Market Frictions and Consumption-Based Asset Pricing

by

Hua He and David M. Modest

July 1992
RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

The Research Program in Finance in the Walter A. Haas School of Business at the University of California has as its purpose the conduct and encouragement of research in finance, investments, banking, securities markets, and financial institutions. The present reprint and working paper series were established in 1971 in conjunction with a grant from the Dean Witter Foundation.

INSTITUTE OF BUSINESS AND ECONOMIC RESEARCH
Carl Shapiro, Director

The Institute of Business and Economic Research is an organized research unit at the University of California, Berkeley, whose mission is to promote research by faculty and graduate students in the fields of business and economics. The Institute carries out its mission by organizing programs and activities that enrich the research environment, adminstering extramural research awards, publishing working papers, and making direct grants for research.
Market Frictions
and
Consumption-Based Asset Pricing

Hua He and David M. Modest
Haas School of Business
University of California at Berkeley

July 1992

Finance Working Paper #223

We are very grateful to Gail Belonsky and Wei Shi for superb research assistance, and to the Berkeley Program in Finance for financial support. The paper has benefited from comments from Wayne Ferson and seminar participants at Berkeley, Carnegie Mellon, the University of Colorado, and NBER's Summer Institute. We would especially like to thank George Constantinides and Ravi Jagannathan for several very helpful conversations. All errors remain our own.
Abstract

A fundamental equilibrium condition underlying most utility-based asset pricing models, in a world without trading frictions, is the equilibration of intertemporal marginal rates of substitution (IMRS) of consumers. Previous empirical research has found that the co-movements of consumption and asset return data fail to satisfy the restrictions imposed by this equilibrium condition. In this paper, we examine whether market frictions can explain previous findings. Our results suggest that a combination of short-sale, borrowing, and trading cost frictions can drive a large enough wedge between IMRS so that the apparent violations may not be inconsistent with market equilibrium.
1. Introduction

A simple intuition underlying most utility-based asset pricing models\(^1\) is that, in a world without trading frictions, the marginal utility of current consumption, \(u'(c_t)\), multiplied by the current price of an asset \((P_t)\) should equal the discounted expected marginal utility of future consumption at any date \(t + \tau\), \(u'(c_{t+\tau})\), multiplied by the expected price of the asset \textit{cum} any cash disbursements \((P_{t+\tau})\). Formally, this can be written as:

\[
u'(c_t) \cdot P_t = \beta \mathbb{E}_t[u'(c_{t+\tau}) \cdot P_{t+\tau}]
\]

where \(\beta\) is a discount factor representing the rate of time preference and \(\mathbb{E}_t[.]\) denotes the \textit{expectation} conditional on information available at date \(t\). The basic intuition that discounted expected marginal utilities should be equilibrated across time is at the heart of the capital asset pricing model of Sharpe (1964) and Lintner (1965), the intertemporal capital asset pricing model of Merton (1973), the multi-period valuation model of Rubinstein (1976), the equilibrium asset pricing model of Lucas (1978), and the consumption-based CAPM of Breeden (1979).

Hansen and Singleton (1982, 1983), in two seminal papers, parameterized the utility function of a representative investor\(^2\) and tested the over-identifying restrictions implied by the first-order condition (1) for different asset classes. Their results yield “economically plausible estimates” of the representative investor’s coefficient of relative risk aversion and time-preference parameter. However, in general, they are able to reject the restrictions implied by (1).\(^3\) Numerous researchers have attempted to explain the apparent failure of co-movements of per-capita consumption and asset returns to satisfy this first-order condition which equilibrates intertemporal marginal rates of substitution. Among the explanations are: non-separable preferences\(^4\), problems associated with the consumption of durable goods and seasonalities in consumption\(^5\), investor heterogeneity\(^6\), non-stationarities due to the business cycle\(^7\), and statistical problems associated with the small sample properties of the relevant asymptotic test statistics.\(^8\) While the above mentioned studies have partially accounted for the failure of co-movements of per-capita consumption and asset returns to

\(^{1}\)That presume state independent and time separable utility functions.

\(^{2}\)Assuming a constant relative risk aversion utility function.

\(^{3}\)This literature is also closely related to the equity premium puzzle as discussed in Mehra and Prescott (1985).


\(^{8}\)Kocherlakota (1990).
satisfy (1), these studies have all adopted a framework in which asset markets operate without any frictions for trading.

In this paper, we examine whether the presence of market frictions can explain the failure of consumption and asset return data to satisfy the restrictions imposed by the equilibration of intertemporal marginal rates of substitution (IMRS). We consider three types of market frictions. The first market friction is a no short-sale constraint which prevents the short-selling of the riskless asset and hence borrowing at the risk-free rate. The second type of portfolio constraint is a borrowing constraint, which precludes investors' current consumption from exceeding their current financial wealth. This amounts to preventing investors from borrowing against future labor income. Finally, we consider the impact of transaction costs. Transaction costs include paying bid-ask spreads as well as commissions that can be either fixed or proportional to the size of the trade. For simplicity, we model transaction costs by a simple commission structure that is proportional to the size of the trade. Combinations of no short-sale constraints, borrowing constraints, and transaction costs are also considered in the paper.

The paper is organized as follows. In the next section, we derive necessary conditions for market equilibrium in the presence of trading frictions. The necessary conditions amount to a set of first order conditions which must hold for every investor in the economy in the presence of market frictions. They are the natural analog to the first order condition (1) which holds for a representative investor in a frictionless world. Section 3 extends the framework developed in Hansen and Jagannathan (1991), and develops diagnostic tests of the impact of market frictions on the equilibrium relation between asset returns and IMRS. We discuss the data and the empirical results in section 4. The results suggest that neither transaction costs or portfolio constraints, by themselves, can explain the apparent failure of co-movements of per-capita consumption and asset returns to satisfy the first-order conditions which equilibrate intertemporal marginal rates of substitution. However, the results suggest the combination of these market frictions can drive a large enough wedge between IMRS so that these apparent violations may not be inconsistent with market equilibrium. Section 5 concludes.

---

5 A closely related analysis of the impact of market frictions has been independently performed by Luttmer (1991). Our results and Luttmer's are surveyed in Cochrane and Hansen (1992). Luttmer derives equilibrium conditions from a no arbitrage argument, while we derive equilibrium conditions directly from a set of first order conditions that individuals' intertemporal marginal rates of substitution must satisfy. For purposes of testing utility-based asset pricing models, these two approaches are essentially equivalent.

10 Zeldes (1989) has shown empirically that borrowing constraints affect the consumption of a significant portion of the population.
2. Equilibrium with Market Frictions

Consider a discrete-time securities market economy in which there are \( N + 1 \) assets available for investing at dates 0, 1, 2, \( \cdots \). The first \( N \) assets are risky securities. We denote the \( N \) vector of their dollar returns (principal plus any capital gain or loss and cash disbursement), from dates \( t \) to \( t + 1 \), by \( \bar{R}_{t+1} \). These assets are assumed to be in non-negative net supply in the economy. The \( N + 1^{st} \) asset is a riskless bond, which earns a certain return of \( R_{f,t+1} \) (principal plus interest) between dates \( t \) and \( t + 1 \). The bond is assumed to be a financial asset and hence in zero net total supply.

This economy is populated with many investors. A typical investor is endowed with a non-negative stream of labor income \( \bar{y}_t, t = 0, 1, 2, \cdots \).\(^{11}\) This income can be used for either consumption or investment. We assume that at each point in time every investor makes consumption and investment decisions in order to maximize expected lifetime discounted utility:

\[
\max_{c_t \geq 0} E_t \left[ \sum_{j=0}^{\infty} \beta^j u(\bar{c}_{t+j}) \right], \quad 0 < \beta < 1
\]

subject to the dynamic budget constraint:

\[
\bar{W}_{t+1} = (W_t - c_t)(\omega_t^f \bar{R}_{t+1} + (1 - \omega_t^f)R_{f,t+1}) + \bar{y}_{t+1}
\]

for \( j = 0, 1, 2, \cdots \), where \( \omega_t \) is an \( N \) vector of portfolio weights, \( \mathbf{1} \) is an \( N \) vector of ones, and \( E_t[\bullet] \) denotes the conditional expectation based upon information available as of date \( t \).\(^{12}\) Financial wealth \( W_t \) is the amount of wealth available at time \( t \) for current consumption and investment.\(^{13}\) We will assume state independent and time separable utility functions throughout this paper. More general preferences could easily be introduced. We would expect that equilibrium conditions under more general preferences will be similar to those derived here, except that we would have to replace the ratio of marginal utilities by an appropriately defined IMRS.

The investor's optimal consumption and investment problem (2) can be solved by dynamic programming. Let \( J(W_t, t) \) be the indirect utility function for time \( t \), where we have omitted the dependence of \( J \) on the information set at time \( t \). The investor solves:

\[
J(W_t, t) = \max_{c_t \geq 0} u(c_t) + \beta E_t J(\bar{W}_{t+1}, t+1)
\]

\(^{11}\) For ease of notation, we will not index investors even though investors can have different utility functions and endowments. We presume, in a Walrasian spirit, that all investors face identical asset menus. Investors are not able to signal individual traits — such as whether they are likely to be more or less liquidity constrained than average.

\(^{12}\) The expectation symbol \( E_t[\bullet] \) (without the \( t \) subscript) will subsequently be used to denote the unconditional expectation operator.

\(^{13}\) It does not include the present value of future labor income.
where $\tilde{W}_{t+1}$ is determined by (3). If there are no market frictions, the equilibrium conditions for $(\tilde{R}_t, R_{ft})$ require: \(^{14}\)

\[ E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{t+1} \right] = 1, \]

\[ E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} R_{ft+1} \right] = 1, \quad t = 0, 1, 2, \ldots \] \(^{16}\)

where we have presumed the price of the asset (denominated in terms of the consumption good) equals one. Equations (5) and (6) are necessary conditions for equilibrium and should hold for every investor in this economy. That is, if $(\tilde{P}_t, R_{ft}, t = 0, 1, 2, \ldots)$ is an equilibrium sequence of returns, the returns must satisfy (5) and (6) for all investors — regardless of their utility functions and intertemporal consumption allocations.

Equilibrium conditions (5) and (6) are obtained when there are no market frictions. However, in practice, investors face market frictions in implementing their optimal consumption and investment policies. For example, short-selling fixed income securities, such as treasury bills, is often difficult and costly as exemplified by significant differences between treasury bill yields and repo rates. In addition, some assets, such as labor income, are nontradable and hence cannot be used as collateral for borrowing purposes. Finally, investors face sizable bid–ask spreads in many markets where trades, at least for relatively impatient traders, must be transacted through dealers. In this paper, we consider five combinations of market frictions: (i) Short-sale constraints which preclude investors from short-selling the riskless asset. (ii) Borrowing constraints which require the investor’s current consumption not to exceed total financial wealth. This is equivalent to the condition that the value of financial holdings be non-negative at each point in time. Borrowing constraints thus prevent investors from consuming by borrowing against future labor income. \(^{15}\) (iii) The combination of borrowing and short-sale constraints. (iv) Transaction costs which affect investors through direct commissions and bid–ask spreads. (v) The combination of transaction costs, and borrowing and short-sale constraints.

The equilibrium conditions for all five cases can be derived by writing down the first–order conditions for the investor’s consumption and investment rules at the optimum. We will simply state the first–order conditions for the first three cases. A detailed proof is provided for the last two cases. In the presence of these market frictions, the equilibrium conditions for $(\tilde{R}_t, R_{ft})$ can be summarized as follows:

\(^{14}\) Implicit in these conditions are the assumptions that there exists a solution to each investor’s consumption and portfolio problem and that the optimal consumption allocation for each individual is strictly positive.

\(^{15}\) Borrowing constraints are closely related to solvency constraints (see Hakansson, 1970) which allow consumption to exceed current financial wealth as long as bankruptcy does not occur in the end. Thus, solvency constraints are weaker than borrowing constraints. The distinction between borrowing and solvency constraints is discussed further in footnote 17.
Short-sale Constraints on the Riskless Asset:

In this case, an investor solves (4) subject to the constraint that the holdings of the riskless asset cannot be negative. This yields:

\[ E_t \left[ \beta \frac{u'(\hat{c}_{t+1})}{u'(c_t)} \hat{R}_{t+1} \right] = 1, \]  
(7)

\[ E_t \left[ \beta \frac{u'(\hat{c}_{t+1})}{u'(c_t)} R_{f_{t+1}} \right] \leq 1. \]  
(8)

The returns on the risky assets satisfy the same equality first-order conditions as in (1). The inequality restriction for the riskless asset may be strict in (8) since in equilibrium the investor may hold a zero amount in the riskless bond. This is the corner solution.

Borrowing Constraints:

In this case, an investor's objective function becomes:

\[ J(W_t, t) = \max_{c_t \geq 0} u(c_t) + \beta E_t J(W_{t+1}, t+1) + \varphi_t(W_t - c_t), \]  
(9)

where \( \varphi_t \) is the Lagrangian multiplier for the borrowing constraint \( c_t \leq W_t \). This yields the following first order conditions:

\[ E_t \left[ \beta \frac{u'(\hat{c}_{t+1})}{u'(c_t)} (\hat{R}_{t+1} - R_{f_{t+1}}) \right] = 0, \]  
(10)

\[ E_t \left[ \beta \frac{u'(\hat{c}_{t+1})}{u'(c_t)} \hat{R}_{t+1} \right] \leq 1, \quad E_t \left[ \beta \frac{u'(\hat{c}_{t+1})}{u'(c_t)} R_{f_{t+1}} \right] \leq 1. \]  
(11)

Strict inequalities may hold in (11) since the consumption plan at the optimum may be the corner solution, i.e., \( c_t = W_t \) for some \( t \).

16 These first order conditions also imply that:

\[ E_t \left[ \beta \frac{u'(\hat{c}_{t+1})}{u'(c_t)} (\hat{R}_{t,i_{t+1}} - \hat{R}_{j_{t+1}}) \right] = 0, \quad \forall \text{ risky assets } i, j \]

17 One form of a solvency constraint amounts to requiring that \( W_{t+1} \geq L_{t+1} \) for some pre-determined lower bound \( L_{t+1} \). The first order condition now becomes:

\[ u'(c_t) = E_t \left[ (\beta u'(\hat{c}_{t+1}) + \varphi_{t+1}) \hat{R}_{t+1} \right], \]

for some \( \varphi_{t+1} \geq 0 \). If risky assets have limited liability, then the above condition implies that (11) holds in equilibrium. However, (10) may not hold if the Lagrangian multiplier varies over time and is not independent of asset returns. The distinction between solvency and borrowing constraints is discussed further in Cochrane and Hansen (1992). The equilibrium conditions (11), derived under solvency constraints, are also identical to the first order conditions which must hold when: (i) there are short-sale constraints on all risky and riskless assets and (ii) in the heterogenous agent model of Constantinides and Duffie (1991).
Short-sale and Borrowing Constraints:

When short-sale and borrowing constraints are both imposed, we can combine the above two cases to get the following first order conditions:

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} (R_{ft+1} - \tilde{R}_{t+1}) \right] \leq 0, \tag{12}
\]

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{t+1} \right] \leq 1, \tag{13}
\]

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} (\tilde{R}_{t+1} - \tilde{R}_{j,t+1}) \right] = 0 \quad \forall \text{ risky assets } i, j \tag{14}
\]

where \( \mathbf{0} \) denotes an \( N \) vector of zeros.

Transaction Costs:

The above equilibrium conditions are obtained when there are no costs associated with trading. In practice, however, market frictions such as transaction costs may affect equilibrium expected returns. For instance, the bid-ask spread for the smallest decile of NYSE and AMEX stocks is approximately 7.5% — compared to a bid-ask spread of approximately 0.5% for the largest decile.\(^{18}\) In addition, stock traders face direct commission costs on the order of 0.25%. On the fixed income side, transaction costs are approximately 0.015% for actively-traded treasury bills, 0.03% for active treasury bonds, and 0.5% for long-term corporate bonds. In the following proposition, we summarize the equilibrium conditions on expected returns in the presence of transaction costs. We assume that transaction costs are paid in proportion to the amount traded, and use \( \rho_j \) to denote the proportional cost for asset \( j \).

In order to avoid potential technical problems that can arise for utility functions that have infinite marginal utility at a zero consumption level, we make the following two assumptions: the consumption plan at the optimum is strictly positive and bounded away from zero; and all of the returns \( \tilde{R}_{j,t+1} \) have compact support.

**Proposition 1** In a world with proportional transaction costs, the returns earned on assets in equilibrium must satisfy:

\[
\left( \frac{1 - \rho_j}{1 + \rho_j} \right) \leq E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right] \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right) \tag{15}
\]

\(^{18}\) All transaction cost figures are presented as order of magnitude estimates. They are estimates of current costs. These numbers should thus serve as lower bounds to the costs incurred by traders in earlier years. Some investors, however, who are forced to purchase or liquidate assets, may effectively face zero marginal transaction costs. The equity numbers were obtained from the ISSM database for the month of September 1997.
where the subscript \( j \) denotes all assets including the nominally riskless asset. The above inequalities must hold for every investor in the economy. These inequalities may be strict, and also hold in unconditional form.

**Proof.** Consider shifting \( \Delta > 0 \) dollars from current consumption to next period's consumption through investing in asset \( j \). This results in a net investment of \( \Delta/(1 + \rho_j) \) dollars in asset \( j \). The additional consumption generated by this strategy for the next period is \( \frac{1 - \rho_j}{1 + \rho_j} \Delta \tilde{R}_{j,t+1} \). Since any deviation from the optimal consumption plan (in the presence of transaction costs) diminishes expected utility, we get the following:

\[
-u'(c_t) + \beta E_t \left[ u'(\tilde{c}_{t+1}) \left( \frac{1 - \rho_j}{1 + \rho_j} \tilde{R}_{j,t+1} \right) \right] \leq 0.
\]

and hence:

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right] \leq \frac{1 + \rho_j}{1 - \rho_j}.
\]

Similarly, shifting \( \Delta < 0 \) dollars from current consumption to consumption next period yields:

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right] \geq \frac{1 - \rho_j}{1 + \rho_j}.
\]

\[ \Box \]

**Short-sale, Borrowing Constraints and Transaction Costs:**

When portfolio constraints and trading costs are both imposed, we can combine both sets of constraints to get the relevant first order conditions.

**Proposition 2** In a world with short-sale and borrowing constraints, and proportional transaction costs, the returns earned on assets in equilibrium must satisfy:

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right] \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right), \quad (16)
\]

\[
E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right] \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right), \quad (17)
\]

\[
\left( \frac{1 - \rho_j}{1 + \rho_j} \right) \left( \frac{1 - \rho_j}{1 + \rho_j} \right) \leq \frac{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right]}{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right]} \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right) \left( \frac{1 + \rho_j}{1 - \rho_j} \right), \quad (18)
\]

\[
\frac{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} R_{f,t+1} \right]}{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \tilde{R}_{j,t+1} \right]} \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right) \left( \frac{1 + \rho_j}{1 - \rho_j} \right), \quad (19)
\]
where subscripts \(i\) and \(j\) denote risky assets, and \(f\) the riskless asset. All of the above inequalities must hold for every investor in the economy. These inequalities may be strict, and also hold in unconditional form.

**Proof.** Similar to the previous proof, consider shifting \(\Delta > 0\) dollars from current consumption to next period's consumption through investing in asset \(j\) or the riskless asset. Using this argument, (16) and (17) are obtained. Note that these two inequality conditions are one-sided, as we may not be able to shift \(\Delta < 0\) dollars from current consumption to next period's consumption due to the borrowing constraints.

Next, consider shifting \(\Delta > 0\) dollars from asset \(i\) to asset \(j\). To do so, the investor pays a transaction cost of \(\Delta \rho_i\) to get out of asset \(i\), which leaves \((1 - \rho_i)\Delta\) to invest in asset \(j\). Since the investor has to pay \(\rho_j\) proportion of the new investment in asset \(j\) as a transaction cost, the total net investment in \(j\) is \(\frac{1 - \rho_i}{1 + \rho_j} \Delta\).

To bring the asset position back to the original holding, the investor has to add \(\Delta (1 + \rho_i) \tilde{R}_{i,t+1}\) into asset \(i\) and withdraw \(\frac{1 - \rho_i}{1 + \rho_j} \Delta (1 - \rho_j) \tilde{R}_{j,t+1}\) out of asset \(j\). The additional consumption generated from this strategy for the next period is:

\[
\frac{1 - \rho_i}{1 + \rho_j} \Delta (1 - \rho_j) \tilde{R}_{j,t+1} = \Delta (1 + \rho_i) \tilde{R}_{i,t+1}
\]

Since any deviation from the original optimal holdings is not optimal, we obtain

\[
E_t \left[ u'(\tilde{c}_{t+1}) \left( - (1 + \rho_i) \tilde{R}_{i,t+1} + \frac{1 - \rho_i}{1 + \rho_j} (1 - \rho_j) \tilde{R}_{j,t+1} \right) \right] \leq 0 \quad (20)
\]

This implies that

\[
\frac{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} \tilde{R}_{j,t+1} \right]}{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} \tilde{R}_{i,t+1} \right]} \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right) \left( \frac{1 + \rho_i}{1 - \rho_i} \right) \quad (21)
\]

Note that (20) holds in unconditional form, i.e.,

\[
E \left[ u'(\tilde{c}_{t+1}) \left( - (1 + \rho_i) \tilde{R}_{i,t+1} + \frac{1 - \rho_i}{1 + \rho_j} (1 - \rho_j) \tilde{R}_{j,t+1} \right) \right] \leq 0
\]

Thus, (21) also holds in unconditional form:

\[
\frac{E \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} \tilde{R}_{j,t+1} \right]}{E \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} \tilde{R}_{i,t+1} \right]} \leq \left( \frac{1 + \rho_j}{1 - \rho_j} \right) \left( \frac{1 + \rho_i}{1 - \rho_i} \right)
\]

Similarly, we have

\[
\left( \frac{1 - \rho_j}{1 + \rho_j} \right) \left( \frac{1 - \rho_i}{1 + \rho_i} \right) \leq \frac{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} \tilde{R}_{j,t+1} \right]}{E_t \left[ \beta \frac{u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} \tilde{R}_{i,t+1} \right]}
\]
which holds in unconditional form as well.

Finally, consider shifting $\Delta > 0$ dollars from asset $i$ to the riskless asset. This yields (19). Note that (19) is one-sided due the short-sale constraints for the riskless asset. \[ \square \]

3. Market Frictions and Intertemporal Substitution: Diagnostic Tests

In this section, we discuss diagnostic tests of the impact of market frictions on the equilibrium relation between asset returns and intertemporal marginal rates of substitution (IMRS). We make use of the framework developed by Hansen and Jagannathan (1991), who derive the relation between historical IMRS (based on consumption data) and the mean–standard deviation frontier for IMRS implied by security return data.

Historical IMRS will be computed using per-capita consumption data. Aggregation is difficult to achieve in economies with general incomplete markets. Hence, we make the following simplifying assumption. There exists an *average* investor in our economy who, at each time $t$, consumes an amount equal to average per-capita consumption. Our *average investor* assumption is somewhat different from the usual representative agent assumption. Consequently, we do not have to require that the average investor holds the per-capita net supply of assets. This implies that in equilibrium the *average* investor may hold either a long or short position in the $i^{th}$ risky asset. The existence of an average consumer ensures that all of the equilibrium conditions derived in section 2 must be satisfied for per-capita consumption data.

3.1. IMRS Frontiers

3.1.1. Short-Sale and Borrowing Constraints

Let us define: $\tilde{m}_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$, where $c_t$ denotes per-capita consumption at time $t$. In the presence of *short-sale* and *borrowing* constraints (but in the absence of transaction costs), we can generally re-write the Euler equations derived above as:

$$\Delta^L \leq E_t \left[ \tilde{m}_{t+1} \tilde{R}_{t+1} \right] \leq \Delta^U \tag{22}$$

where $\Delta^L$ and $\Delta^U$ are $N$-vectors whose elements are restricted by the equilibrium relations presented in section 2.

For estimation purposes, we assume the time series sequences of the growth rates of per-capita consumption and asset returns are jointly stationary and ergodic. This implies the sequence $\tilde{m}_t, \tilde{R}_t$
is also stationary and ergodic. These stationarity assumptions allow us to invoke the unconditional expectation operator and drop the time index in (22) such that:

$$ \mathbb{E} \left[ \tilde{m}_t \tilde{R}_t \right] = \lambda \quad \Delta^L \leq \lambda \leq \Delta^U $$

Equation (23) is the theoretical basis for our construction of the mean–standard deviation frontier for IMRS in the presence of market frictions.\(^{19}\)

We now discuss the construction of a proxy portfolio to mimic \( \tilde{m}_t \). As there may not exist riskless, in terms of the consumption good, unit discount bonds, we let \( v \) be the implicit price of a unit discount bond, i.e., \( v = \mathbb{E}[\tilde{m}_t] \).\(^{20}\) Following Hansen and Jagannathan (1991), let us project the payoffs \( \tilde{m}_t \) onto the space generated by the \( N \)-vector of returns \( \tilde{R}_t \) and a unit vector:

$$ \tilde{m}_t = \alpha + \tilde{R}'_t \beta_v + \tilde{\epsilon}_{v,t} $$

(24)

where \( \alpha \) is a scalar, \( \beta_v \) is an \( N \)-vector, and \( \sigma(\tilde{m}) \) denotes the standard deviation of \( \tilde{m}_t \). Since \( \tilde{m}_t \) is unobservable this regression can't be run in the usual fashion. However, one can make use of the population pricing restrictions:

$$ \mathbb{E} \left[ \tilde{m}_t \tilde{R}_t \right] = \lambda $$

$$ \mathbb{E}[\tilde{m}_t] = v $$

and the least squares normal equations to construct the minimum-variance mimicking portfolio. It is straightforward to show that the portfolio which has payoffs \( \tilde{m}_{v,t} \), that are most highly correlated with \( \tilde{m}_t \) among the class of portfolios with linear unbiased payoffs (i.e. those satisfying the previous two conditions), is:\(^{21}\)

$$ \tilde{m}_{v,t} = v + (\tilde{R}_t - \mathbb{E}[\tilde{R}_t])' \Sigma^{-1}(\lambda - v \mathbb{E}[\tilde{R}_t]) $$

(25)

where \( \Sigma \) is the covariance matrix of \( \tilde{R}_t \). This portfolio’s returns have standard deviation:

$$ \sigma(\tilde{m}_v) = [\lambda - v \mathbb{E}[\tilde{R}_t]' \Sigma^{-1}(\lambda - v \mathbb{E}[\tilde{R}_t])]^{1/2} $$

(26)

Since \( \tilde{m}_{v,t} \) is the orthogonal projection of \( \tilde{m}_t \) onto \( \tilde{R}_t \) and a vector of ones:

$$ \sigma(\tilde{m}_v)^2 = \sigma(\tilde{m}_v)^2 + \sigma(\tilde{\epsilon}_v)^2 \geq \sigma(\tilde{m}_v)^2 $$

(27)

\(^{19}\)Hansen and Jagannathan (1991) derive the theoretical frontier for IMRS assuming \( \Delta^L = \lambda = \Delta^U = 1 \).

\(^{20}\)In the absence of market frictions, the existence of a riskless unit discount bond would allow investors to observe the equilibrium value of \( \mathbb{E}[\tilde{m}_{t+1}] \). However, even if the riskless unit discount bond existed, market frictions might cause the observed price to differ from \( \mathbb{E}[\tilde{m}_{t+1}] \). We denote \( v \) to be the price at which a riskless unit discount bond would trade. Since \( v \) is not observable, the bounds on IMRS will be computed for a range of values for \( v \).

\(^{21}\)\( \beta_v = \Sigma^{-1}(\lambda - v \mathbb{E}[\tilde{R}_t]) \) and \( \alpha = v - \mathbb{E}[\tilde{R}_t] \beta_v \). For notational simplicity, we do not explicitly show the dependence of \( \tilde{m}_{v,t} \) on \( \Delta \).
Thus, the region defined by
\[ \Omega_{\lambda} = \{(v, \sigma) \in \mathbb{R}^2 : \sigma \geq \sigma(\bar{m}_v)\} \]
is the feasible region for the mean and standard deviation of IMRS. Or equivalently, \( \sigma(\bar{m}_v) \) is a lower bound for the standard deviation of IMRS for a given \( v \).

The lower bound for the volatility of IMRS depends on the population value of \( \lambda \). In practice, it is difficult to obtain sample estimates of \( \lambda \) without explicitly solving for the general equilibrium. To avoid this problem, we take the worst case to find the lowest possible bound for IMRS:

\[ \bigcup_{\lambda^L \leq \lambda \leq \lambda^U} \Omega_{\lambda} \quad (28) \]

Specifically for a given \( v \), the lower bound can be found by choosing \( \lambda \) to minimize (26). This feasible region can be estimated using sample means and covariances of asset returns and does not require consumption data. Given the sharp rejections of the bounds found by Hansen and Jagannathan (1991) and others, the focus on the lowest possible bound provides a minimum hurdle for the consumption and return data.\(^{22}\)

3.1.2. Transaction Costs

In the presence of transaction costs, our restrictions are of the form:

\[ \lambda_i^L \leq E_t[\bar{m}_{t+1}\bar{R}_{t+1}] \leq \lambda_i^U \quad (29) \]

\[ \lambda_{ij}^L \leq \frac{E_t[\bar{m}_{t+1}\bar{R}_{j,t+1}]}{E_t[\bar{m}_{t+1}\bar{R}_{i,t+1}]} \leq \lambda_{ij}^U, \quad i \neq j \quad (30) \]

where \( \lambda_i^L, \lambda_i^U, \lambda_{ij}^L \) and \( \lambda_{ij}^U \) are the constant bounds derived in propositions 1 and 2. These restrictions can be rewritten in unconditional form by invoking the unconditional expectation operator:

\[ E_t[\bar{m}_t\bar{R}_t] = \Lambda, \quad \lambda_i^L \leq \lambda_i \leq \lambda_i^U \]

\[ \lambda_{ij}^L \leq \frac{\lambda_i}{\lambda_j} \leq \lambda_{ij}^U \quad (31) \]

Again, we can take the worst case to find the lowest possible bound for IMRS:

\[ \bigcup_{\lambda \in \Lambda} \Omega_{\lambda} \quad (32) \]

where

\[ \Lambda = \{ \lambda : \lambda_i^L \leq \lambda_i \leq \lambda_i^U, \lambda_{ij}^L \leq \frac{\lambda_i}{\lambda_j} \leq \lambda_{ij}^U \} \]

\(^{22}\)The estimation procedure outlined above does not utilize the a priori information that the intertemporal marginal rates of substitution must be positive. In their paper, Hansen and Jagannathan (1991) suggest a procedure for imposing this constraint. In general, this leads to a sharper test. Their results, however, suggest there are negligible differences in the relevant regions. Hence, we ignore the positivity constraint.
Given the sharp rejections of the IMRS bounds found by Hansen and Jagannathan (1991) and others, the focus on the lowest possible bound, defined by (28) and (32), provides a minimal hurdle for the consumption and return data. Hence, we can reject unambiguously the model if historical IMRS lie outside the feasible region.

There is an important difference between testing models with frictions and models without frictions. In economies without market frictions, equilibrium conditions usually take an equality form as in (1). Rejection or acceptance of these equality conditions hence leads to an unambiguous conclusion as to whether the equilibrium conditions are statistically satisfied — abstracting from the classical statistical inference problem that one can never ‘accept’ a null hypothesis but only ‘fail to reject.’ In economies with market frictions as in this paper, however, the equilibrium conditions are expressed in terms of a series of inequality relations which do not fully specify all of the equilibrium conditions. Rejection of these inequality conditions thus clearly leads to the unambiguous conclusion that the equilibrium conditions are not satisfied. However, failure to reject the inequality conditions does not necessarily lead to an unambiguous conclusion that the equilibrium conditions are satisfied for two reasons. First, we can only ‘reject’ or ‘fail to reject’ the null hypothesis under classical statistical inference. Second, unless a general equilibrium solution is solved explicitly, the equilibrium conditions are still not fully tested. Thus, in general, failure to reject the inequality conditions can only be interpreted as “there is no evidence that the data is inconsistent with market equilibrium” — for more than the usual reasons.

3.2. IMRS Frontiers with Conditioning Instruments

The feasible regions derived above can be sharpened if we incorporate conditioning instruments into the analysis. Specifically, let $z_t$ be a non-negative instrumental variable observable at time $t$. Define the re-normalized variable $r^*_i = z_t/E[z_t]$ which has an unconditional mean equal to one. In the absence of transaction costs, the following equilibrium restriction must also hold:

$$
\Delta_i^L \leq E \left[ \tilde{m}_{t+1} \tilde{R}_{t+1} r^*_i \right] \leq \Delta_i^U
$$

(33)

for each asset and all eligible instruments. This relation reduces to (23) in the special case of a single instrument $z$, which is equal to a constant.

Similarly, in the presence of transaction costs:

$$
\Delta_i^{L*} \leq E \left[ \tilde{m}_{t+1} \tilde{R}_{t+1} r^*_i \right] \leq \Delta_i^{U*}
$$

(34)

---

The portfolio approach used by Hansen and Jagannathan (1991) can be thought of as constructing IMRS frontiers with conditioning instruments.
\[ \Delta_{ij}^L \leq \frac{E \left[ \tilde{r}_{t+1} \Delta_i \tilde{r}_{t+1} \right]}{E \left[ \tilde{r}_{t+1} \tilde{r}_{t+1} \right]} \leq \Delta_{ij}^U \] (35)

The feasible IMRS regions, based on the above relations, can be drawn using the same techniques discussed in the previous subsection. They will undoubtedly lie inside the feasible regions based on (23) and (31) — presuming one of the instruments is a constant vector, and hence sharpen the volatility bound for IMRS.

4. Market Frictions and Intertemporal Substitution: Empirical Results

4.1. Data

All of the empirical results reported in this paper utilize the SBBI and Ibbotson monthly returns data files supplied by CRSP. The SBBI data consists of the returns on a representative U.S. treasury bond with “a term of approximately 20 years and a reasonably current coupon”, the returns associated with the Salomon Brothers Long-Term High-Grade Corporate Bond index, and the returns on a one-month U.S. treasury bill.\(^{24}\) The series used from the Ibbotson data files are the returns on a value-weighted portfolio of all stocks on the NYSE, and an equally-weighted portfolio of NYSE stocks. Monthly seasonally adjusted real consumption\(^{25}\), the implicit consumption deflator, and population data were obtained from the Citibase database.\(^{26}\) All returns series were deflated using the implicit consumption deflator for non-durables and services. The monthly consumption data are only available since 1959 and hence our results are based on data from the 1959–1990 period.

Tables 1–3 contain summary statistical information on the consumption and asset return data used in the diagnostic tests. Breeden, Gibbons and Litzenberger (1989) and Heaton (1990) present evidence suggesting that monthly consumption data have quite different statistical properties than quarterly consumption series. Hence, summary statistics and diagnostic tests are presented for both monthly and quarterly data. The summary tables present estimates of the means, standard deviations, skewness, kurtosis, and autocorrelation coefficients of the natural logarithm of one plus.

\(^{24}\)The Salomon Corporate Bond index is used to derive the returns on corporate bonds since 1969. Prior to 1969, the series was constructed by Ibbotson and Sinquefeld. The return on the treasury bill, is the one-month return on a U.S. treasury bill with a minimum maturity of at least one month. For a more complete description of the SBBI series, see Stocks, Bonds, Bills, and Inflation (1991).

\(^{25}\)In tests not reported, we found that the U.S. Commerce Department's X11 seasonal filter does not remove all of the seasonal dependencies in the consumption data. For instance, strong monthly seasonals remain. There also appear to be calendar dependencies based on the number of days in the month, and the number of Mondays, Tuesdays etc. in a month.

\(^{26}\)The population series (POPRES) is a first of the month estimate of the total resident population excluding armed forces overseas.
the growth rate in nondurables and services consumption (table 1), and the natural logarithm of one plus the asset return series (tables 2 and 3). The tables also contain modified Box–Pierce statistics which test the joint hypothesis that all of the presented autocorrelation coefficients are equal to zero. Table 1 contains summary statistics on monthly consumption and two different quarterly measures of consumption: \(\ln[(c_t + c_{t-1} + c_{t-2})/(c_{t-3} + c_{t-4} + c_{t-5})]\) (denoted by 'sum' in the table) and \(\ln[c_t/c_{t-3}]\) (denoted by 'point' in the table). The statistical differences between the behavior of the monthly and quarterly consumption data is quite striking. The monthly series display strong first-order negative autocorrelation \((\rho_1 = -.252)\) whereas the quarterly consumption series display significant positive autocorrelation. The first-order autocorrelation of .248 for the time-averaged quarterly consumption series is very close to the .250 figure which would be predicted by the results of Working (1960) — if actual consumption is a martingale in continuous time but we only observe time-averaged consumption.\(^27\) Most of the autocorrelation coefficients for consumption past the first lag are rather small in size and less than one standard error from zero. The joint hypothesis that the first twelve autocorrelation coefficients for the monthly consumption data are equal to zero can be rejected at very high significance levels (p-value of less than .001). Similarly, the joint hypothesis that the first four autocorrelation coefficients for the time-averaged consumption data are equal to zero can be rejected at very high significance levels. The \(p\)-value for the same test using the quarterly growth rates of monthly consumption is equal to 0.078. One can not reject the hypothesis that the correlation coefficients past the first lag are equal to zero for all of the monthly and quarterly series.

Most of the monthly real returns series display small positive first-order autocorrelation. The first-order autocorrelation coefficients for these series are: .090 for the value-weighted index, .182 for the equally-weighted portfolio, .053 treasury bond returns, .161 for corporate bonds, and .408 for the monthly returns on treasury bills. Modified Box–Pierce tests, testing whether the first twelve autocorrelation coefficients are jointly different from zero, suggest highly significant autocorrelation for all monthly asset return series except the value-weighted index of NYSE stocks.\(^28\) The quarterly returns series, which are defined as \((1 + r_t)(1 + r_{t-1})(1 + r_{t-2}) - 1\), are less autocorrelated than the monthly series. The corresponding first-order serial correlation coefficients are: .069, -.004, -.073, -.005, and .534. Modified Box–Pierce tests of the quarterly series suggest only the treasury bill series has significant autocorrelation.

The sample skewness and kurtosis statistics suggest that the return series are not lognormally

---

\(^{27}\) The first-order autocorrelation coefficient of .160 for the quarterly growth rate of monthly consumption is within two standard errors of .250.

\(^{28}\) The autocorrelation coefficients which are presented do not correct for the small sample downward bias which is of the order of \((-1/\text{number of observations})\). The standard errors are valid asymptotically as is the modified Box–Pierce test.
distributed. All of the returns series have fatter tails than would be expected from a normal distribution. In addition, the equity return series are skewed to the left with evidence of long tails in the positive direction. There is some tendency for fixed income returns to be skewed to the right with long tails in the opposite direction. This evidence of non-normal distributions may impinge on the small sample reliability of the confidence intervals presented below, since they are computed using Monte Carlo techniques which presume lognormality.

4.2. Graphical Results

In this section, we plot the mean-standard deviation frontier for IMRS implied by returns and compare it to the IMRS computed from historical consumption data. All of the graphs contain the frontier for IMRS in the absence of market frictions. This curve serves as a reference for examining the impact of market frictions. The monthly figures utilize data from the 1959:2 - 1990:12 period and the quarterly figure uses data from the 1959:2 - 1990:4 period.

Figures 1 and 2 depict the failure of the IMRS bounds, implied by asset returns in a frictionless world, to satisfy the first-order conditions given by (1) using monthly (figure 1) and quarterly (figure 2) data. The solid lines in figures 1 and 2 plot the mean and standard deviation frontier of IMRS implied by the returns on a value-weighted portfolio of NYSE stocks, an equally-weighted portfolio of the same stock universe, long-term U.S. treasury bonds, an index of high-grade corporate bonds, and one-month treasury bills. The dashed lines surrounding the returns-based IMRS frontier define the 95% confidence interval for the point estimates of the frontier. These confidence bounds were computed using Monte Carlo techniques which assume that the natural logarithm of one plus the return series are multivariate normal.29 Also plotted in these figures are the time series sample means and standard deviations of IMRS implied by an average investor with a CRRA utility function of the form:

\[ u(c_t) = \frac{c_t^{1-\alpha} - 1}{1 - \alpha}, \quad \alpha > 0 \]

A time preference discount factor \( \beta \) of 0.99754 is used for figures based on the monthly data and 0.99241 for the quarterly data.30 The black diamond symbols represent mean-standard deviation pairs of IMRS computed using consumption of non-durables and services for alternative values of \( \alpha \) ranging from 0 to 60 in one unit increments. We have restricted the visible region of the graphs, and hence all 61 mean and standard deviation IMRS points will not generally be shown. In all cases, the most southeast point corresponds to alpha equal to zero.

29 The returns were generated assuming a VAR system with lags 1-6.9, and 12 for the monthly series and lags 1-4 for the quarterly runs. The confidence bounds are based on 3000 simulations.
30 This corresponds to an annual discount factor of 0.97.
Figure 1 shows that the mean and standard deviation pairs of IMRS based on monthly data do not fall into the admissible region for any choice of relative risk aversion coefficient between 0 and 60. The failure manifests itself in that the historical volatility of IMRS based on consumption data is significantly less than the lower bound for the volatility of IMRS computed from asset return data. Figure 2 reproduces figure 1 using the two quarterly consumption series. Despite the fact that quarterly growth rates in consumption behave very differently from monthly consumption growth rates in terms of their autocorrelation and kurtosis properties, the two figures are very similar qualitatively. Figure 2 contains the mean-standard deviation pairs of IMRS computed using two measures of quarterly growth rates in consumption: time-averaged quarterly growth rates (denoted by the black diamonds) and quarterly growth rates of monthly consumption (denoted by the black circles). The results for the two quarterly consumption series are so similar it is often difficult to distinguish visually between the IMRS computed using the two different series.

Figures 3–8 are similar to figure 1 in that they plot the historical mean and standard deviation frontiers of IMRS based on monthly non-durable and services consumption and the mean and standard deviation frontier for IMRS implied by monthly returns on our five assets (solid curve). These figures also plot the mean and standard deviation frontier for IMRS implied by asset returns in the presence of market frictions (dotted and dashed curves) and their associated 95% confidence intervals (dashed curves). Figure 3 plots the IMRS frontier assuming the riskless asset (treasury bills) can’t be shorted. Figures 4 and 5 plot the frontiers assuming a borrowing constraint and a solvency constraint respectively. Figure 6 plots the returns-based frontier when there are both borrowing constraints and restrictions on shorting the riskless asset. Figure 3 shows that the no-short sale constraint shifts the admissible IMRS region leftward although not sufficiently downward to explain the IMRS based on consumption data. The relevant first-order conditions in the presence of this constraint are: $\mathbb{E} \left[ \beta^{\frac{\gamma}{\sigma(c\gamma)}} T_{ft+1} \right] = 1$ and $\mathbb{E} \left[ \beta^{\frac{\gamma}{\sigma(c\gamma)}} R_{ft+1} \right] = \lambda_f \leq 1$. Our diagnostic empirical procedure searches for the $\lambda_f$ which minimizes the volatility of IMRS based on the return data. A different value of $\lambda_f$ is found for each possible value of $v$, the implicit price of a unit

---

31 Figures similar to figure 1 have also been constructed using three alternative measures of consumption: total consumption including durables, non-durables consumption, and the consumption of services. The historical mean-standard deviation pairs of IMRS only lie within the returns-based IMRS frontier when consumption of services is used as the measure of consumption, and in that case only for extremely large coefficients of relative risk aversion. To conserve space, we do not present those figures.

32 Unlike asset return data which is very precisely measured, the available consumption data suffers from many serious problems which potentially can have important econometric consequences and could explain the apparent anomalies evident in figures 1 and 2. Among the most important problems are: (i) the time aggregation bias introduced by measuring consumption over an interval rather than at a point in time, (ii) measurement errors in consumption due to infrequent sampling of many of the components and difficulties in imputing service flows from durables, and (iii) seasonal patterns in many types of consumption (e.g. home heating oil and automobile gasoline). In this paper, however, we focus on the ability of market frictions to explain the anomalies.

33 The lambdas for the risky assets, in the presence of only a short-sale constraint, are equal to one.
discount bond. The volatility-minimizing \( \lambda \) corresponding to a mean \( v \) of .9976 is equal to .9987 in figure 3 and results in a lower bound in the volatility of monthly IMRS of 0.1002. We focus on this value of \( v \) in our discussion in the text since this approximately corresponds to a real riskless rate of 3% per annum, which would seem to be a reasonable upper bound for riskless real rates if such securities existed.

The imposition of borrowing constraints results in a more dramatic shift in the IMRS frontier, which is shown in figure 4, although it is still insufficient to be consistent with the low historical volatility of IMRS based on consumption data. For an implicit price of a unit discount bond \( v \) equal to .9976, \( \lambda \) equal to .9987 for all assets\(^{34}\) minimizes the standard deviation of the portfolio with payoffs \( \tilde{m}_{n,t} \) which is equal to .1062. The volatility of ex-post IMRS based on monthly consumption data in this region, where the real riskless rate is approximately 3% per year, is less than one-tenth of the volatility bound based on returns. For completeness, we present in figure 5 the returns-based IMRS frontier in the presence of solvency constraints. The primary difference between solvency and borrowing constraints is that solvency constraints place a restriction on future wealth, e.g. \( \tilde{W}_{t+1} \geq L_{t+1} \), and hence current financial wealth can in principle be negative (given positive future labor income), whereas borrowing constraints place a restriction on current financial wealth, i.e. \( c_t \leq W_t \).\(^{35}\) For implementation purposes, the primary distinction is that in the presence of borrowing constraints the \( \lambda \) for all risky assets will be the same and less than or equal to one, whereas they need not be equal under solvency constraints. Relaxation of the equality restriction results in a smaller lower bound on the volatility of IMRS in the presence of solvency constraints than that depicted in figure 4 in the presence of borrowing constraints. For values of the coefficient of relative risk aversion larger than approximately two, the historical IMRS using consumption data now lie inside the returns-based IMRS frontier. The associated lambdas for the value- and equally-weighted stock indices, treasury bonds, corporate bonds, and treasury bills are respectively: .9993, 1.000, .9984, .9987, and .9986. The lower bound on the volatility of IMRS corresponding to \( v \) equal to .9976 is .0811. Figure 6 examines the combined impact of borrowing and no-short sale constraints. The picture differs little from figure 4 and results in a lower bound on the volatility of IMRS of .0992 for a value of \( v \) equal to .9976.

Figures 7 and 8 examine the impact of transaction costs. Figure 7 assumes the following transaction costs: 0.75% for the value-weighted portfolio, 1.5% for the equally-weighted portfolio, 0.015% for treasury bills, 0.03% for treasury bonds, and 0.5% for corporate bonds. As is readily apparent, the introduction of transaction costs by itself does little to explain the disparity between

\(^{34}\)For this case, the lambdas for all the asset are constrained to be equal although they may differ from one.

\(^{35}\)The first order conditions in the presence of solvency constraints are discussed in footnote 17.
historical IMRS based on consumption data and the returns-based IMRS frontier plotted in figure 1. Finally, figure 8 combines the impact of transaction costs, borrowing constraints, and no short-sale restrictions. The figure indicates that the combination of these markets friction sufficiently shifts the lower bound of the admissible IMRS region implied by asset returns so that historical IMRS based on consumption data lie in the region. For \( v = .9976 \), the elements of \( \lambda \) are 1.0024 (value-weighted index), 1.0047 (equally-weighted index), .9989 (treasury bond), .9993 (corporate bond), and .9986 (treasury bill). This figure thus suggests that market frictions may in part explain the apparent deviations between historical IMRS based on consumption data and IMRS implied by asset returns.

One of the limitations of the diagnostic tests presented in this paper is that they provide minimal conditions for asset market equilibrium. Hence, the tests may have low power to reject the null hypothesis, especially in the presence of all three market frictions. In an attempt to improve the power of the diagnostic procedure, we also examined whether conditioning instruments could be used to sharpen the restrictions in the presence of market frictions — following the discussion in Section 3.2. Toward that end, we effectively expanded the number of assets from 5 to 15 by using the absolute values of the lagged real treasury bill returns and the lagged real consumption growth rates as instruments. This did not, however, substantially sharpen the bounds. For instance, for an implicit price of a unit discount bond \( v \) equal to .9976, the use of the two instruments did not measurably increase the lower bound above zero. Hence to conserve space, we do not present an additional graph.

5. Concluding Remarks

In this paper, we have performed diagnostic tests for consumption-based asset pricing models in the presence of market frictions. In particular, we have examined theoretically and empirically the impact of short-sale restrictions on the risk-free asset, borrowing constraints which prevent borrowing against future labor income, and transaction costs on the equilibrium relation between co-movements in consumption and asset returns. Our results show that none of the market frictions alone can explain the apparent rejection of the first order equilibrium conditions between consumption and asset returns, discovered by many researchers. However, a combination of short-sale and borrowing constraints, and trading costs does not yield a rejection of the model. The primary limitation of our analysis is that our diagnostic tests, which generally take the form of inequality restrictions, are likely to be significantly weaker than the standard tests of equality restrictions.
References


Table 1
Summary Statistics for the Natural Logarithm of the Ratios of Real Per-Capita Consumption
\((\ln[C_t/C_{t-1}])\)
1959 – 1990

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Nondurables and Services</th>
<th>Monthly</th>
<th>Quarterly</th>
<th>Sum</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annual Mean (%)</td>
<td>1.910</td>
<td>1.901</td>
<td>1.882</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annual Standard Deviation</td>
<td>1.522</td>
<td>1.037</td>
<td>1.1187</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
<td>-.102</td>
<td>-.234</td>
<td>-.343</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>.417</td>
<td>.287</td>
<td>.119</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kurtosis</td>
<td>.684</td>
<td>.843</td>
<td>.071</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>.007</td>
<td>.059</td>
<td>.874</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_1)</td>
<td>-.252</td>
<td>.248</td>
<td>.160</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_2)</td>
<td>.064</td>
<td>.132</td>
<td>.039</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_3)</td>
<td>.083</td>
<td>.209</td>
<td>.188</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_4)</td>
<td>-.057</td>
<td>.139</td>
<td>.059</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_5)</td>
<td>.023</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_6)</td>
<td>.018</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_9)</td>
<td>-.035</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\rho_{12})</td>
<td>-.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\sigma_\rho)</td>
<td>.052</td>
<td>.090</td>
<td>.090</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modified Box-Pierce</td>
<td>38.69</td>
<td>21.75</td>
<td>8.41</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>.000</td>
<td>.000</td>
<td>.078</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Notes for Table 1

The $p$-values of the mean were computed under the assumption that the natural logarithm of consumption ratios are independently and identically normally distributed random variables. The annual standard deviations were computed by multiplying the monthly (quarterly) standard deviations by $\sqrt{12}$ ($\sqrt{4}$). The monthly (quarterly) moment statistics were computed using data over the 1959:2–1990:12 (1959:2–1990:4) period. The monthly (quarterly) autocorrelation coefficients were computed using data over the 1960:2–1990:12 (1960:2–1990:4) period. The asymptotic standard error of the correlation coefficients was computed under the null hypothesis of zero autocorrelation. The skewness, kurtosis and their associated test statistics were computed using the RATS computer package.
Table 2
Summary Statistics for the Natural Logarithm of the Ratios of Monthly Real Asset Prices
\((\ln[P_t/P_{t-1}])\)
1959 – 1990

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Asset</th>
<th>VW NYSE</th>
<th>EW NYSE</th>
<th>Treasury Bond</th>
<th>Corporate Bond</th>
<th>Treasury Bill</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>NYSE</td>
<td>Treasury</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annual</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (%)</td>
<td></td>
<td>4.595</td>
<td>6.406</td>
<td>1.151</td>
<td>1.593</td>
<td>1.254</td>
</tr>
<tr>
<td>p-value</td>
<td></td>
<td>.094</td>
<td>.074</td>
<td>.515</td>
<td>.328</td>
<td>.000</td>
</tr>
<tr>
<td>Annual</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deviation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
<td></td>
<td>-.659</td>
<td>-.506</td>
<td>.512</td>
<td>.426</td>
<td>-.017</td>
</tr>
<tr>
<td>p-value</td>
<td></td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td>.001</td>
<td>.893</td>
</tr>
<tr>
<td>Kurtosis</td>
<td></td>
<td>2.918</td>
<td>4.182</td>
<td>2.625</td>
<td>3.273</td>
<td>.866</td>
</tr>
<tr>
<td>p-value</td>
<td></td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td>.001</td>
<td>.001</td>
</tr>
<tr>
<td>(\rho_1)</td>
<td></td>
<td>.090</td>
<td>.182</td>
<td>.053</td>
<td>.161</td>
<td>.408</td>
</tr>
<tr>
<td>(\rho_2)</td>
<td></td>
<td>-.056</td>
<td>-.035</td>
<td>.005</td>
<td>-.009</td>
<td>.406</td>
</tr>
<tr>
<td>(\rho_3)</td>
<td></td>
<td>-.002</td>
<td>-.018</td>
<td>-.126</td>
<td>-.066</td>
<td>.296</td>
</tr>
<tr>
<td>(\rho_4)</td>
<td></td>
<td>.011</td>
<td>-.001</td>
<td>.066</td>
<td>.004</td>
<td>.317</td>
</tr>
<tr>
<td>(\rho_5)</td>
<td></td>
<td>.109</td>
<td>.046</td>
<td>.067</td>
<td>.107</td>
<td>.325</td>
</tr>
<tr>
<td>(\rho_6)</td>
<td></td>
<td>-.042</td>
<td>.006</td>
<td>.073</td>
<td>.096</td>
<td>.339</td>
</tr>
<tr>
<td>(\rho_7)</td>
<td></td>
<td>-.002</td>
<td>-.008</td>
<td>.040</td>
<td>.076</td>
<td>.332</td>
</tr>
<tr>
<td>(\rho_{12})</td>
<td></td>
<td>.032</td>
<td>.120</td>
<td>.012</td>
<td>.022</td>
<td>.242</td>
</tr>
<tr>
<td>(\sigma_\varphi)</td>
<td></td>
<td>.052</td>
<td>.052</td>
<td>.052</td>
<td>.052</td>
<td>.052</td>
</tr>
<tr>
<td>Modified</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-Pierce</td>
<td></td>
<td>14.23</td>
<td>28.01</td>
<td>26.54</td>
<td>35.52</td>
<td>483.0</td>
</tr>
<tr>
<td>p-value</td>
<td></td>
<td>.286</td>
<td>.006</td>
<td>.009</td>
<td>.000</td>
<td>.000</td>
</tr>
</tbody>
</table>
Notes for Table 2

*VW NYSE* and *EW NYSE* refer to value-weighted and equally-weighted portfolios of NYSE stocks respectively. The *p-values* of the mean were computed under the assumption that the natural logarithm of one plus asset returns are independently and identically normally distributed random variables. The annual standard deviations were computed by multiplying the monthly standard deviations by $\sqrt{12}$. The monthly moment statistics were computed using data over the 1959:2–1990:12 period. The monthly autocorrelation coefficients were computed using data over the 1960:2–1990:12 period. The asymptotic standard error of the correlation coefficients was computed under the null hypothesis of zero autocorrelation. The skewness, kurtosis and their associated test statistics were computed using the *RATS* computer package.
Table 3
Summary Statistics for the Natural Logarithm of the Ratios of Quarterly Real Asset Prices
\((\ln[P_t/P_{t-1}])\)
1959 – 1990

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Asset</th>
<th>VW NYSE</th>
<th>EW NYSE</th>
<th>Treasury Bond</th>
<th>Corporate Bond</th>
<th>Treasury Bill</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Annual Mean (%)</strong></td>
<td></td>
<td>4.572</td>
<td>6.317</td>
<td>1.119</td>
<td>1.591</td>
<td>1.252</td>
</tr>
<tr>
<td><strong>p-value</strong></td>
<td></td>
<td>.140</td>
<td>.142</td>
<td>.568</td>
<td>.402</td>
<td>.000</td>
</tr>
<tr>
<td><strong>Annual Standard Deviation</strong></td>
<td></td>
<td>17.44</td>
<td>24.27</td>
<td>11.05</td>
<td>10.71</td>
<td>1.36</td>
</tr>
<tr>
<td><strong>Skewness</strong></td>
<td></td>
<td>-.947</td>
<td>-.303</td>
<td>.513</td>
<td>.315</td>
<td>.165</td>
</tr>
<tr>
<td><strong>p-value</strong></td>
<td></td>
<td>.000</td>
<td>.168</td>
<td>.020</td>
<td>.152</td>
<td>.452</td>
</tr>
<tr>
<td><strong>Kurtosis</strong></td>
<td></td>
<td>1.784</td>
<td>1.341</td>
<td>2.873</td>
<td>2.637</td>
<td>.922</td>
</tr>
<tr>
<td><strong>p-value</strong></td>
<td></td>
<td>.000</td>
<td>.003</td>
<td>.000</td>
<td>.000</td>
<td>.039</td>
</tr>
<tr>
<td>(\rho_1)</td>
<td></td>
<td>.069</td>
<td>-.004</td>
<td>-.073</td>
<td>-.005</td>
<td>.534</td>
</tr>
<tr>
<td>(\rho_2)</td>
<td></td>
<td>-.139</td>
<td>-.064</td>
<td>.108</td>
<td>.101</td>
<td>.562</td>
</tr>
<tr>
<td>(\rho_3)</td>
<td></td>
<td>-.066</td>
<td>-.064</td>
<td>.107</td>
<td>.118</td>
<td>.488</td>
</tr>
<tr>
<td>(\rho_4)</td>
<td></td>
<td>-.008</td>
<td>.100</td>
<td>.118</td>
<td>.099</td>
<td>.421</td>
</tr>
<tr>
<td>(\sigma_p)</td>
<td></td>
<td>.090</td>
<td>.090</td>
<td>.090</td>
<td>.090</td>
<td>.090</td>
</tr>
<tr>
<td><strong>Modified Box-Pierce</strong></td>
<td>3.62</td>
<td>2.32</td>
<td>5.44</td>
<td>4.37</td>
<td>129.38</td>
<td></td>
</tr>
<tr>
<td><strong>p-value</strong></td>
<td></td>
<td>.460</td>
<td>.676</td>
<td>.245</td>
<td>.358</td>
<td>.000</td>
</tr>
</tbody>
</table>
Notes for Table 3

VW NYSE and EW NYSE refer to value-weighted and equally-weighted portfolios of NYSE stocks respectively. The p-values of the mean were computed under the assumption that the natural logarithm of one plus asset returns are independently and identically normally distributed random variables. The annual standard deviations were computed by multiplying the quarterly standard deviations by \( \sqrt{4} \). The quarterly moment statistics were computed using data over the 1959:2–1990:4 period. The quarterly autocorrelation coefficients were computed using data over the 1960:2–1990:4 period. The asymptotic standard error of the correlation coefficients was computed under the null hypothesis of zero autocorrelation. The skewness, kurtosis and their associated test statistics were computed using the RATS computer package.
Figure 1
Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

*No Market Frictions*

(1959:2 – 1990:12)
Notes for Figures

Figures 1–2: The solid lines in figures 1 and 2 plot the mean and standard deviation frontier of IMRS implied by the returns on a value-weighted portfolio of NYSE stocks, an equally-weighted portfolio of the same stock universe, long-term U.S. treasury bonds, an index of high-grade corporate bonds, and one-month treasury bills. The dashed lines surrounding the returns-based IMRS frontier define the 95% confidence interval for the point estimates of the frontier. These confidence bounds were computed using Monte Carlo techniques which assume that the natural logarithm of one plus the return series are multivariate normal. The returns were generated assuming a VAR system with lags 1–6,9, and 12 for the monthly series and lags 1–4 for the quarterly runs. The confidence bounds are based on 3000 simulations. The black diamond symbols represent mean-standard deviation pairs of IMRS computed using consumption of non-durables and services for alternative values of the average investor's coefficient of relative risk aversion. The average investor is assumed to have a CRRA utility function of the form: \( u(c_t) = \frac{c_t^{-\alpha} - 1}{1-\alpha} \), \( 0 \leq \alpha \leq 60 \). We have restricted the visible region of the graphs, and hence all 61 mean and standard deviation IMRS points will not generally be shown. In all cases, the most southeast point corresponds to \( \alpha \) equal to zero.

Figures 3–8: These figures are similar to figure 1 in that they plot the historical mean and standard deviation frontiers of IMRS based on monthly non-durable and services consumption and the mean and standard deviation frontier for IMRS implied by monthly returns on our five assets (solid curve). These figures also plot the mean and standard deviation frontier for IMRS implied by asset returns in the presence of market frictions (dotted and dashed curves) and their associated 95% confidence intervals (dashed curves).
Figure 2
Intertemporal Marginal Rates of Substitution
Based on Quarterly Real Per-Capita Consumption
and Asset Return Data

No Market Frictions

(1959:2 - 1990:4)
Figure 3
Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

No Short-Sale Constraints

(1959:2 – 1990:12)
Figure 4.
Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

* Borrowing Constraints

(1959:2 – 1990:12)
Figure 5
Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

Solvency Constraints

(1959:2 – 1990:12)
Figure 6
Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

Borrowing and No Short-Sale Constraints

(1959:2 – 1990:12)
Figure 7
Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

*Transaction Cost Frictions*

(1959:2 – 1990:12)
Figure 8

Intertemporal Marginal Rates of Substitution
Based on Monthly Real Per-Capita Consumption
and Asset Return Data

_Borrowing, No Short-Sale, and Transaction Cost Frictions_

_(1959:2 – 1990:12)_

---

**Diagram Description:**
- **Historical IMRS Using Monthly Consumption Data**
- **Returns—Based IMRS Frontier in the Absence of Frictions**
- **Frontier w/ Short-Sale, Borrowing, and Transaction Cost Frictions**
- **2.5% Confidence Level**
- **97.5% Confidence Level**