### 76.00 PT2: TWO-COLOR PORTFOLIO EUROPEAN OPTIONS

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### Values

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### Summary

- The tables above provide a comprehensive view of the pricing of European options under different scenarios.
- The tables include parameters such as volatility, strike prices, interest rates, dividend yields, correlations, and years to maturity.
- The results are presented in a tabular format, showing the values for different correlation levels and YrsToExp.
Value: Better of Two Risky Assets European Option (Lognormal)

Index=2, Index=Strike=100, IntRate=.1, Vol=2, Vol=.2, Yld=.05, Yld=.05
2-Asset Portfolio Binomial European Call Values

Index=2, Strike=100, IntRate=.1, 1Vol=.1, 2Vol=.3, Corr=0
2-Asset Portfolio Binomial Minus Standard Call Lognormal Values

Index=2Index, Strike=100, IntRate=.1, 1Vol=.1, 2Vol=.3, Corr=0
2-Asset Portfolio Binomial Minus Standard Call Lognormal 1Deltas

Index=2Index, Strike=100, IntRate=.1, 1Vol=.1, 3Vol=.3, Corr=0
American Rainbow Options

Mark Rubinstein

September 10, 1994

Two-color rainbow options are here defined as options on two risky underlying assets which cannot by some trick of reasoning be valued as if they were options on one underlying asset. In this essay, my objective is to value a variety of these options in an American Black-Scholes environment; that is:

(1) where the two underlying assets can be assumed to follow a jointly lognormal random walk, and

(2) where arbitrage arguments allow us to use a risk-neutral working backwards recursive valuation approach – set the value of the option at any period equal to the discounted expected value of the option one period latter or its early exercise value, whichever is greater, and where the riskless interest rate is used as the discount rate and both underlying asset prices are expected to appreciate at the same riskless rate less payouts.

Examples of American options which can be valued in this way, include:

- **options delivering the best of two risky assets and cash**
  \[
  \max[S_1^*, S_2^*, K]
  \]

- **calls and puts on the maximum of two risky assets**
  \[
  \max[0, \phi \max(S_1^*, S_2^*) - \phi K]
  \]

- **calls and puts on the minimum of two risky assets**
  \[
  \max[0, \phi \min(S_1^*, S_2^*) - \phi K]
  \]

- **spread options**
  \[
  \max[0, \phi (S_2^* - S_1^*) - \phi K]
  \]

- **portfolio options**

---

1 Mark Rubinstein is Professor of Finance at the University of California at Berkeley. A version of this paper has appeared under the title, "Return to Oz," in RISK, November 1994.

2 For examples of these, see my article about American exchange options which appeared as "One for Another," RISK 4 (July-August 1991) and Eric Reiner's article about currency-translated options which appeared under the title "Quanto Mechanics," in RISK 5 (March 1992), pp. 59-63. In both of these cases, closed-form European valuation formulas can be derived in terms only of univariate normal distribution functions. This paper is a sequel to my earlier article, "Somewhere Over the Rainbow," RISK 4 (November 1991), which examined European versions of all the options considered here.
\[
\max[0, \phi(n_1 S_1^* + n_2 S_2^*) - \phi K]
\]

**dual-strike options**

\[
\max[0, \phi_1(S_1^* - K_1), \phi_2(S_2^* - K_2)]
\]

Here, the mathematical expressions are the payoff functions for each of these options at their expiration, where \( S_1^* \) and \( S_2^* \) represent the expiration date values of two underlying assets assumed to have a risk-neutral joint lognormal distribution, \( K \) represents a fixed number of dollars (striking price), and \( \phi, \phi_1, \phi_2 \) are either equal to 1 or -1 depending on whether the payoff is like a call or a put.

While many of these options have been traded for some time in the over-the-counter markets, the New York Mercantile Exchange plans to list the first exchange-traded rainbow options during the fourth quarter of 1994. They plan to list options on the heating oil crack spread (heating oil minus crude oil prices) and the gasoline oil crack spread (unleaded gasoline minus crude oil prices), and both types of options will be American.

American versions of all of these options can be efficiently valued by constructing a square recombining binomial pyramid where the move sizes and move probabilities are chosen with mean returns to match a given annualized discrete interest return \( r \) and given annualized discrete payout returns \( \delta_1 \) and \( \delta_2 \) for each of the two underlying assets, with standard deviations to match given annualized standard deviations \( \sigma_1 \) and \( \sigma_2 \) for the (logarithmic) returns of each of the two underlying assets, and to match a given correlation \( \rho \) between their (logarithmic) returns.

I first outline the structure of the binomial pyramid and then define the move sizes in the pyramid in terms of the above given parameters. The pyramid can then be used to value any of the two-color rainbow options given above.

I. Constructing the Binomial Pyramid

The returns of the underlying assets are currently standardized to 1 for the first and 1 for the second, which we write as the pair (1,1). During the first move, the first asset return is assumed to be either \( d \) or \( u \), with equal probability, mimicking the behavior of a single underlying asset in a standard binomial tree, commonly used to value options on a single underlying asset. Simultaneously, if the first asset moves to \( u \), then the second asset has returns either \( A \) or \( B \), with equal probability; or if the first asset moves to \( d \), then the second asset has returns either \( C \) or \( D \), with equal probability. Assume that the second asset returns are chosen so that \( AD = BC \). Figure 1 summarizes the first move, indicating that the pair of underlying assets starting at (1,1) moves to either (u,A), (u,B), (d,C), or (d,D), each with probability \( \frac{1}{4} \).

![Figure 1: Move 1](image-url)

2
Allowing \( A = C \) and \( B = D \) makes it possible to build in a non-zero correlation between the returns of the two assets. To see this, suppose \( A = C \) and \( B = D \), then knowing the realized return of the first asset, tells us nothing about what happened to the second, since whether the first asset moves up or down, the second asset can still move to A or B with equal probability. To embody non-zero correlation, it is necessary for the realized return the first asset to change the possible realized returns (or probabilities of returns) for the second.

Figure 2 summarizes the second move. For example, given that the first return was \((u,A)\), then the second return can again be either \((u,A)\), \((u,B)\), \((d,C)\) or \((d,D)\). Multiplied together the total return over the first two moves is then either \((u^2,A^2)\), \((u^2,AB)\), \((ud,AC)\), or \((ud,AD)\), each with the equal probability \(1/4 \times 1/4 = 1/16\).

Try to picture the possible total returns after each move as a horizontal slice of a square pyramid, with \((1,1)\) at the apex and the last move at the bottom. With this structure, several paths through the pyramid lead to the same node. For example, the paths \((1,1) \rightarrow (u,A) \rightarrow (u,B)\) and \((1,1) \rightarrow (u,B) \rightarrow (u,A)\) both lead to the same node \((u^2,AB)\) after the second move, so this has a total nodal probability of \(1/16 + 1/16 = 1/8\). Indeed, as shown in Figure 2, four paths reach the central node of the pyramid since by assumption \(AD = BC\). Thus, the total probability for this node is \(1/4\).

Figure 2: Move 2

\[
\begin{array}{ccc}
(u^2, A^2) & (u^2, AB) & (u^2, B^2) \\
(u, A) & (u, B) & \\
(ud, AD) & (du, BC) & \\
(d, C) & (d, D) & \\
(d^2, C^2) & (d^2, CD) & (d^2, D^2) \\
\end{array}
\]
Continuing in this same fashion, starting from the X nodes at the end of move 2, Figure 3 shows the possible nodes at the end of the third move.

Figure 3: Move 3

Clearly, if the pyramid consists of $n$ moves, then the total number of distinct nodes at the bottom would be $(1 + n)^2$. 
II. Matching the Move Sizes to the Given Parameters

It will be convenient to begin by matching the natural logarithms of the asset returns to the given parameters. At the apex of the pyramid, we then start with \((0,0)\) since \(\log 1 = 0\). Denote \(x\) and \(y\) as the random returns over the first move for the first and second assets, respectively. First, suppose we want to replicate the move structure from \((0,0)\) to \((\log x, \log y)\), with means 0, standard deviations 1 and correlation 0.

Figure 4: Step 1
Standard bivariate binomial move \((\log x, \log y)\) for \(\rho = 0\)

\[
\begin{array}{c c c c}
(1, 1) & \frac{1}{2} & \frac{1}{2} & (1, -1) \\
\frac{1}{2} & (0,0) & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & (1, 1) & (1, -1) \\
(-1, 1) & (-1, -1) & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

Observe that: \(E[\log x] = E[\log y] = 0\), \(\text{Var}[\log x] = \text{Var}[\log y] = 1\), \(\text{Cov}(\log x, \log y) = 0\)
Now build in the given logarithmic correlation $\rho$ between the two asset returns:

**Figure 5: Step 2**
Standard bivariate binomial move $(\log x, \log y)$ for $\rho \neq 0$

\[
(1, [\rho + \sqrt{1-\rho^2}]) \quad (1, [\rho - \sqrt{1-\rho^2}])
\]

\[
(-1, -[\rho + \sqrt{1-\rho^2}]) \quad (-1, -[\rho - \sqrt{1-\rho^2}])
\]

\[
(0,0)
\]

Note that:

\[
E[\log x] = E[\log y] = \frac{1}{4}(\rho + \sqrt{1-\rho^2}) + [\rho + \sqrt{1-\rho^2}] - [\rho - \sqrt{1-\rho^2}] - [\rho - \sqrt{1-\rho^2}] = 0
\]

\[
\text{Var}[\log x] = \text{Var}[\log y] = \frac{1}{4}([\rho + \sqrt{1-\rho^2}]^2 + [\rho - \sqrt{1-\rho^2}]^2 + [\rho + \sqrt{1-\rho^2}]^2 + [\rho - \sqrt{1-\rho^2}]^2) = 1
\]

\[
\text{Cov}[\log x, \log y] = \frac{1}{4}(1 \cdot [\rho + \sqrt{1-\rho^2}] + 1 \cdot [\rho - \sqrt{1-\rho^2}] - 1 \cdot [\rho + \sqrt{1-\rho^2}] - 1 \cdot [\rho - \sqrt{1-\rho^2}]) = \rho
\]
Now build in given logarithmic means $(\mu, h, \mu \cdot h)$ and logarithmic standard deviations $(\sigma_1 \cdot h, \sigma_2 \cdot h)$, where $t$ is the time to expiration in years, $n$ is the number of moves in the pyramid, and $h = t/n$ is the elapsed time per move:

\begin{align*}
&\text{Figure 6: Step 3} \\
&\text{Bivariate binomial move (log x, log y) with means and variances} \\
&(\mu_1 h + \sigma_1 \sqrt{h}, \mu_2 h + \sigma_2 \sqrt{h} \left[ \rho + \sqrt{1 - \rho^2} \right]), \quad (\mu_1 h + \sigma_1 \sqrt{h}, \mu_2 h + \sigma_2 \sqrt{h} \left[ \rho - \sqrt{1 - \rho^2} \right])
\end{align*}

\begin{align*}
&\text{Again, it is easy to see that:} \\
&E[\log x] = \mu_1 h \quad E[\log y] = \mu_2 h \\
&\text{Var}[\log x] = \sigma_1^2 h \quad \text{Var}[\log y] = \sigma_2^2 h
\end{align*}

and that the correlation between $\log x$ and $\log y$ is unaffected by these changes and remains equal to $\rho$. 

\begin{align*}
&\frac{\sigma_1 \sqrt{h}}{\mu_1 h} \\
&\frac{\sigma_2 \sqrt{h}}{\mu_2 h}
\end{align*}
To switch this from the logarithmic returns \((\log x, \log y)\), to the returns themselves \((x, y)\), simply exponentiate each of the above move sizes since \(x = \exp(\log x)\) and \(y = \exp(\log y)\):

\[
\begin{array}{c}
\begin{aligned}
(\exp(\mu_1 h + \sigma_1 \sqrt{h}), \\
\exp(\mu_2 h + \sigma_2 \sqrt{h}[\rho + \sqrt{1 - \rho^2}]))
\end{aligned}
\end{array}
\]

\[
\begin{array}{c}
\begin{aligned}
(\exp(\mu_1 h + \sigma_1 \sqrt{h}), \\
\exp(\mu_2 h + \sigma_2 \sqrt{h}[\rho - \sqrt{1 - \rho^2}]))
\end{aligned}
\end{array}
\]

Finally, the risk neutrality approach to option pricing allows us to express the logarithmic means in terms of the discrete interest return, so that:

\[
\mu_1 = (\log r/\delta_1) - \frac{1}{2} \sigma_1^2 \quad \text{and} \quad \mu_2 = (\log r/\delta_2) - \frac{1}{2} \sigma_2^2
\]

III. Summary and Example

To recapitulate: given as inputs \(S_1, S_2\) (the current underlying asset prices), \(r, t, \sigma_1, \sigma_2, \delta_1, \delta_2, \rho, n\), define:

\[
h = t/n
\]

\[
\mu_1 = (\log r/\delta_1) - \frac{1}{2} \sigma_1^2 \quad \text{and} \quad \mu_2 = (\log r/\delta_2) - \frac{1}{2} \sigma_2^2
\]

\[
u = \exp\{\mu_1 h + \sigma_1 \sqrt{h}\} \quad \text{and} \quad d = \exp\{\mu_1 h - \sigma_1 \sqrt{h}\}
\]

\[A = \exp\{\mu_2 h + \sigma_2 \sqrt{h}[\rho + \sqrt{1 - \rho^2}]\}\]

\[B = \exp\{\mu_2 h + \sigma_2 \sqrt{h}[\rho - \sqrt{1 - \rho^2}]\}\]

\[C = \exp\{\mu_2 h - \sigma_2 \sqrt{h}[\rho + \sqrt{1 - \rho^2}]\}\]

\[D = \exp\{\mu_2 h - \sigma_2 \sqrt{h}[\rho - \sqrt{1 - \rho^2}]\}\]

Note that, as required for the tree to recombine properly, \(AD = BC\).

These definitions of \((u,d)\) and \((A,B,C,D)\) can be used to construct the appropriate move sizes in a square binomial pyramid. Starting at the end of the tree, the value of the option can be written down at each node.
Then, work backwards discounting 4 nodes into 1 at each move, using the same probability of $1/4$ for each node.

For example, consider an American call option on a spread with payoff at expiration:

$$\max[0, (S_2^*-S_1^*)-K]$$

For a $n=2$ move tree, at expiration:

$$C(u^*,A^*) = \max[(S_2A^*-S_1u^*)-K,0]$$
$$C(u^*,AB) = \max[(S_2AB-S_1u^*)-K,0]$$
$$C(u^*,B^*) = \max[(S_2B^*-S_1u^*)-K,0]$$
$$C(ud,AC) = \max[(S_2AC-S_1ud)-K,0]$$
$$C(ud,BC) = C(du,AD) = \max[(S_2BC-S_1ud)-K,0]$$
$$C(ud,BD) = \max[(S_2BD-S_1ud)-K,0]$$
$$C(d^*,C^*) = \max[(S_2C^*-S_1d^*)-K,0]$$
$$C(d^*,CD) = \max[(S_2CD-S_1d^*)-K,0]$$
$$C(d^*,D^*) = \max[(S_2D^*-S_1d^*)-K,0]$$

Working backwards one move:

$$C(u,A) = \max\{(S_2A-S_1u)-K, \frac{1}{4}[C(u^*,A^*)+C(u^*,AB)+C(ud,AC)+C(ud,AD)]/r^2\}$$
$$C(u,B) = \max\{(S_2B-S_1u)-K, \frac{1}{4}[C(u^*,BA)+C(u^*,B^*)+C(ud,BC)+C(ud,BD)]/r^2\}$$
$$C(d,C) = \max\{(S_2C-S_1d)-K, \frac{1}{4}[C(du,CA)+C(du,CB)+C(d^*,C^*)+C(d^*,CD)]/r^2\}$$
$$C(d,D) = \max\{(S_2D-S_1d)-K, \frac{1}{4}[C(du,DA)+C(du,DB)+C(d^*,DC)+C(d^*,D^*)]/r^2\}$$

And, finally, working from backwards one more move to the beginning of the tree:

$$C = \max\{(S_2-S_1)-K, \frac{1}{4}[C(u,A)+C(u,B)+C(d,C)+C(d,D)]/r^2\}$$

IV. Significance of Early Exercise

The following tables compare the values of otherwise identical European and American calls on spreads on assets where the inputs, in terms of the notation in this article are:

$$S_1 = S_2 = 100, \ K = 5, \ t = (0.05 .28 .50 .73 .95), \ \sigma_1 = \sigma_2 = .3, \ \delta_1 = (1.025 1.050 1.075), \ \delta_2 = (1.025 1.050 1.075), \ \text{and} \ n = 50. \ \text{In Table 1,} \ \rho = -.40; \ \text{in Table 2,} \ \rho = .00; \ \text{and in Table 3,} \ \rho = .50.$$ 

As can be seen from the tables, early exercise seems to increase in importance the longer the time to expiration and the higher the dividend yield of underlying asset two relative to underlying asset one. The most extreme differences between American and European values occur when $t = .95, \ \delta_1 = 1.025$ and $\delta_2 = 1.075$. 

9
### Table 1: $\rho = -0.50$

<table>
<thead>
<tr>
<th>Value</th>
<th>Amer/Euro=Amer</th>
<th>Amer/Euro=Euro</th>
</tr>
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<tr>
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<td>YrsToExp</td>
</tr>
<tr>
<td>1DivY1 2DivY1</td>
<td>0.05 0.28 0.50 0.73 0.95</td>
<td>0.05 0.28 0.50 0.73 0.95</td>
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<tr>
<td>0.025</td>
<td>2.56 8.52 12.17 15.02 17.40</td>
<td>2.56 8.52 12.16 14.98 17.35</td>
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<tr>
<td>0.075</td>
<td>2.49 7.99 11.17 13.55 15.48</td>
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<td>0.050</td>
<td>2.60 8.77 12.61 15.63 18.18</td>
<td>2.60 8.77 12.61 15.63 18.18</td>
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<tr>
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<td>2.56 8.46 12.03 14.78 17.05</td>
<td>2.56 8.45 11.98 14.69 16.91</td>
</tr>
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<td>2.52 8.20 11.53 14.05 16.10</td>
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<td>2.64 9.01 13.05 16.27 18.99</td>
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<td>2.59 8.69 12.43 15.34 17.76</td>
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<td>0.075</td>
<td>2.55 8.40 11.90 14.56 16.74</td>
<td>2.55 8.38 11.82 14.40 15.49</td>
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### Table 2: $\rho = 0.00$

<table>
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<th>Amer/Euro=Euro</th>
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</thead>
<tbody>
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<td>YrsToExp</td>
<td>YrsToExp</td>
</tr>
<tr>
<td>1DivY1 2DivY1</td>
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<td>0.05 0.28 0.50 0.73 0.95</td>
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<td>1.81 6.57 9.55 11.89 13.85</td>
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<td>1.77 6.33 9.05 11.15 12.89</td>
<td>1.77 6.29 8.96 11.00 12.66</td>
</tr>
<tr>
<td>0.075</td>
<td>1.74 6.10 8.62 10.50 12.03</td>
<td>1.73 6.02 8.44 10.21 11.60</td>
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<td>1.84 6.82 10.03 12.56 14.69</td>
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<tr>
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<td>1.80 6.52 9.41 11.65 13.49</td>
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<tr>
<td>0.075</td>
<td>1.77 6.29 8.96 11.00 12.66</td>
<td>1.76 6.24 8.85 10.79 12.36</td>
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<td>1.88 7.07 10.49 13.19 15.51</td>
</tr>
<tr>
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<td>1.84 6.75 9.87 12.29 14.29</td>
</tr>
<tr>
<td>0.075</td>
<td>1.80 6.48 9.34 11.54 13.34</td>
<td>1.80 6.46 9.28 11.42 13.15</td>
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Table 3: $p = .50$

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</tr>
</thead>
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<td>.05 .28 .50 .73 .95</td>
<td>.05 .28 .50 .73 .95</td>
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<tr>
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<td>.86 3.87 5.74 7.17 8.35</td>
<td>.86 3.84 5.67 7.05 8.17</td>
</tr>
<tr>
<td>.075</td>
<td>.84 3.66 5.34 6.59 7.60</td>
<td>.84 3.60 5.19 6.34 7.24</td>
</tr>
<tr>
<td>.025</td>
<td>.92 4.34 6.64 8.50 10.11</td>
<td>.92 4.34 6.64 8.50 10.11</td>
</tr>
<tr>
<td>.050</td>
<td>.89 4.07 6.11 7.72 9.06</td>
<td>.89 4.07 6.09 7.67 8.99</td>
</tr>
<tr>
<td>.075</td>
<td>.86 3.84 5.68 7.07 8.21</td>
<td>.86 3.81 5.59 6.91 7.97</td>
</tr>
<tr>
<td>.025</td>
<td>.95 4.57 7.09 9.18 11.00</td>
<td>.95 4.57 7.09 9.18 11.00</td>
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<tr>
<td>.075</td>
<td>.89 4.04 6.03 7.58 8.87</td>
<td>.89 4.03 6.00 7.51 8.75</td>
</tr>
</tbody>
</table>
Appendix: Useful Facts about Normal Distributions

**normal density and distribution**

\[ n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \]

\[ f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \]

\[ N(h) = \int_{-\infty}^{h} n(x) \, dx \]

(1.1) \[ \int_{-\infty}^{h} n(x) \, dx = N(-h) \]

(1.2) \[ \int_{-\infty}^{\infty} f(x) \, dx = \mu \]

(1.3) \[ \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \, dx = \sigma^2 \]

(1.4) \[ \int_{-\infty}^{h} f(x) \, dx = N \left( \frac{h-\mu}{\sigma} \right) \]

(1.5) \[ \int_{-\infty}^{h} e^{x} f(x) \, dx = e^{\mu+\frac{1}{2}\sigma^2} N \left( \frac{h-\mu}{\sigma} - \sigma \right) \]

(1.6) \[ \int_{-\infty}^{\infty} n(x) N(a+bx) \, dx = N \left( \frac{a}{\sqrt{1+b^2}} \right) \]

(1.7) **if** \( \mu = 0 \), **then** \( E[|x|] = \sigma \sqrt{\frac{2}{\pi}} \)

(1.8) **if** \( \mu = 0 \), **then** \( E[x^2|x|] = 2\sigma^2 \sqrt{\frac{2}{\pi}} \)
\[
\frac{\partial N(h)}{\partial a} = n(h) \frac{\partial h}{\partial a}
\]

\[
\int_{b}^{k} e^{ax} N(bx) \, dx = \frac{1}{a} \left[ e^{ax} N(bx) - e^{\frac{a^2}{2b^2}} N(bx - \frac{a}{b}) \right]^k, \quad ab \neq 0
\]

\[
\int_{b}^{k} x e^{ax} N(x) \, dx = \left[ \left( \frac{ax - 1}{a^2} \right) e^{ax} N(x) - \left( \frac{a^2}{a^2} \right) e^{\frac{a^2}{2}} N(x-a) + \frac{1}{a} e^{\frac{a^2}{2}} n(x-a) \right]^k, \quad a \neq 0
\]

\[
\int_{b}^{k} x N(a+bx) \, dx = \left[ \left( \frac{b^2 x^2 - a^2 - 1}{2b^2} \right) N(a+bx) + \left( \frac{bx-a}{2b^2} \right) n(a+bx) \right]^k, \quad b \neq 0
\]

\[
\int_{b}^{k} N(a+bx) \, dx = \left[ \left( \frac{a+bx}{b} \right) N(a+bx) + \frac{1}{b} n(a+bx) \right]^k, \quad b \neq 0
\]

\[
\int_{b}^{k} N(x) N(bx) \, dx = \left[ x N(x) N(bx) + \frac{1}{b} N(x) n(bx) + n(x) N(bx) - \frac{\sqrt{1+b^2}}{b \sqrt{2\pi}} N(x \sqrt{1+b^2}) \right]^k, \quad b \neq 0
\]

\[
\int_{b}^{k} x N(x) N(a+bx) \, dx = \left[ \frac{b}{\sqrt{1+b^2}} n \left( \frac{a}{\sqrt{1+b^2}} \right) N \left( x \sqrt{1+b^2} + \frac{ab}{\sqrt{1+b^2}} \right) - N(a+bx) n(x) \right]^k
\]

\[
\int_{b}^{k} x^2 n(x) N(x) \, dx = \left[ \frac{[N(x)]^2}{2} - \frac{[n(x)]^2}{2} - x N(x) n(x) \right]^k
\]

\[
\int_{b}^{k} e^{-\frac{1}{2} (ax+b)^2} \, dx = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2} \left( \frac{b^2}{a^2} \right)} \left[ N \left( \sqrt{a} x + \frac{b}{2\sqrt{a}} \right) \right]^k
\]

\[
\int_{b}^{k} e^{-\frac{1}{2} (ax+b)^2} \, dx = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2} \left( \frac{b^2}{a^2} \right)} N \left( \sqrt{\frac{a}{a+b^2}} \left( e^{-\frac{b^2}{2a}} \right) \right)
\]

\[
\int_{b}^{k} x^p n(x) \, dx = \left[ x^p N(x) \right]^k - p \int_{b}^{k} x^{p-1} N(x) \, dx
\]
\[
(1.20) \quad \int_{h}^{k} x^p \ n(x) \ dx = [-x^{p-1} \ n(x)]_{h}^{k} + (p-1) \int_{h}^{k} x^{p-2} \ n(x) \ dx
\]

\[
(1.21) \quad \text{if } p > 0 \text{ is odd: } \quad \int_{h}^{k} x^p \ n(x) \ dx =
\]

\[
[-(x^{p-1}+\frac{1}{2}(p-1)x^{p-3}+\frac{1}{3}(p-1)(p-3)x^{p-5}+\ldots+(p-1)(p-3)\ldots6\cdot4\cdot2)n(x)]_{h}^{k}
\]

\[
(1.22) \quad \text{if } p > 0 \text{ is even: } \quad \int_{h}^{k} x^p \ n(x) \ dx =
\]

\[
[((p-1)(p-3)\ldots5\cdot3)N(x) - (x^{p-1}+(p-1)x^{p-3}+(p-1)(p-3)x^{p-5}+\ldots+(p-1)(p-3)\ldots5\cdot3\cdot1)n(x)]_{h}^{k}
\]

\[
(1.23) \quad \int_{h}^{k} x^p \left[n(x)\right]^q \ dx = \frac{1}{q} \left[-x^{p-1} \left[n(x)\right]^q\right]_{h}^{k} + \frac{p-1}{q} \int_{h}^{k} x^{p-2} \left[n(x)\right]^q \ dx, \quad q > 0
\]

\[
(1.24) \quad \int_{h}^{k} e^{ax} \left[n(bx)\right]^q \ dx = \left(\frac{2\pi}{b\sqrt{q}}\right)^{\frac{1}{2}} e^{\frac{a^2}{4b^2}} \left[N\left(bx\sqrt{q} - \frac{a}{b\sqrt{q}}\right)\right]_{h}^{k}, \quad b > 0, \quad q > 0
\]

\[
(1.25) \quad \int_{h}^{k} \left[n(a+bx)\right]^q \ dx = \left(\frac{2\pi}{b\sqrt{q}}\right)^{\frac{1}{2}} \left[N(bx\sqrt{q} + a\sqrt{q})\right]_{h}^{k}, \quad b > 0, \quad q > 0
\]

\[
(1.26) \quad \int_{h}^{k} \left[n(x)\right]^q \left[n(a+bx)\right]^q \ dx = \left(\frac{2\pi}{\sqrt{1+q}b}\right)^{\frac{1}{2}} n\left(\frac{a\sqrt{q}}{\sqrt{1+q}b}\right) \left[N\left(x\sqrt{1+q}b^2 + \frac{abq}{\sqrt{1+q}b^2}\right)\right]_{h}^{k}, \quad q > -\frac{1}{b^2}
\]

\[
(1.27) \quad \int_{h}^{k} x^p \ n(x) \ n(a+bx) \ dx =
\]

\[
\frac{1}{1+b^2} \left[-n\left(\frac{a}{\sqrt{1+b^2}}\right) n\left(x\sqrt{1+b^2} + \frac{ab}{\sqrt{1+b^2}}\right) - \frac{ab}{\sqrt{1+b^2}} \left(\frac{a}{\sqrt{1+b^2}}\right) n\left(x\sqrt{1+b^2} + \frac{ab}{\sqrt{1+b^2}}\right)\right]_{h}^{k}
\]

\[
(1.28) \quad \int_{h}^{k} x^p \ n(x) \ dx = \frac{1}{p+1} \left[x^{p-1} n(x) + x^p n(x)\right]_{h}^{k} - \frac{p}{p+1} \int_{h}^{k} x^{p-2} \ n(x) \ dx, \quad p > -1
\]

\[
(1.29) \quad \int_{h}^{k} x^p \ n(x) \ dx = \frac{1}{p+1} \left[(x^{p+1} - px^{p-1}) n(x) + x^p n(x)\right]_{h}^{k} + \frac{p(p-1)}{p+1} \int_{h}^{k} x^{p-2} \ n(x) \ dx, \quad p > -1
\]
(1.30) if $p \geq 0$ is even: \[
\int_h^x P \; n(x) \; dx =
\]
\[
\frac{1}{p+1} \left[ x^{p+1} \; n(x) + (x^{p+1}+p(p-1)x^{p-2}+\ldots+p(p-2)\ldots+p) \; n(x) \right]_h^x
\]

(1.31) if $p > 0$ is odd: \[
\int_h^x P \; n(x) \; dx =
\]
\[
\frac{1}{p+1} \left[ x^{p+1}+p(p-1)x^{p-2}+\ldots+p(p-2)\ldots+p \; n(x) \right]_h^x
\]

(1.32) \[
\int_h^x \left[ \frac{2}{3} \left( N(x) \right)^2 + 2N(x) \; n(x) \right] \; dx
\]

(1.33) \[
\int_h^x \left[ \frac{2}{3} \left( N(x) \right)^2 + \frac{5}{6\sqrt{\pi}}\; N(x) \; n(x) \right] \; dx
\]

(1.34) \[
\int_h^x \left[ \frac{2}{3} \left( N(x) \right)^2 + \frac{2}{3} \; N(x) \; n(x) \right] \; dx
\]

(1.35) \[
\int_h^x \; x^{(-b+1)x+b} \; dx = \left[ \frac{a}{b} \right] e^{(b+1)x} \left( N \left( \frac{x}{\sqrt{2a}} \right) \right)^k, \; a, x > 0
\]

(1.36) \[
\int_h^x \left( n(x) \; n(a+bx) \right) \; dx = \int_h^x \left( N(k/b^2+1+b) \; dx - \int_h^x \left( \frac{a}{b} \right) \; n(x) \; N(h/b^2+1+b) \; dx
\]

(1.37) \[
\int_h^x \int_y n(ax) \; n(by) \; dy \; dx = \frac{1}{2ab}
\]

(1.38) \[
\int_h^x \int_h^y n(x) \; n(y) \; N(a+bx+cy) \; dy \; dx = N \left( \frac{a}{\sqrt{1+b^2+c^2}} \right)
\]

(1.39) \[
\int_h^x \int_h^y n \left( \frac{x-a}{a} \right) \; n \left( \frac{y-a}{a} \right) \; dx \; dy = \frac{a^2 (a^2+x^2)}{a^2+y^2} \; n \left( \frac{x}{a^2+y^2} \right) \; n \left( \frac{y}{a^2+y^2} \right)
\]

(1.40) \[
\int_0^x N \left( \sqrt{a^2-x^2} \right) \; n(x) \; dx = \frac{1}{2} \; N(a) - \frac{1}{4} \; e^{-\frac{1}{4} x^2}
\]
(1.41) \[ \int_{-\infty}^{\infty} [N(ax)]^2 [n(x)]^p \, dx = \left( \pi - \arccos \frac{a^2}{p+a^2} \right) p^{-\frac{1}{2}} (2\pi)^{-\frac{p+1}{2}}, \ p \neq 0 \]

(1.42) \[ \int_{-\infty}^{\infty} [N(ax)]^3 [n(x)]^p \, dx = \frac{1}{2} \left( 2\pi - 3 \arccos \frac{a^2}{p+a^2} \right) p^{-\frac{3}{2}} (2\pi)^{-\frac{p+3}{2}}, \ p \neq 0 \]
bivariate normal density and distribution

\[ n(x, y) = \frac{1}{2\pi \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2)} \]

\[ f(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left[ \frac{(x-x_0)^2}{\sigma_x^2} - 2\rho \frac{(x-x_0)(y-y_0)}{\sigma_x \sigma_y} + \frac{(y-y_0)^2}{\sigma_y^2} \right]} \]

\[ N_2(h, k; \rho) = \int_{-\infty}^{h} \int_{-\infty}^{k} n(x, y) \, dx \, dy \]

(2.1) \( N_2(0, 0; \rho) = \frac{1}{2\pi} \arcsin(\rho) + \frac{1}{4} \)

(2.2) \( N_2(h, \infty; \rho) = N(h) \)

(2.3) \( \rho = 0 \) if and only if \( N_2(h, k; \rho) = N(h)N(k) \)

(2.4) \( N_2(-h, -k; \rho) = N_2(h, k; \rho) + 1 - N(h) - N(k) \)

(2.5) \( N_2(h, k; \rho) = \int_{-\infty}^{h} n(x) \frac{N \left( \frac{k-x}{\sqrt{1-\rho^2}} \right)}{\sqrt{1-\rho^2}} \, dx \)

(2.6) \( \int_{-\infty}^{\infty} n(x) N(a_1+b_1 x) N(a_2+b_2 x) \, dx = N_2 \left( \frac{a_1}{\sqrt{1+b_1^2}}, \frac{a_2}{\sqrt{1+b_2^2}}; \frac{b_1 b_2}{\sqrt{1+b_1^2 \sqrt{1+b_2^2}}} \right) \)

(2.7) \( \int_{-\infty}^{h} \int_{-\infty}^{k} e^{yf(x, y)} \, dy \, dx = e^{\frac{1}{2}\frac{1}{\rho^2} \sigma_y} N \left( \frac{h-\mu_x}{\sigma_x} - \rho \sigma_y \right) \)

(2.8) \( \int_{-\infty}^{h} \int_{-\infty}^{k} e^{x f(x, y)} \, dx \, dy = e^{\frac{1}{2}\frac{1}{\rho^2} \sigma_x \sigma_y + \frac{1}{2} \sigma_x^2 - \frac{1}{2} \sigma_y^2} N \left( \frac{h-\mu_x}{\sigma_x} - \rho \sigma_y - \sigma_x \right) \)

(2.9) if \( \frac{\partial \rho}{\partial a} = 0 \), then \( \frac{\partial N_2(h, k; \rho)}{\partial a} = N \left( \frac{k-p h}{\sqrt{1-\rho^2}} \right) \frac{n(h)}{\sqrt{1-\rho^2}} + N \left( \frac{h-p k}{\sqrt{1-\rho^2}} \right) \frac{n(k)}{\sqrt{1-\rho^2}} \)
\[
(2.10) \quad \frac{\partial N_z(h,k;\rho)}{\partial \rho} = \frac{1}{2\pi \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(h^2 + 2h \rho k + k^2)}
\]

\[
(2.11) \quad E(xy) = \mu_x \mu_y + \rho \sigma_x \sigma_y
\]

\[
(2.12) \quad Var(xy) = \sigma^2_x \sigma^2_y (1-\rho^2) + \mu^2_x \sigma^2_y + \mu^2_y \sigma^2_x + 2 \mu_x \mu_y \rho \sigma_x \sigma_y
\]

\[
(2.13) \quad Cov(xy, z) = \mu_x Cov(x, z) + \mu_y Cov(y, z)
\]

\[
(2.14) \quad \text{if } \mu_x = \mu_y = 0, \text{ then } E[x|y] = 0
\]

\[
(2.15) \quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 + bx + c)} N(\delta x + e) dx = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{4}\left(c - \frac{b^2}{4a}\right)} N_2 \left( b\sqrt{a} + \frac{b}{2\sqrt{a}}, \sqrt{\frac{a}{a+\delta^2}} \left( e - \frac{b\delta}{2a} \right); \frac{\delta}{\sqrt{a+\delta^2}} \right)
\]

\[
(2.16) \quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 + bx + c)} N(\delta x + e, fx + g; p) dx = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{4}\left(c - \frac{b^2}{4a}\right)} N_2 \left( \frac{a}{a+\delta^2} \left( e - \frac{b\delta}{2a} \right), \frac{a}{a+f^2} \left( g - \frac{bf}{2a} \right); \frac{\delta f + ap}{\sqrt{(a+\delta^2)(a+f^2)}} \right)
\]
conditional bivariate normal density and distribution

\[ f(y|x) = \frac{f(x,y)}{f(x)} = \frac{1}{\sigma_y \sqrt{2\pi(1-p^2)}} \exp \left(-\frac{1}{2(1-p^2)} \left(\frac{(y-\mu_y - \rho \sigma_x}{\sigma_y} (x-\mu_x)^2\right)\right) \]

(3.1) \[ \int_{-\infty}^{\infty} f(y|x) \, dy = \mu_y + \rho \frac{\sigma_y}{\sigma_x} (x-\mu_x) = \mu_{y|x} \]

(3.2) \[ \int_{-\infty}^{\infty} (y-\mu_{y|x})^2 f(y|x) \, dy = (1-p^2) \sigma_y^2 = \sigma_{y|x}^2 \]

(3.3) \[ \int_{-\infty}^{\infty} e^{y} f(y|x) \, dx = e^{\mu_y + \rho \frac{\sigma_y}{\sigma_x} (x-\mu_x) + \frac{1}{2} (1-p^2) \sigma_y^2} \]

(3.4) \[ \text{Cov}(x, g(y)) = E[g'(y) \, \text{Cov}(x, y)] \]

trivariate normal density and distribution

(4.1) \[ N_3(h, k, \omega; p_{12}, p_{13}, p_{23}) = N_2(h, k; p_{12}) \]

(4.2) \[ N_3(h, k, \lambda; p_{12}, p_{13}, p_{23}) = \int_{-\infty}^{\infty} n(x) N_2 \left( \frac{h-p_{12}x}{\sqrt{1-p_{12}^2}}, \frac{k-p_{23}x}{\sqrt{1-p_{23}^2}}, \frac{p_{13} - p_{12}p_{23}}{\sqrt{1-p_{12}^2 \cdot \sqrt{1-p_{23}^2}}} \right) \, dx \]

(4.3) \[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) f(z) N(a_1 + b_1 x + c_1 y) N(a_2 + b_2 x + c_2 y) \, dx \, dy \, dz = \]

\[ N_3 \left( \frac{a_1}{\sqrt{1 + b_1^2 + c_1^2}}, \frac{a_2}{\sqrt{1 + b_2^2 + c_2^2}}, \frac{b_1 b_2}{\sqrt{(1 + b_1^2)(1 + b_2^2 + c_2^2)}} \right) \]

(4.4) \[ \int_{-\infty}^{\infty} n(x) N(a_1 + b_1 x) N(a_2 + b_2 x) N(a_3 + b_3 x) \, dx = \]

\[ N_3 \left[ \frac{a_1}{\sqrt{1 + b_1^2}}, \frac{a_2}{\sqrt{1 + b_2^2}}, \frac{a_3}{\sqrt{1 + b_3^2}}, \frac{b_1 b_2}{\sqrt{(1 + b_1^2)(1 + b_2^2 + b_3^2)}}, \frac{b_2 b_3}{\sqrt{(1 + b_2^2)(1 + b_3^2)}}, \frac{b_1 b_3}{\sqrt{(1 + b_1^2)(1 + b_3^2)}} \right] \]
The multivariate normal density function with mean vector \( \mu \) and covariance matrix \( \Omega \)

\[ f_x(\mu, \Omega) = (2\pi)^{-\frac{n}{2}} |\Omega|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (x-\mu)' \Omega^{-1} (x-\mu) \right) \]

is the multivariate normal density function with mean vector \( \mu \) and covariance matrix \( \Omega \).

\[ N_n(h_1; \{\rho_{ij}\}) \]

is the \( m \)-dimensional multivariate normal integral with upper limits of integration \( h_1, \ldots, h_n \) and correlation matrix \( \{\rho_{ij}\} \).

(5.1) \( \exp(w'x) f_x(\mu, \Omega) = \exp(w'x + \frac{1}{2} w' \Omega w) f_x(\mu + \Omega w, \Omega) \)

(5.2) \( N_n(h; \{\rho\}; \text{ all } h's = h \text{ and } \rho's = \rho > 0) = \int_0^h \left[ N\left( \frac{h+y}{\sqrt{1-p^2}} \right) \right] n(x) \, dx \)

(5.3) \( N_n(h_1; \{\rho_{ij}\}; \text{ all } \rho_{ij} \text{ are of the form } \frac{\gamma_i}{\gamma_j} \text{ for } i < j \text{ where } |\gamma_i| < |\gamma_j| \text{ for } i < j ) = \int_0^{h_j} N_{n-1}(h_i'; \{\rho_{ij}\}) n(y) \, dy, \ i, j \neq s \)

where \( \rho_{ij} = \frac{\rho_{ij} - \rho_{is} \rho_{js}}{\sqrt{1-\rho_{is}^2} \sqrt{1-\rho_{js}^2}} \) and \( h_i' = \frac{h_i - \rho_{is} y}{\sqrt{1-\rho_{is}^2}} \), \( i, j \neq s \)

Partition \( x, \mu, \) and \( \Omega \) conformably into: \( x = (x_1, x_2) \), \( \mu = (\mu_1, \mu_2) \), \( \Omega = (\Omega_{11}, \Omega_{12}; \Omega_{21}, \Omega_{22}) \).

(5.4) The marginal density of \( x_1 \) is \( f_{x_1}(\mu_1, \Omega_{11}) \).

(5.5) The conditional density of \( x_1 \) given \( x_2 \) is \( f_{x_1}(\mu_1 + \Omega_{12} \Omega_{22}^{-1} (x_2 - \mu_2), \Omega_{11} - \Omega_{12} \Omega_{22}^{-1} \Omega_{21}) \).
(6.1) Stochastic Differential Equation: \(dX(t) = \mu \, dt + \sigma \, dW(t)\), \(\sigma > 0\)

(6.2) Integrated Version: \(X(t) = x_0 + \mu t + \sigma W(t)\)

(6.3) Transition Probability Density Function: \(P(X(T) \in dx \mid X(0) = x_0) = p(x, x_0, T) \, dx = \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp\left(-\frac{1}{2} \frac{(x-x_0-\mu t)^2}{\sigma^2 t}\right)\)

(6.4) Transition Probability Distribution Function: \(\int_{-\infty}^{x} p(x, x_0, T) \, dx = N\left(\frac{X - x_0 - \mu T}{\sigma \sqrt{T}}\right)\)

(6.5) Time Integral of Transition Probability Density Function
\[
\int_{0}^{T} \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp\left(-\frac{1}{2} \frac{(x-x_0-\mu t)^2}{\sigma^2 t}\right) \, dt = \frac{1}{\mu} \left\{ e^{\frac{2\mu(x-x_0)}{\sigma^2}} \left[ N\left(\frac{X-x_0+\mu T}{\sigma \sqrt{T}}\right) - 1_{x>0} \right] - \left[ N\left(\frac{X-x_0-\mu T}{\sigma \sqrt{T}}\right) - 1_{x<0} \right] \right\}
\]

(6.6) Time Integral of Discounted Transition Probability Distribution Function
\[
\int_{0}^{T} e^{-\lambda t} N\left(\frac{X-x_0-\mu T}{\sigma \sqrt{T}}\right) \, dt = \frac{1}{\lambda} \left[ 1_{x>x_0} e^{-\lambda t} N\left(\frac{X-x_0-\mu T}{\sigma \sqrt{T}}\right) \right] + \frac{1}{2\lambda} \left\{ (\mu-\lambda) e^{\frac{\mu+\lambda}{\sigma^2} (x-x_0)} \left[ 1_{x<x_0} N\left(\frac{X-x_0+\lambda T}{\sigma \sqrt{T}}\right) \right] - (\mu+\lambda) e^{\frac{\mu-\lambda}{\sigma^2} (x-x_0)} \left[ 1_{x>x_0} N\left(\frac{X-x_0-\lambda T}{\sigma \sqrt{T}}\right) \right] \right\}, \lambda = \sqrt{\mu^2 + 2\sigma^2}.
\]
(7.1) **First Passage Time Density Function,** $x_0 > a$

$$P(T_a \geq t) = f(t) \; dt = \frac{x_0 - a}{\sqrt{2 \pi \sigma^2 t^3}} \exp\left(-\frac{1}{2} \left(\frac{x_0 - a + \mu t}{\sigma \sqrt{t}}\right)^2\right) \; dt$$

(7.2) **First Passage Time Distribution Function,** $x_0 > a$

$$p(T_a \leq t) = \int_0^t f(t) \; dt = N\left(\frac{a - x_0 - \mu T}{\sigma \sqrt{T}}\right) + \exp\left(\frac{2 \mu (a - x_0)}{\sigma^2}\right) N\left(\frac{a - x_0 + \mu T}{\sigma \sqrt{T}}\right)$$

(7.3) **Laplace Transform of f(t),** $x_0 > a$:  

$$\int_0^\infty e^{-st} f(t) \; dt = \exp\left[(a - x_0) \frac{\mu + \sqrt{\mu^2 + 2 \lambda \sigma^2}}{\sigma^2}\right]$$

(7.4) **Absorbing Brownian Motion Density Function,** $x_0 > a$

$$p_s(x, x_0, T) \; dx = \frac{\exp\left(-\frac{1}{2} \left(\frac{x-x_0-\mu T}{\sigma \sqrt{T}}\right)^2\right)}{\sqrt{2 \pi \sigma^2 T}} - \exp\left(\frac{2 \mu (a - x_0)}{\sigma^2}\right) \frac{\exp\left(-\frac{1}{2} \left(\frac{x+x_0-2a-\mu T}{\sigma \sqrt{T}}\right)^2\right)}{\sqrt{2 \pi \sigma^2 T}} \; dx$$

(7.5) **Absorbing Brownian Motion Distribution Function,** $x_0 > a$

$$\int_x^\infty p_s(x, x_0, T) \; dx = N\left(\frac{X - x_0 - \mu T}{\sigma \sqrt{T}}\right) - N\left(\frac{a - x_0 - \mu T}{\sigma \sqrt{T}}\right)$$  

$$- \exp\left(\frac{2 \mu (a - x_0)}{\sigma^2}\right) \left[N\left(\frac{X + x_0 - 2a - \mu T}{\sigma \sqrt{T}}\right) - N\left(\frac{x_0 - a - \mu T}{\sigma \sqrt{T}}\right)\right]$$
(8.1) **First Passage Time Density Function,** \( x_0 < b \)

\[
P(T_{x_0} < t) = f(t) dt = \frac{b - x_0}{\sqrt{2\pi \sigma^2 t}} \exp\left(-\frac{1}{2} \left(\frac{b - x_0 - \mu T}{\sigma \sqrt{t}}\right)^2\right) dt
\]

(8.2) **First Passage Time Distribution Function,** \( x_0 < b \)

\[
P(T_{x_0} < T) = \int_0^T f(t) dt = N\left(\frac{x_0 - b + \mu T}{\sigma \sqrt{T}}\right) + \exp\left(\frac{2\mu (b - x_0)}{\sigma^2}\right) N\left(\frac{x_0 - b - \mu T}{\sigma \sqrt{T}}\right)
\]

(8.3) **Laplace Transform of** \( f(t) \), \( x_0 < b \):

\[
\int_0^\infty e^{-sT} f(t) dt = \exp\left(\frac{b - x_0}{\sigma^2}\right) \frac{\mu - \sqrt{\mu^2 + 2\lambda \sigma^2}}{\sigma^2}
\]

(8.4) **Absorbing Brownian Motion Density Function,** \( x_0 < b \)

\[
p_b(x, x_0, T) dx = \exp\left(-\frac{1}{2} \left(\frac{x - x_0 - \mu T}{\sigma \sqrt{T}}\right)^2\right) \frac{\exp\left(-\frac{1}{2} \left(\frac{x + x_0 - 2b - \mu T}{\sigma \sqrt{T}}\right)^2\right)}{\sqrt{2\pi \sigma^2 T}} dx
\]

(8.5) **Absorbing Brownian Motion Distribution Function,** \( x_0 < b \)

\[
\int_{-\infty}^x p_b(x, x_0, T) dx = N\left(\frac{x - x_0 - \mu T}{\sigma \sqrt{T}}\right) \exp\left(\frac{2\mu (b - x_0)}{\sigma^2}\right) N\left(\frac{x + x_0 - 2b - \mu T}{\sigma \sqrt{T}}\right)
\]
Let $x_0 \in (a, b)$ and $T_{ab} = \min(T_a, T_b)$.

(9.1) Transition Probability Density Function

$$p_{ab}(x, x_0, T)dx = \frac{1}{\sqrt{2\pi \sigma^2 T}} \sum_{n=-\infty}^{\infty} \left[ \exp\left(\frac{\mu x_n'}{\sigma^2} - \frac{1}{2}\left(\frac{x - x_n'}{\sigma \sqrt{T}}\right)^2\right) - \exp\left(\frac{\mu x_n''}{\sigma^2} - \frac{1}{2}\left(\frac{x - x_n''}{\sigma \sqrt{T}}\right)^2\right) \right]$$

where $x_n' = x_0 + 2n(b - a)$ and $x_n'' = 2b - x_n'$.

(9.2) Probability of Hitting Upper Barrier First: $P(X_{T_{ab}} = b) = \frac{\exp\left(-\frac{2\mu x_0}{\sigma^2}\right) - \exp\left(-\frac{2\mu a}{\sigma^2}\right)}{\exp\left(-\frac{2\mu b}{\sigma^2}\right) - \exp\left(-\frac{2\mu a}{\sigma^2}\right)}$

(9.3) Probability of Hitting Lower Barrier First: $P(X_{T_{ab}} = a) = \frac{\exp\left(-\frac{2\mu b}{\sigma^2}\right) - \exp\left(-\frac{2\mu x_0}{\sigma^2}\right)}{\exp\left(-\frac{2\mu b}{\sigma^2}\right) - \exp\left(-\frac{2\mu a}{\sigma^2}\right)}$