Corporate Debt Value, Bond Covenants, and Optimal Capital Structure

by

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Corporate Debt Value, Bond Covenants, and Optimal Capital Structure

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Abstract

This paper examines corporate debt values and capital structure in a unified analytical framework. It derives closed form results for the value of long-term risky debt and yield spreads, and for optimal capital structure, when firm asset value follows a diffusion process with constant volatility.

Debt values and optimal leverage are explicitly linked to firm risk, taxes, bankruptcy costs, riskfree interest rates, payout rates, and bond covenants. The results elucidate the different behavior of junk bonds vs. investment grade bonds, and aspects of asset substitution, debt repurchase, and debt renegotiation.
CORPORATE DEBT VALUE, BOND COVENANTS, AND OPTIMAL CAPITAL STRUCTURE

1. INTRODUCTION

The value of corporate debt and capital structure are interlinked variables. Debt values (and therefore yield spreads) cannot be determined without knowing the firm's capital structure, which affects the potential for default and bankruptcy. But capital structure cannot be optimized without knowing the effect of leverage on debt value.

This paper examines corporate debt values and optimal capital structure in a unified analytical framework. It derives closed form results relating the value of long term corporate debt and optimal capital structure to firm risk, taxes, bankruptcy costs, bond covenants, and other parameters when firm asset value follows a diffusion process with constant volatility.

Received capital structure theory, pioneered by Modigliani and Miller [1958], holds that taxes are an important determinant of optimal capital structure. As leverage increases, the tax advantage of debt eventually will be offset by an increased cost of debt, reflecting the greater likelihood of financial distress. While identifying some prime determinants of optimal capital structure, this theory has been less useful in practice because it provides qualitative guidance only.

1 Personal as well as corporate taxes will affect the tax benefits to leverage (Miller [1977]). Disagreement remains as to the precise value of net tax benefits.

2 The costs of financial distress include bankruptcy costs and agency problems associated with risky debt (for example, see Altman [1984]; Asquith, Gertner, and Shleifer [1991]; Harris and Raviv [1991]; Jensen and Meckling [1976]; Myers and Majluf [1984]; Titman and Wessels [1988]); and Warner [1977].

3 Baxter [1967], Kraus and Litzenberger [1973], and Scott [1976] offer general analyses balancing tax advantages with the costs of financial distress. But their results have not provided directly usable formulas to determine optimal capital structure. For an alternative view on the determinants of capital structure, see Myers [1984].
Brennan and Schwartz [1978] provide the first quantitative examination of optimal leverage. They utilize numerical techniques to determine optimal leverage when a firm's unlevered value follows a diffusion process with constant volatility.⁴ Although an important beginning, the Brennan and Schwartz analysis has three limitations.

First and most importantly, their numerical approach precludes general closed form solutions for the value of risky debt and optimal leverage. Numerical examples suggest some possible comparative static results but cannot claim generality.

Second, their analysis focuses on the special case in which bankruptcy is triggered when the firm's asset value falls to the debt's principal value. This provision approximates debt with a positive net worth covenant. But it is by no means the only—or even the typical—situation.⁵ We shall show that alternative bankruptcy-triggering conditions, including endogenously determined ones, lead to very different debt values and optimal capital structure.

Finally, Brennan and Schwartz consider changes in financial structure which last only until the bonds mature. A maturity date is required for their numerical algorithm; permanent capital structure changes are not explicitly analyzed.⁶

This paper considers two possible bankruptcy determinants. The first is when bankruptcy is triggered (endogenously) by the inability of the firm to raise sufficient equity capital to

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⁴ Kim [1978] also presents numerical examples of optimal capital structure, based on a mean-variance model. His model is less parsimonious, in that knowledge of the joint distribution of market and firm returns is required.

⁵ Minimum net-worth requirements are not uncommon in short-term debt contracts, but are rare in long-term debt instruments. In a later and more complex model, Brennan and Schwartz [1984] offer some examples with alternative bankruptcy conditions.

⁶ Brennan and Schwartz do look at some examples when T becomes large. The relative insensitivity of these examples to T as T exceeds 25 years suggests that our limiting closed form results for infinite maturity debt will be good approximations for debt with long but finite maturity.
meet its current debt obligations. The second is the Brennan and Schwartz case with a positive net worth covenant. Debt with such a covenant will be termed *protected debt*.

We can derive closed form results by examining corporate securities which depend on underlying firm value but are otherwise *time-independent*. Yet debt securities generally have a specified maturity date and therefore have time-dependent cash flows and values. Time independence nonetheless can be justified, perhaps as an approximation, in at least two ways. First, if debt has sufficiently long maturity, the return of principal effectively has no value and can be ignored.\(^7\) Very long time horizons for fixed obligations are not new—either in theory or in practice. The original Modigliani and Miller argument assumes debt with infinite maturity. Merton [1974] and Black and Cox [1976] look at infinite maturity debt in an explicitly dynamic model. Since 1752 the Bank of England has on occasion issued Consols, bonds promising a fixed coupon with no final maturity date. And preferred stock typically pays a fixed dividend without time limit.

An alternative time-independent environment is when, at each moment, the debt matures but is rolled over at a fixed interest rate (or fixed premium to a reference riskfree rate) unless terminated because of failure to meet a minimum value, such as a positive net worth covenant. As we later discuss, this environment bears resemblance to some revolving credit agreements.

Time independence permits the derivation of closed form solutions for risky debt value, given capital structure. These results extend those of Merton [1974] and Black and Cox [1976] to include taxes, bankruptcy costs, and protective covenants (if any). They are then

\(^7\) For 30-year debt, the final repayment of principal represents 1.5% of debt value when the interest rate is 15%, and 5.7% of value when the interest rate is 10%. Recently, a number of firms have issued 50-year debt, and one firm (Disney) has issued 100-year debt.
used to derive closed form solutions for optimal capital structure. The analysis addresses the following questions:

> **How do yield spreads on corporate debt depend on leverage, firm risk, taxes, payouts, protective covenants, and bankruptcy costs?**

> **Do high-risk ("junk") bond values behave in qualitatively different ways than investment-grade bond values?**

> **What is the optimal amount of leverage, and how does this depend on risk-free interest rates, firm risk, taxes, protective covenants, and bankruptcy costs?**

> **How does a positive net worth covenant affect the potential for agency problems between bondholders and stockholders?**

> **When can debt renegotiation be expected prior to bankruptcy, and can renegotiation achieve results which debt repurchase cannot?**

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8 Since this work was completed, I have become aware of important work by Anderson and Sundaresan [1992], Longstaff and Schwartz [1992], and Mella and Perraudin [1993]. Anderson and Sundaresan focus on risky debt in a binomial contingent claims framework. Using numerical examples, they examine the choice of debtors to discontinue coupon payments prior to bankruptcy, and show this may explain the sizable default premiums found in bond prices (see Jones et al. [1984], and Sarig and Warga [1989]). They do not examine optimal capital structure.

Longstaff and Schwartz [1992] derive closed form solutions for risky debt values with finite maturity and with stochastic risk-free interest rates. Their key assumption is that bankruptcy is triggered whenever firm value $V$ falls to an exogenously given level $K$ (our $V_b$) which is time-independent. This is a strong assumption for finite maturity debt, where debt service payments are time-dependent. Equation (11) below also shows that $V_b$ depends on the riskfree interest rate, suggesting that an endogenously determined $K$ will depend upon the stochastic interest rate. Longstaff and Schwartz do not consider optimal capital structure.

Mella and Perraudin's approach more closely parallels this paper, with endogenously determined bankruptcy levels. However, firm value is driven by a random product selling price whose drift as well as volatility must be specified, as must the firm's cost structure. (See also Fries, Miller, and Perraudin [1993]). Like Anderson and Sundaresan, the paper considers an endogenous decision to continue service debt.
The model follows Modigliani and Miller [1958], Merton [1974], and Brennan and Schwartz [1978] in assuming (i) that the activities of the firm are unchanged by financial structure, and (ii) that capital structure decisions, once made, are not subsequently changed.

Much of the recent literature in corporate finance examines possible variants to assumption (i).\(^9\) A particularly important variant is the "asset substitution" problem, where shareholders of highly leveraged firms may transfer value to themselves from bondholders by choosing riskier activities. If the appropriate functional form were known, feedback from capital structure to volatility could be captured in an extension of our model, at the likely cost of losing closed form results.\(^10\) But a simpler model which ignores such potential feedback still serves some important purposes:

1) Taxes and bankruptcy costs will importantly condition optimal capital structure even if asset substitution can occur; knowing these relationships in a basic model will provide useful insights for more complex situations.

2) The potential magnitude of the asset substitution problem can be identified by knowing how sensitive are debt and equity values to the risk of the activities chosen.

3) Bond covenants may directly limit opportunities for firms to alter the risk of their activities.\(^11\) In other cases, bond covenants may indirectly limit asset

\(^9\) See, for example, the survey by Harris and Raviv [1991]).

\(^{10}\) Mello and Parsons [1992], using a numerical approach similar to Brennan and Schwartz [1978] but including operating decisions of a (mining) firm, contrast decisions which maximize equity value with those which maximize the total value of the firm. They associate the difference in resulting values with agency costs, and present an example showing the effect of these costs on optimal leverage. Mauer and Triantis [1993] also use the Brennan and Schwartz [1978] approach to examine the interaction of investment decisions and corporate financing policies.

\(^{11}\) For example, bond covenants may specify that the firm adheres to a stated line(s) of business.
substitution by reducing potential conflicts of interest between stockholders and bondholders. Section VII below shows that a positive net worth requirement can eliminate the firm's incentive to increase risk.

Our second major assumption is that the face value of debt, once issued, remains static through time. This is not as unreasonable as it might appear: in Section IX, we show that additional debt issuance will hurt current debtholders; it is typically proscribed by bond covenants. We further show that marginal debt reductions via repurchases will hurt current stockholders. These considerations may preclude continuous changes in the outstanding amount of debt, even if refinancing costs are zero.

However, large (discontinuous) debt repurchases via tender offers may under certain circumstances benefit both stock and bondholders--if refinancing costs are not excessive. A dynamic model of capital structure capturing these possibilities is desirable but considerably more difficult. First steps in this direction have been made in important work by Kane, Marcus and McDonald [1984] and Fischer, Heinkel, and Zechner [1989]. Their analyses pose several difficulties which we avoid by adopting the static assumption shared with earlier authors. And some aspects of debt repurchase and renegotiation can be examined within the context of our model, as discussed in Section IX.

The structure of the paper is as follows. Section II develops a simple dynamic model of a levered firm, and derives values for time-independent securities. Sections III and IV consider debt value and optimal leverage when bankruptcy is determined endogenously. Sections V and VI consider debt value and optimal leverage when bankruptcy is triggered by a positive net worth covenant. Section VII considers some alternative assumptions about tax deductibility, cash payouts by the firm, and absolute priority of payments in bankruptcy.

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12 In Fischer, Heinkel, and Zechner [1989], the value of an unlevered firm (their A) cannot be exogenous, since it depends on the optimally levered value less costs of readjustment. (See their p. 25). Additionally, since closed form solutions are not available for the restructuring boundaries, they do not offer closed form equations for risky debt value and optimal capital structure.
Section VIII addresses agency problems and asset substitution, while Section IX considers aspects of debt repurchase and renegotiation. Section X concludes.

II. A MODEL OF TIME-INDEPENDENT SECURITY VALUES

Consider a firm whose activities have value \( V \) which follows a diffusion process with constant volatility of rate of return:

\[
\frac{dV}{V} = \mu(V,t)dt + \sigma dW, \tag{1}
\]

where \( W \) is a standard Brownian motion. We shall refer to \( V \) as the "asset value" of the firm.\(^{13}\) The stochastic process of \( V \) is assumed to be unaffected by the financial structure of the firm. Thus any net cash outflows associated with the choice of leverage (e.g., coupons after tax benefits) must be financed by selling additional equity.\(^{14}\)

Following Modigliani and Miller [1957], Merton [1974], Black and Cox [1976], and Brennan and Schwartz [1978], we also assume that a riskless asset exists which pays a constant rate of interest \( r \). This permits us to focus on the risk structure of interest rates directly.\(^{15}\)

Now consider any claim on the firm which continuously pays a nonnegative coupon \( C \) per
instant of time, financed by equity issuance, when the firm is solvent. Denote the value of such a claim by \( F(V,t) \). It is well known (e.g. Black and Cox [1976]) that any such asset's value must satisfy the partial differential equation

\[
(1/2)\sigma^2 \Delta^2 F(V,t) + rVF(V,t) - rF(V,t) + F_t(V,t) + C = 0
\]

with boundary conditions determined by payments at maturity, and by payments in bankruptcy should this happen prior to maturity.\(^{16}\) In general, there exist no closed form solutions to equation (2) for arbitrary boundary conditions. Hence Brennan and Schwartz [1978] resort to computer analysis of some examples. However, when securities have no explicit time dependence, the term \( F_t = 0 \) and the equation of value (2) becomes an ordinary differential equation with \( F(V) \) satisfying

\[
(1/2)\sigma^2 V^2 F_{VV}(V) + rVF_V(V) - rF(V) + C = 0,
\]

with boundary conditions determined (in part) by bankruptcy conditions, which will be shown to be time independent.\(^{17}\) The differential equation (3) has general solution

\[
F(V) = A_0 + A_1 V + A_2 V^{-X},
\]

where

\[
X = 2r/\sigma^2
\]

\(^{16}\) More generally, if net payouts by the firm not financed by further equity issuance are denoted \( P(V,t) \), and \( C(V,t) \) represents the payout flow to security \( F \), then

\[
(1/2)\sigma^2 (V,t)^2 F_{VV}(V,t) + [rV-P(V,t)]F_V(V,t) - rF(V,t) + F_t(V,t) + C(V,t) = 0.
\]

Note that \( \sigma^2 (V,t) \) could be of form \( \sigma^2 [C(V,t),V,t] \), reflecting possible asset substitution.

Equation (2) requires that \( V \), or an asset perfectly correlated with \( V \) (such as equity) be traded. See also footnote 13.

\(^{17}\) Observe that the drift term in equation (1) does not enter (3), in contrast with the volatility term \( \sigma^2 \). Thus time independence requires that the volatility term in (1) be independent of \( t \), but not the drift term.
and the constants $A_0$, $A_1$, and $A_2$ are determined by boundary conditions. Any time-independent claim with an equity-financed constant payout $C \geq 0$ must have this functional form. We turn now to examining specific securities.

Debt promises a perpetual coupon payment $C$ whose level remains constant unless the firm declares bankruptcy. The value of debt can be expressed as $D(V;C)$. For simplicity, however, we will suppress the coupon as an argument and simply write debt value as $D(V)$. Let $V_B$ denote the level of value at which bankruptcy is declared. (Note we again suppress the argument $C$). If bankruptcy occurs, a fraction $0 \leq \alpha \leq 1$ of value will be lost to bankruptcy costs, leaving debtholders with value $(1-\alpha)V_B$ and stockholders with nothing.\footnote{We focus on bankruptcy costs which are proportional to asset value when bankruptcy is declared. Alternatives such as constant bankruptcy costs could readily be explored within the framework developed. Deviations from absolute priority (in which bondholders do not receive all remaining value) can also be incorporated in the boundary conditions; we do so in Section VIIc. Franks and Torous [1989] and Eberhart, Moore, and Roenfeldt [1990] document deviations from the absolute priority rule.}

Later we shall show how the bankruptcy level $V_B$ is determined, given alternative debt covenants. For the moment regard it as fixed. Since the value of debt must be of form (4), we must determine the constants $A_0$, $A_1$, and $A_2$. Boundary conditions are:

1. At $V = V_B$, $D(V) = (1-\alpha)V_B$
2. As $V \to \infty$, $D(V) \to C/r$.

Condition (ii) holds because bankruptcy becomes irrelevant as $V$ becomes large, and the value of debt approaches the value of the capitalized coupon (and therefore the value of riskfree debt).

From (4), it is immediately apparent using (ii) that $A_1 = 0$. Because $V^{-X} \to 0$ as $V \to \infty$, this with (ii) implies that $A_0 = C/r$. Finally, $A_2 = [(1-\alpha)V_B - C/r)V_B^{-X}]$, using (i). Thus
(6) \[ D(V) = \frac{C}{r} + [(1-\alpha)V_B - \frac{C}{r}][V/V_B]^{-\lambda}. \]

Equation (6) can also be written as \( D(V) = [1 - p_B]\frac{C}{r} + p_B[(1-\alpha)V_B] \), where \( p_B \equiv (V/V_B)^{-\lambda} \) has the interpretation of the present value of $1 contingent on future bankruptcy (i.e. \( V \) falling to \( V_B \)).\textsuperscript{19}

Equation (6) represents a straightforward extension of Black and Cox [1976] to include bankruptcy costs.\textsuperscript{20} But we shall later see that taxes affect the value \( V_B \) when bankruptcy is determined endogenously. Both taxes and bankruptcy costs are important determinants of debt value in this case.

Debt issuance affects the total value of the firm in two ways. First, it reduces firm value because of possible bankruptcy costs. Second, it increases firm value due to the tax deductibility of the interest payments \( C \). The value of both these effects will depend upon the level of firm value \( V \), and are time-independent.

Imagine a security which pays no coupon, but has value equal to the bankruptcy costs \( \alpha V_B \) at \( V = V_B \). This security has current value denoted \( BC(V) \), which reflects the market value of a claim to \( \alpha V_B \) should bankruptcy occur. Because its returns are time-independent, it too must satisfy (4) with boundary conditions

\[(i') \quad \text{At } V = V_B, \quad BC(V) = \alpha V_B \]

\[(ii') \quad \text{As } V \to \infty, \quad BC(V) \to 0 \]

\textsuperscript{19} More exactly, \( p_B = \int_{0}^{\infty} \exp(-rt)f(t;V,V_B)dt \), where \( f(t;V,V_B) \) is the density of the first passage time from \( V \) to \( V_B \), when the process for \( V \) has drift equal to the riskfree interest rate \( r \).

\textsuperscript{20} Merton [1974] derives a different formula in the case he examines (\( \alpha = 0 \)). This is because he assumes the firm liquidates assets to pay coupons.
In this case (4) has solution

(7) \[ BC(V) = \alpha V_{B0}(V/V_{B0})^{X}. \]

BC is a decreasing, strictly convex function of V. Again, note the reinterpretation of (7) as \[ BC = p_{B}[\alpha V_{B0}]: \] The current value of bankruptcy costs is their magnitude if bankruptcy occurs, times the present value of $1 conditional on future bankruptcy. Subsequent expressions will have similar interpretations.

Now imagine a security which pays a constant coupon equal to the tax-sheltering value of interest payments \((\tau C)\) as long as the firm is solvent, and nothing in bankruptcy. This security's value, \(TB(V)\), equals the value of the tax benefit of debt. It too is time independent and therefore must satisfy (4) with boundary conditions

\[
\begin{align*}
(i^*) & \quad \text{At } V = V_{B}, \quad TB(V) = 0 \\
(ii^*) & \quad \text{As } V \to \infty, \quad TB(V) = \tau C/r. 
\end{align*}
\]

\((i^*)\) reflects the loss of the tax benefits if the firm declares bankruptcy.\(^{21}\) \((ii^*)\) reflects the fact that as bankruptcy becomes increasingly unlikely in the relevant future, the value of tax benefits approaches the capitalized value of the tax benefit flow \(\tau C\). Using (4) and the boundary conditions above gives

(8) \[ TB(V) = \tau C/r - (\tau C/r)(V/V_{B0})^{X}. \]

Tax benefits are an increasing, strictly concave function of V.

\(^{21}\) Reorganizations under Chapter 11 of the Bankruptcy Code may carry forward some tax benefits. This could be reflected by a boundary condition \((i^*)\) with a positive value. Also, in Section VIIa below, we consider the case where the loss of tax benefits occurs at values of V exceeding the bankruptcy level \(V_{B}.\)
The total value of the firm, \( v(V) \), reflects three terms: the firm's asset value plus the value of the tax deduction of coupon payments, less the value of bankruptcy costs:

\[
(9) \quad v(V) = V + TB(V) - BC(V) = V + (\tau C/\tau)[1 - (V/V_B)^{\alpha}] - \alpha V_B(V/V_B)^{\alpha}.
\]

Note that the total value of the firm \( v \) is strictly concave in asset value \( V \), when \( C > 0 \) and either \( \alpha > 0 \) or \( \tau > 0 \). Note also that with \( \alpha > 0 \) and \( \tau > 0 \), then \( v(V) < V \) as \( V \to V_B \), and \( v(V) > V \) as \( V \to \infty \). This coupled with concavity implies that \( v \) is (proportionately) more volatile than \( V \) at low values of \( V \), and less volatile at high values.\(^{22}\)

The value of equity is the total value of the firm less the value of debt:

\[
(10) \quad E(V) = v(V) - D(V) = V - (1 - \tau)C/\tau + [(1 - \tau)C/\tau - V_B][V/V_B]^{\alpha}.
\]

We shall see from (11) below that when \( V_B \) is endogenously determined, \( [(1 - \tau)C/\tau - V_B] > 0 \), implying that \( E(V) \) is a convex function of \( V \). This reflects the "option-like" nature of equity, even when debt has an infinite horizon. When \( V_B \) is determined by a minimum net worth requirement, however, we show in Section VI that equity may be a concave function of \( V \). This has important ramifications for agency problems associated with asset substitution, which are examined in Section VIII. Finally, Ito's Lemma can be used to show that the volatility of equity return \( (dE/E) \) declines as \( V \) (and therefore \( E \)) rises. Stock option pricing models would need to reflect this nonconstant volatility, as well as the possibility that \( E \) reaches zero with positive probability.

Equations (6) - (10) indicate the importance of \( V_B \) in determining the values of debt and

\[\text{\footnotesize{22 By Ito's Lemma, it can be shown that}}\]
\[dV = (\mu V + [X\mu - \delta X(X+1)^{\alpha}][\tau C/\tau + \alpha V_B][V/V_B]^{\alpha})dt + (V + X[\tau C/\tau + \alpha V_B][V/V_B]^{\alpha})d\zeta.\]
equity. In the following sections, we consider alternative bankruptcy-triggering scenarios.

III. DEBT WITH NO PROTECTIVE COVENANTS:
THE ENDOGENOUS BANKRUPTCY CASE

If the firm is not otherwise constrained by covenants, bankruptcy will occur only when the firm cannot meet the required (instantaneous) coupon payment by issuing additional equity: i.e., when equity value falls to zero. However, any level of \( V_B \) (less than the debt's principal value) will imply that the value of equity is zero at that asset value, given the absolute priority rule.

When \( V_B \) can be chosen by the firm (rather than imposed by a covenant such as a positive net worth requirement), it can be seen from (9) that total firm value \( V \) will be maximized by setting \( V_B \) as low as possible. Limited liability of equity, however, prevents \( V_B \) from being arbitrarily small: \( E(V) \) must be nonnegative for all values of \( V \geq V_B \). From (10), \( E(V) \) is strictly convex in \( V \) when \( V_B < (1 - \tau)C/r \). Thus the lowest possible value for \( V_B \) consistent with positive equity value for all \( V > V_B \) is such that \( dE/dV|_{V = V_B} = 0 \): a "smooth-pasting" or "low contact" condition at \( V = V_B \). This choice of bankruptcy level can also be seen to maximize the value of equity at any level of \( V \): \( dE/dV_B = 0 \). Differentiating (10) with

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23 Note that in continuous time, the instantaneous coupon to be paid \((C dt)\) is zero. In discrete time, the value of equity would have to exceed the actual coupon to be paid.

It is sometimes assumed that bankruptcy is triggered by a cashflow shortage. This can be criticized because, if equity value remains, a firm will always be motivated to issue additional equity rather than declare bankruptcy because of an immediate cashflow problem. Value rather than cashflow seems to be the essential element if bankruptcy is endogenously determined.

24 See also Merton [1973; footnote 60]. The equivalence of the two conditions suggests that the endogenously-set \( V_B \) is incentive compatible in the following sense. Ex ante (before debt issuance), stockholders will wish to maximize firm value subject to the limited liability of equity. \( V_B \) achieves this by satisfying the smooth-pasting condition. Ex post, equity holders will have no incentive to declare bankruptcy at a different \( V_B \). If \( V_B \) were set at a higher level than this ex ante--perhaps because of protective covenants--then stockholders would have an incentive ex post to violate the covenants if possible and declare bankruptcy at the lower, equity value maximizing level.
respect to $V$, setting this expression equal to zero with $V = V_B$, and solving for $V_B$ gives

\begin{align*}
(11) \quad V_B &= [(1-\tau)C/r][X/(1+X)] \\
&= (1-\tau)C/(r + .5\sigma^2),
\end{align*}

where the second line uses (5). Note that $V_B < (1-\tau)C/r$, implying that equity is indeed convex in $V$. Also observe that $V_B$ is proportional to $C$ and

- Is independent of current asset value $V$
- Decreases as the corporate tax rate $\tau$ increases
- Is independent of bankruptcy costs $\alpha$
- Decreases as the riskfree interest rate $r$ rises\(^{25}\)
- Decreases with increases in the riskiness of the firm $\sigma^2$.

The results above also describe the behavior of total firm value at bankruptcy $v_B \equiv v[V_B] = (1-\alpha)V_B$, except that $v_B$ falls as bankruptcy cost $\alpha$ increases. The fact that asset value $V$ does not affect $v_B$ means that the bankruptcy level of total firm market value can be estimated from the coupon $C$ (plus parameters $r, \sigma^2$, and $\tau$), without needing to know the firm's current asset value.\(^{26}\)

Substituting (11) into (6), (9), and (10) gives

\begin{align*}
(12) \quad D(V) &= (C/r)[1 - (C/V)^xk] \\
(13) \quad v(V) &= V + (\tau C/r)[1 - (C/V)^xh] \\
(14) \quad E(V) &= V - (1-\tau)(C/r)[1 - (C/V)^xm]
\end{align*}

\(^{25}\) This suggests that if the riskfree rate $r$ were random, the bankruptcy value $V_B$ would itself be random.

\(^{26}\) Knowledge of the market value of equity $E$ and debt $D$ in addition to $C$, combined with equations (6), (10), and (11), permits calculation of a unique $V$ and $\alpha$ given $r$, $\tau$, and $\sigma$. Alternatively, $\alpha$ and $\sigma$ can be recovered given $r$, $\tau$, and $V$. 

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where
\[ m = [(1 - \tau)X/\tau(1 + X)]^X/(1 + X) \]
\[ h = [1 + X + \alpha(1 - \tau)X/\tau]m \]
\[ k = [1 + X - (1 - \alpha)(1 - \tau)X]m. \]

The interest rate paid by risky debt, \( R(C/V) \), can be derived directly from dividing \( C \) by \( D(V) \), giving

\[ R(C/V) = C/D(V) = rK(C/V), \] \hspace{1cm} (15)

where
\[ K(C/V) = [1 - (C/V)^Xk]^{-1}. \]

The interest rate depends positively on the ratio of the coupon \( C \) to firm asset value \( V \). Note \( K(C/V) \) has the interpretation of a risk adjustment factor (multiplying the riskfree rate) which the firm must pay to compensate bondholders for the risks assumed. The yield spread is \( R(C/V) - r \).

The values above are derived for an arbitrary level of coupon \( C \). Section IV examines the optimal choice of coupon (and leverage) for unprotected debt. But first we examine the behavior of unprotected debt values and yield premiums, for arbitrary coupon level.

**IIIa. The Comparative Statics of Debt Value \( D(V) \)**

Equation (12) extends Black and Cox's [1976] results to include the effects of taxes and bankruptcy on debt value. **Row a** of **Table A** summarizes the comparative statics of debt value. Not surprisingly, **a.8** shows that larger bankruptcy costs decrease the value of debt. **Less obvious is a.9**, that an increase in the corporate tax rate will raise debt value, through
lowering the bankruptcy level $V_B$.\footnote{These comparative static results presume that other parameters (including $V$) remain at their current level, the usual ceteris paribus assumption. Note, however, that a change in the corporate tax rate might affect $V$ as well.}

More surprising still are the results a.5, a.6, and a.7 when taxes or bankruptcy costs are positive. As firm asset value $V$ nears the bankruptcy level $V_B$, the effects of increases in the coupon, firm riskiness, and the riskfree rate become reversed from what is expected. An increase in coupon can lower debt value. An increase in firm risk can raise debt value, as can an increase in the riskfree rate. Thus \textit{the behavior of "junk" bonds (or "fallen angels") differs significantly from the behavior of investment-grade bonds when bankruptcy costs and/or taxes are positive.}\footnote{The ratios of $V/V_B$ (or $C/V$) at which the various behaviors are reversed are not identical. Of course, these ratios may not correspond to Wall Street's definition of "junk" bonds.}

To understand these results, first consider the presence of positive bankruptcy costs. If $V$ is close to $V_B$, the value of debt will be very sensitive to such costs. Lowering $V_B$ will raise the value of debt since bankruptcy costs will be less imminent. From (11), higher asset volatility, higher riskfree interest rates, or lower coupon $C$, will all serve to lower $V_B$. For values of $V$ close to $V_B$, this positive effect on $D(V)$ will dominate. Even if there are no direct bankruptcy costs, the loss of the tax shield when $\tau > 0$ creates an opportunity loss if bankruptcy is declared, and the previous conclusions continue to hold.

An implication of a.5 is that the value of debt $D$ reaches a maximum $D_{\max}(V)$ for a finite coupon $C_{\max}(V)$, and declines for greater $C$. We can naturally think of $D_{\max}$ as the debt capacity of the firm. Differentiating (12) with respect to $C$, setting the resulting equation equal to zero, and solving for $C$ gives

\begin{equation}
C_{\max}(V) = V[(1+X)k]^{-1/X}
\end{equation}

\footnote{These comparative static results presume that other parameters (including $V$) remain at their current level, the usual ceteris paribus assumption. Note, however, that a change in the corporate tax rate might affect $V$ as well.}

\footnote{The ratios of $V/V_B$ (or $C/V$) at which the various behaviors are reversed are not identical. Of course, these ratios may not correspond to Wall Street's definition of "junk" bonds.}
Substituting this into (12) and simplifying gives

\begin{equation}
D_{\text{max}}(V) = V[Xk^{-1/X}(1+X)^{-1+1/X}] / r
\end{equation}

The debt capacity of a firm is proportional to asset value \( V \), and falls with increases in firm risk \( \sigma^2 \) and bankruptcy costs \( \alpha \). Debt capacity rises with increases in the corporate tax rate \( \tau \) and the riskfree rate \( r \).

**Figures 1 and 2** show the relationship between debt value and the coupon for varying firm volatility and bankruptcy costs, when \( V = $100 \) and \( r = 6\% \). Our normalization implies that the coupon level (in dollars) also represents the coupon rate as a percentage of asset value \( V \). Note that at high coupon levels, the debt of riskier firms has higher value than that of less risky firms. The peak of each curve indicates the maximum debt capacity \( D_{\text{max}} \), with corresponding leverage level. **Figure 3** repeats **Figure 1**, but with leverage \([D/V]\) rather than coupon level on the X-axis. The reversals seen in **Figure 1** do not appear in **Figure 3**. This is because leverage itself depends on the value of debt.

**IIIb. Yield spreads: The Risk Structure of Interest Rates**

**Row b of Table A** indicates the behavior of risky interest rates and yield spreads. Increasing the coupon \( C \) always raises the yield spread. An increase in bankruptcy costs \( \alpha \) also raises the spread, although a rise in the corporate tax rate will lower the spread because (from a.9) debt value will rise. Related to our earlier discussion, **b.6** indicates the surprising result that **junk bond yield spreads may actually decline when firm riskiness increases**. Of course, this holds only for junk bonds: the yield spread on investment-grade debt will increase when firm risk rises. Also note that junk bond interest rates may actually fall when the riskfree rate increases, from **b.7**. **Figures 4 and 5** plot yield spreads against coupon level and leverage, respectively, as asset value risk changes.

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29 Again, this assumes \( V \) remains unchanged when the riskfree rate rises.
b.4 implies that \( R(C/V) - (r + .5\sigma^2) \) as \( V - V_b \) when \( \alpha = \tau = 0 \). That is, long-term risky debt will never have a yield exceeding the riskfree rate by more than \( 0.5\sigma^2 \) if there are no bankruptcy costs or tax benefits to debt.\(^{30}\) Observing a yield spread greater than this on corporate long-term debt implies the presence of bankruptcy costs, taxes, or both.\(^{31}\)

IIIc. The Comparative Statics of Firm Value \( v(V) \) and Equity Value \( E(V) \)

Row c of Table A indicates the comparative statics of total firm value. Again observe the perverse behavior of total firm value for firms with junk debt. In the presence of bankruptcy costs and/or corporate taxes, c.6 indicates that total firm value may rise as firm riskiness increases. Rising riskfree rates may also lead total firm value to increase. The values of firms with investment-grade debt will not exhibit such behavior. Figure 6 and Figure 7 illustrate total firm value \( v \) as a function of the coupon level \( C \) and the leverage \( D/V \), respectively. Optimal leverage is the ratio at which each curve reaches its peak. Observe that the maximal gains to leverage are substantial, relative to an unlevered firm.

Row d of Table A indicates the behavior of equity value. Unlike debt, there are no reversals of comparative static results when \( V \) is close to \( V_b \). The fact that bankruptcy costs do not affect equity value (d.8) is perhaps surprising. But it reflects the fact that, given the coupon \( C \), debtholders bear all bankruptcy costs. In Section IV we shall see that the optimal coupon and debt-equity ratio does depend upon \( \alpha \), and that initial equity holders ultimately are hurt by greater bankruptcy costs.

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\(^{30}\) A firm whose asset value has annual standard deviation 20%, for example, would have debt whose yield spread never exceeded two percent. It has been argued that the tax advantage to debt may be nil (Miller [1977]). For arguments that bankruptcy costs may be small, see Warner [1977] (who focuses on direct costs only), and Haugen and Senbet [1988].

\(^{31}\) When the firm has several debt issues, junior debt could have higher rates. But the weighted average cost of debt will be limited to \( \tau + .5\sigma^2 \) in this case.
IV. OPTIMAL LEVERAGE WITH UNPROTECTED DEBT

Consider now the coupon rate \( C \) which maximizes the total value \( v \) of the firm, given current asset value \( V \). Differentiating (13) with respect to \( C \), setting the derivative equal to zero and solving for the optimal coupon \( C^* \) as a function of asset value \( V \) gives:

\[
C^*(V) = V[(1+X)h]^{-1/X}
\]

Note that \( h > k \), implying \( C^*(V) < C_{\text{max}}(V) \). Substituting \( C^*(V) \) into (12), (13), (15), and (11) gives

\[
D^*(V) = V[(1+X)h]^{-1/X} \left\{ 1 - k[(1+X)h]^{-1} \right\}/r
\]

\[
v^*(V) = V(1 + (r/r)[(1+X)h]^{-1/X}[X/(1+X)])
\]

\[
R^* = r[(1+X)h]/[(1+X)h - k]
\]

\[
V_B^*(V) = V(m/h)^{1/X}
\]

Table B indicates the comparative statics of these variables plus optimal leverage \( L^* = D^*/v^* \) and equity \( E^* = v^* - D^* \). While most results are consistent with what might be expected, a few merit comment.

The optimal coupon \( C^* \) is a U-shaped function of firm riskiness, as illustrated in Figure 8. Firms with little activity risk or very large activity risk will optimally commit to pay sizable coupons. Firms with intermediate levels of risk will promise smaller coupons. However, the optimal leverage ratios of riskier firms will always be less than that of less risky firms, as can be seen in Figure 9. The potential gains in moving from no leverage to optimal leverage (where \( v = v^* \)) are considerable. For reasonable parameter levels, optimizing financial structure can increase firm value by as much as 35-40 percent (see also Figure 6).

Our results confirm Brennan and Schwartz's [1978] observation that optimal leverage is less
than 100% even when bankruptcy costs are zero. Too high leverage risks bankruptcy--and while there are no bankruptcy costs, the tax deductibility of coupon payments is lost.

Leverage of about 75-95% is optimal for firms with low-to-moderate levels of asset value risk and moderate bankruptcy costs. Me Even firms with high risks and high bankruptcy costs should have leverage on the order of 50-60%, when the effective tax rate is 35%. Optimal leverage ratios drop by 5-25% when the effective tax rate is 15%, with the more pronounced falls at high volatility levels. Variations of our assumptions which lead to lower optimal leverage ratios are discussed in Section VII.

The behavior of the yield spread at the optimal leverage ratio exhibits one surprise. Increased bankruptcy costs might be thought to increase interest rates. Indeed they do--but only if the coupon is fixed. As bankruptcy costs rise, the optimal coupon $C^*$ falls. The probability of bankruptcy is now less and the yield spread decreases.

Higher riskfree interest rates might also be expected to reduce the optimal amount of borrowing. But they do not: the added tax shield when interest rates are high more than offsets the greater costs of borrowing. This could be destabilizing, since supply would normally be expected to decrease as interest rates rise.

---

32 It is of interest that many of the leveraged buyouts of the 1980's created capital structures that had 95% leverage or more. And targets were often firms with relatively stable value (low $\sigma^2$). Our analysis indicates these firms will reap maximal benefits from increased leverage. Subsequent reductions in leverage by many of these firms could be explained by the substantial fall in interest rates, which reduces the optimal leverage ratio.

33 Following Miller [1977], if the effective personal tax rate on stock returns (reflecting tax deferral) were 20%, the tax rate on bond income were 40%, and the corporate tax rate 35%, the effective tax advantage of debt is $[1 - (1-.35)(1-.20)/(1-.40)] = .133$, or slightly less than 15%.

34 Again, we note that an increase in $r$ might well cause a decline in $V$. If so, it is possible that the desired amount of borrowing (which is proportional to $V$) could decline even though optimal leverage rises.

20
V. POSITIVE NET WORTH COVENANTS AND THE VALUE OF PROTECTED DEBT

Consider now the case in which debt remains outstanding without time limit unless bankruptcy is triggered by the value of the firm's asset value falling beneath the principal value of debt, denoted $P$. We presume the principal value coincides with the market value of the debt when it is issued, denoted $D_0$. Thus $V_B = D_0$.\footnote{We focus on net worth requirements equal to the bond's principal value. But alternative levels could equally be considered, with minimum net worth being some multiple of principal value. It must be verified that the $V_B$ is consistent with the value of equity remaining positive at all levels $V > V_B$. This requires that $V_B$ exceed the level (11) satisfying the smooth-pasting conditions. In fact, this is always the case at optimal coupon levels, and is satisfied at all but extremely high coupon levels.}

Are there contractual arrangements in which this is a realistic description of bankruptcy? One possibility would be long-term debt as examined previously, with a protective covenant stipulating that the asset value of the firm always exceed the principal value of the debt: a positive net worth requirement. Such covenants are not common in long-term bond contracts, however.

An alternative contractual arrangement approximating this case would be a continuously renewable line of credit, in which the borrowing amount and interest rate are fixed at inception.\footnote{In contrast with a normal credit line, we assume the firm will never choose to borrow less than the stipulated loan amount. The fact that most credit lines are tied to a floating rate is not important here, since by assumption the riskfree rate is constant. It is important that the interest rate paid by the firm be independent of the firm's asset value $V$ (providing $V > V_B$) after the initial agreement is reached.} At each instant the debt will be extended ("rolled over" at a fixed interest rate) if and only if the firm has sufficient asset value $V$ to repay the loan's principal $P$; otherwise bankruptcy occurs.\footnote{Many lines of credit have a "paydown" provision, requiring that the amount borrowed must be reduced to zero at least once per year. A firm will fail to meet this provision if its (market) value of assets is less than the loan principal. Also note Merton [1974] requires $V > P$ at maturity to avoid default on a pure-discount bond: the firm must have positive net worth at maturity or bankruptcy occurs.} Thus the roll-over process proxies for a positive net worth requirement. With this latter interpretation, the differences between the unprotected debt
analyzed above and protected debt analyzed below may capture many of the differences between long term debt and (rolled over) short term financing.

From (6) with $V_B = D_0$, we can write the value of protected debt as a function of the value of assets $V_0$ at the time the loan is initiated:

$$D_0(V_0) = \frac{C}{r} + [(1-\alpha)D_0(V_0) - \frac{C}{r}][V_0/D_0(V_0)] - X$$

Except when $\alpha = 0$, closed form solutions for the function $D_0(V_0)$ satisfying (23) have not been found. However, we can easily solve this equation numerically to determine the value $D_0$ of the debt, given initial value $V_0$ and $C$. Note that the function $D_0(V_0)$ is homogeneous of degree one in $V_0$ and $C$.

Figures 10 and 11 illustrate the behavior of protected debt value as the coupon rate and leverage change. They should be compared with Figures 1 and 2. We observe that the surprising behavior of unprotected "junk" debt does not hold for protected debt, even when the debt exhibits considerable risk. Unlike the unprotected case, the value of debt increases with the coupon at all levels of $C$. And increased firm risk always lowers debt value. This is because the bankruptcy-triggering value $V_B$ is determined exogenously rather than endogenously.

When there are no bankruptcy costs ($\alpha = 0$),

- Protected debt is riskless and pays the riskfree rate $r$.

- For any $C$, the value of the tax shield with protected debt is less than the tax shield with unprotected debt.

Protected debt is riskless because the firm's asset value is constantly monitored. Should asset value fall to the principal value, bankruptcy is declared and, because there are no
bankruptcy costs, debtholders receive their full principal value. In this case, for given coupon \( C \), the value of protected debt always exceeds that of unprotected debt. Further, \( V_B = P = D_0(V_0) = C/r \). This exceeds the bankruptcy-triggering value of assets for unprotected debt, and implies smaller tax benefits from (8).

When bankruptcy costs are positive (\( \alpha > 0 \)), the results change markedly. For a given coupon \( C \), protected debt may have a lesser value than unprotected debt (and therefore may pay a higher interest rate). This follows because bankruptcy will occur more frequently when debt is protected, because \( V_B \) is higher in the protected case, and bankruptcy costs will be incurred when \( \alpha > 0 \). Figure 12, when compared with Figure 5, shows yield premiums to be substantially higher for protected debt when \( \alpha = 0.5 \), except at extreme leverage ratios.

VI. OPTIMAL LEVERAGE WITH PROTECTED DEBT

We now use a simple search procedure to find the coupon \( C^* \) which maximizes the total value \( v \) of the firm with protected debt. Figure 13, compared with Figure 7, illustrates that maximal firm value occurs at lower leverage when debt is protected. For a reasonable range of parameters, we find that

- Optimal leverage for protected debt is substantially less than for unprotected debt

- The interest rate paid at the optimum leverage is less for protected debt, even when bankruptcy costs are positive (\( \alpha > 0 \))

- The maximum value of the firm (and therefore the benefit from leverage) is less when protected debt is used

- The maximal benefits of unprotected over protected debt increase as:
> Corporate taxes increase
> Interest rates are higher
> Bankruptcy costs are lower

A closer examination of numerical results reveals that the optimal bankruptcy level \( V_B^* \) is the same for both protected and unprotected debt, when bankruptcy costs are zero. But we know the closed form solution for unprotected debt's optimal bankruptcy level \( V_B \) from (22). Since \( D_0 = V_B \), this in turn suggests a closed form solution for the optimal value of protected debt and related values when bankruptcy costs are zero, and \( V = V_0 \):

\[
(24) \quad D_0^*(V_0) = V_B^*(V_0) = V_0(m/h)^{1/x}
\]

Because protected debt is riskfree when \( \alpha = 0 \), it also follows that

\[
(25) \quad C^*(V_0) = rD_0^*(V_0) = rV_B^*(V_0) = rV_0(m/h)^{1/x}
\]

\[
(26) \quad v^*(V_0) = V_0 + [\tau C^*(V_0)/r] \{1 - [C^*(V_0)/V_0]^{\lambda h}\}
\]

Recall that equations (24)-(26) hold only for the protected debt case with no bankruptcy costs. Debt is riskfree in this case. We have not been able to find closed form solutions when \( \alpha > 0 \).

Note from (25) that \([1-\tau)(C^*/r) - V_B^*] = -\tau C^*/r < 0\), when \( \alpha = 0 \). From (10), this implies that equity is a strictly concave function of \( V \). In the numerical example considered in Section VIII, equity is strictly concave in \( V \) for all \( \alpha > 0 \) as well.\(^{38}\)

The observed comparative statics of optimal protected debt value (and related values) are

\(^{38}\) While equity is concave in \( V \) for small \( \alpha \), and for all \( \alpha \) in the later examples (with asset volatility of 20%), it will not be under all circumstances. In particular, when bankruptcy costs are large and asset volatility is very high, equity may be convex in \( V \).
given in Table C. There are some important differences with the comparative statics of optimal unprotected debt value. Perhaps most importantly, with protected debt the optimal leverage ratio ($D^*/\nu^*$) declines as the corporate tax rate increases. Optimal debt $D^*$ increases with $\tau$, but (unlike the unprotected debt case) increases less rapidly than $\nu^*$.

VII. DISCUSSION AND VARIATIONS: DEBT VALUE AND CAPITAL STRUCTURE

Our analysis has determined optimal leverage ratios and associated yield spreads for a variety of environments, for both long-term debt and protected (or continuously rolled-over) debt. It is of interest to compare these results with typical leverage ratios and yield spreads in the United States. Yield premiums (relative to Treasuries) of investment-grade bonds have ranged from a minimum of 15 bps to a maximum of 215 bps and averaged 77 bps from 1926-1986.\(^{39}\) Leverage in companies with highly rated debt is generally less than 40%.

These spreads reflect finite-maturity debt and also reflect the fact that corporate debt typically is callable. Call provisions typically add about 25 bps. to the annual cost of corporate debt.\(^{40}\) Subtracting 25 bps. from the average yield spread of 77 bps. to eliminate the impact of call provisions gives an adjusted historical yield spread of about 52 bps.

We examine a base case where the volatility of the firm’s assets is 20%, the corporate tax rate is 35%, the riskfree rate is 6%, and bankruptcy costs are 50%. In this case, optimal leverage with unprotected debt is 75% and the yield spread is 75 basis points. The optimally-levered firm’s equity is volatile, with a 57% annual standard deviation. Reducing the effective tax rate would reduce optimal leverage and the yield spread. For example, with an effective tax rate of 15%, optimal leverage is 59%, and the yield spread is only 35

\(^{39}\) As reported by Kim, Ramaswamy, and Sundaresen [1993]; see also Sarig and Warga [1989]).

\(^{40}\) Kim, Ramaswamy, and Sundaresen estimate a call premium of 22 basis points using a numerical example.
bps. Equity volatility is lower, but still substantial. It is clear that, based on our assumptions thus far, the analysis of unprotected debt suggests optimal leverage considerably in excess of current practice.

Optimal leverage ratios for protected debt seem more consistent with historical ratios. In the base case, optimal leverage is 45% and the yield spread is 45 basis points. Equity has a 34% annual standard deviation, which is a bit higher than the historical average equity risk of a single firm of about 30%.

We now consider how variations in the assumptions may affect the nature of optimal leverage and yield spreads, in both the unprotected and protected cases.

VIIa. No Tax Shelter for Interest Payments When Value Falls

We have assumed that the deductibility of interest payments generates tax savings at all values above the bankruptcy level. But as firm asset value drops, it is quite possible that profits will be negative and tax savings will not be realized (or will be substantially postponed). If lesser tax benefits are available, the optimal leverage ratio declines.

In Appendix A, we extend the analysis to allow for no tax benefits whenever \( V < V_T \), where \( V_T \) is an exogenously-specified level of firm asset value.\(^{43}\) In the base case considered

\[^{41}\text{See footnote 33.}\]
\[^{42}\text{This could be construed as a criticism of current management rather than the model. Managers may be loath to pay out "free cash flow" (see Jensen); the wave of LBOs in the late 1980's suggests that firm value may be raised by using greater leverage. However, the model's predicted yield spreads seem low given the suggested high leverage.}\]
\[^{43}\text{We do not allow tax loss carryforwards in this analysis, since they would introduce a form of time (and path) dependence. Thus we may overstate the loss of tax shields: the "truth" perhaps lies somewhere between the previous results and the results of this analysis.}\]
above, optimal leverage falls from 75% to 70%, and the yield spread at the optimal leverage rises from 75 bps. to 87 bps., when $V_T = 90$, i.e. 90% of the current asset value.

A possible criticism of the above approach is that $V_T$ does not depend upon the amount of debt the firm has issued. Consider an alternative scenario in which higher profit is needed if higher coupon payments are to be fully deductible. For example, assume that the rate of gross profit (or "earnings before interest and taxes" EBIT) is related to asset value as follows:

$$\text{EBIT} = (V - 60)/6.$$ 

Thus gross profit before interest drops to zero when $V$ falls to 60, and represents one-sixth of asset value in excess of 60. Further assume that coupon payments $C$ can be deducted from profit (for tax purposes) only if $\text{EBIT} - C \geq 0$. (We ignore partial deductibility). It then follows that

$$V_T = 60 + 6C.$$ 

In contrast with the previous scenario, greater debt now has a greater likelihood of losing its tax benefits.\textsuperscript{44} Optimal leverage falls to 65%. The yield spread falls to 61 bps., reflecting the lesser leverage. Equity volatility remains high at 51%. In the case of protected debt, the loss of tax deductibility has a much smaller effect on optimal leverage and yield spread. As expected, the loss of tax deductibility reduces the maximum value of the firm in all cases.

\textbf{VIIb. Cash Payouts by the Firm}

\textsuperscript{44} In the base case the optimal coupon falls to $5.08$, implying $V_T$ is about 90, as above.
Following Brennan and Schwartz [1978] and others, we have focused on the case where the firm has no net cash outflows resulting from payments to bondholders or stockholders. We now change this assumption. Cash outflows may occur because dividends are paid to shareholders, and/or because after-tax coupon expenses are being paid without fully offsetting equity financing. In this latter case, assets are being liquidated and the scale of the firm’s activities is clearly affected by the extent of debt financing.

To keep matters analytically tractable, we consider only cash outflows which are proportional to firm asset value, where the proportion $d$ may depend on the coupon paid on debt. Equation (3) is replaced by

\[(27) \quad (1/2)\sigma^2 V^2 F_{vv}(V) + (r-d)V F_v(V) - r F(V) + C = 0,\]

with all solutions for security values as before, but with

\[X = \{(r-d-.5\sigma^2) + [(r-d-.5\sigma^2)^2 + 2\sigma^2 r]^{1/2}\}/\sigma^2.\]

This of course simplifies to $X = 2r/\sigma^2$ when $d = 0$. When $d = .01$, a one-percent dividend on asset value (equivalent to approximately a 3% dividend on equity value, given the leverage of the base case), the optimal leverage to fall from 75% to 74%, and the yield spread rises from 75 bps to 86 bps.

But what if payouts also depend upon the coupon being paid to debtholders? Consider the case where the proportional payout is sufficient to cover the after-tax cost of debt when it is originally offered.\(^{45}\) Normalizing the initial value of $V$ to 100 implies that the payout $d = (1-r)C/100$, or .0065C in the above example. Any dividend payout would be in addition to this amount. For the base case above, we searched over coupon levels C which

\(^{45}\) Note that as value falls, the proportional payout will no longer completely cover the after-tax coupon—some equity financing becomes necessary. This may not be unreasonable, since bondholders will become increasingly sensitive to liquidation of assets as firm value approaches the bankruptcy level.
maximize \( v \), subject to the constraint that \( d = 0.065C + 0.01 \). This reduces optimal leverage from 75% to 64%, and increases the yield spread at the optimum from 75 bps. to 124 bps. The volatility of equity falls from 57% to 42%. In the case of protected debt, optimal leverage falls from 45% to 36%, the yield spread increases from 45 bps. to 49 bps., and the volatility of equity falls from 34% to 29%.

The maximum firm value drops from $128.4 to $122.0 with unprotected debt, and from $113.3 to $110.0 with protected debt. Therefore, ex ante, shareholders benefit from a covenant which prevents them from selling assets to meet coupon payments. It is not surprising that many debt instruments have such a preventive covenant. But if such a covenant cannot be written (or cannot be enforced), shareholders will always benefit from the firm selling assets to pay coupons after the debt has been issued. Recognizing this incentive, debtholders will pay less for debt and the optimal leverage will fall as indicated above.

\[ \text{VIIc. Absolute Priority Not Respected} \]

We have assumed that debtholders receive all assets (after costs) if bankruptcy occurs, and stockholders none: the "absolute priority" rule. Now consider a simple alternative, where debtholders receive some fraction \((1-b)\) of remaining assets \((1-\alpha)V_B\), while equity holders receive \(b(1-\alpha)V_B\).\(^{46}\) This will affect debt value in two ways: debtholders will receive less value if bankruptcy occurs; and bankruptcy will occur at a different level \(V_B\).

It can readily be shown that \((6)\) will be replaced by

\[ (6') \quad D(V) = C/r + [(1-b)(1-\alpha)V_B - C/r][V/V_B]^{-x}, \]

---

\(^{46}\) Franks and Torous [1989] estimate that deviations in favor of equity holders in Chapter 11 reorganizations is only 2.3% of the value of the reorganized firm. Eberhart, Moore, and Roenfeldt [1990] estimate average equity deviations of 7.8% for their sample of Chapter 11 firms. We choose a 10% deviation as an upper bound for this effect.
and the optimal $V_B$ when endogenously determined will be

\[(1') \quad V_B = (1-\tau)C/[r(1-b+\alpha b)][X/(1+X)].\]

For the base case with unprotected debt, deviations from absolute priority of 10% ($b = .1$) cause the optimal leverage ratio to fall from 75% to 72%. The yield spread remains at 75 bps. The effect of deviations from absolute priority are also minor when debt is protected: leverage remains unchanged at 45%, while the yield spread rises from 45 bps to 51 bps.

**VIId. All of the Above**

As a final exercise, consider the case where (i) dividends equal 3% of equity value, and after-tax coupons are not initially financed with additional equity; (ii) coupon payments are not tax deductible when $V < V_T = 60 + 6C$; and (iii) there is a 10% deviation from absolute priority ($b = .1$). When these conditions hold simultaneously, the optimal leverage with unprotected debt falls to 47%, and the yield spread is 69 bps. The annual standard deviation of equity is 36%. For protected debt, the optimal leverage falls to 32%, the yield spread is 52 bps., and the standard deviation of equity is 29%. These last numbers seem quite in line with historical yield spreads, leverage ratios, and equity risks.

**VIII. PROTECTED vs. UNPROTECTED DEBT: POTENTIAL AGENCY PROBLEMS**

Our results suggest that optimal leverage ratios are lower when debt is protected, and that the maximal gains to leverage are less. This raises a key question: Why should firms issue protected debt? The answer may lie with agency problems created by debt, and asset substitution in particular. Jensen and Meckling [1976] argue that equity holders would prefer to make the firm's activities riskier, ceteris paribus, so as to increase the value of equity at the expense of debt. The expected cost to debtholders will be passed back to
equity holders in a rational expectations equilibrium, through lower prices on newly-issued debt.

Higher firm asset risk tends to benefit equity holders when equity is a strictly convex function of firm asset value \( V \). Equation (16) indicates that equity is strictly convex in \( V \), when debt is unprotected. But in Section VI it was shown that equity can be a strictly concave function of \( V \) when debt has a positive net worth covenant.

To illustrate our point, consider the base case above. If debt is unprotected, the optimal coupon is $6.50, firm value is $128.4, and \( V_B \) is $52.8. If debt is protected, the optimal coupon is $3.26, firm value is $113.3, and \( V_B \) is $50.6. Assume that managers can raise the risk of the firm’s activities from its current annual standard deviation of 20%—with no change in current asset value \( V \). Will it be motivated to do so? Using equations (6) and (10), and recalling from (11) that \( V_B \) will change when debt is unprotected but not when debt is protected, gives

<table>
<thead>
<tr>
<th>Asset Volatility</th>
<th>Unprotected Debt</th>
<th></th>
<th>Protected Debt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Debt Value</td>
<td>Equity Value</td>
<td>Debt Value</td>
</tr>
<tr>
<td>20%</td>
<td>$96.3</td>
<td>$32.1</td>
<td>$50.6</td>
</tr>
<tr>
<td>40%</td>
<td>$70.4</td>
<td>$45.9</td>
<td>$36.9</td>
</tr>
<tr>
<td>60%</td>
<td>$52.6</td>
<td>$59.1</td>
<td>$31.2</td>
</tr>
</tbody>
</table>

Debtholders are hurt by higher risk. In the case of unprotected debt, equity value is enhanced by greater risk. Without covenants to prevent such a change, it will always be in the interest of equityholders to increase risk. *But the opposite is true when debt is protected by a positive net worth covenant:* increasing risk lowers equity value as well as debt value.47

47 The difference in behavior as \( \sigma \) changes reflects the convexity (concavity) of equity in \( V \) when debt is unprotected (protected). In addition, \( V_B \) changes in the unprotected case as \( \sigma \) changes. This latter effect explains the curious result in Section III, that (for \( V \) close to \( V_B \)) an increase in firm risk can raise unprotected (continued...
In the absence of protective covenants, investors recognize that shareholders will wish to raise asset volatility to the maximum (60%). They will pay only $52.60 for the debt and total firm value will be $111.70. If the firm offers protected debt, investors recognize that shareholders will have no interest in increasing firm risk, and total firm value will be $113.30. The firm maximizes value by issuing protected rather than unprotected debt.\textsuperscript{48} 

A re-evaluation of the belief that it is always advantageous for equity holders to raise firm risk \textit{ex post} seems warranted. It is true for unprotected debt. But it is false in the case examined here, when debt is protected by a positive net worth covenant. The greater incentive compatibility of protected debt may well explain its prevalence (or the prevalence of short-term financing), despite the fact that \textit{ceteris paribus} it exploits the tax advantage of debt less effectively.

\textbf{IX. RESTRUCTURING VIA DEBT REPURCHASE OR DEBT RENEGOTIATION: SOME PRELIMINARY THOUGHTS} 

The preceding analysis has assumed that the coupon $C$ of the debt issue is fixed through time. In the absence of transactions costs, restructuring by continuous readjustments of $C$ would seem to be desirable to maximize total firm value as $V$ fluctuates. However, we shall see that continuous readjustments of $C$ by debt repurchase (issuance) may be blocked by

\textsuperscript{47}(...continued) 

debt value. Thus there is yet a further anomaly with unprotected debt: at the brink of bankruptcy (and only there), both debtholders and stockholders wish to increase firm risk!

\textsuperscript{48} Even if the firm initially chose to issue the amount of unprotected debt optimal for a 60% volatility, the total firm value would be $112.14$--less than the maximal value with protected debt.
stockholders (debtholders).\textsuperscript{49} Debt renegotiation may be required to maximize total firm value in these cases.

To prove this contention, first consider the firm selling a small amount of additional debt, thereby increasing the current debt service by \( dC \). This will change the total value of debt by

\[
dD = (\partial D/\partial C)dC.
\]

But this total value change will be shared by current and new debtholders. New debtholders will hold a fraction \( dC/C \) of the total debt value, leaving current debtholders with value

\[
(D + dD)(1 - dC/C) = D + dD - (D/C)dC.
\]

(ignoring terms of \( O(dC^2) \)). The change from \( D \), the current debtholders' value before the debt issuance, is

\[
(28) \\
[(\partial D/\partial C) - (D/C)]dC < 0 \quad \text{for} \quad dC > 0,
\]

with the inequality resulting from the concavity of \( D \) in \( C \) and that \( D = 0 \) when \( C = 0 \). This "dilution" result holds for arbitrary initial \( V \) and \( C \), implying current debtholders will always resist increasing the coupon through additional debt issuance, even though such sales may increase the value of equity and the firm. This resistance is frequently codified in debt covenants which restrict additional debt issuance at greater or equal seniority.\textsuperscript{50}

\textsuperscript{49} We consider debt issuance/repurchase for capital restructuring only. Any funds raised by debt issuance will be used to retire equity, and vice-versa. Debt raised for new investment, or retired by asset sales, are asset-changing decisions which lie outside the scope of this paper.

\textsuperscript{50} Our analysis assumes a single class of debt, implying newly issued debt has the same seniority in the event of bankruptcy. Even if the newly issued debt is junior to the current debt, it will reduce the value of the current (continued...)
A related result on debt repurchase is perhaps more surprising: Current shareholders will always resist decreasing the coupon C by repurchasing current debt (in small amounts) on the open market. To prove this, consider a small decrease \( dC < 0 \), and its effect on current shareholders. The total value of equity will change by

\[
dE = \left( \frac{\partial E}{\partial C} \right) dC
\]

The cost of retiring debt will equal the value of the fraction of debt retired, or \(-D(dC/C)\). This cost must be financed with new equity, whose value is included in the change in total equity value above. Current shareholders will therefore have equity value

\[
E + dE + D(dC/C),
\]

implying a change in value to current shareholders of

\[
(29) \quad \left[ \left( \frac{\partial E}{\partial C} \right) + \left( \frac{D}{C} \right) \right] dC.
\]

For unprotected debt, it follows from (12) and (16) that \( \left[ \left( \frac{\partial E}{\partial C} \right) + \left( \frac{D}{C} \right) \right] > 0 \), implying that the change in equity value to the original shareholders is negative when \( dC < 0 \). This result holds for arbitrary initial V and C. Therefore it will never be optimal for the firm’s shareholders to restructure initial V and C. Therefore it will never be optimal for the firm’s shareholders to restructure by retiring unprotected debt via open market repurchases financed by new equity.\(^51\) Equation (16), important in the derivation above, holds only for unprotected debt. In Appendix B, we show that the result also holds for small repurchases of protected debt.

\(^{50}\)(...continued)
debt by raising V. A full analysis of multiple classes of debt securities is beyond the scope of the present paper.

\(^{51}\) This debt repurchase result holds even if there are multiple classes of debt. Stockholders might benefit from retiring debt via asset sales, but this would violate the assumption that the asset value V is independent of firm’s the capital structure.
To illustrate the arguments above, consider the base case with unprotected debt. With $V = $100, the optimal coupon is $6.50 and $V_b = $52.80. Assume this coupon level has been chosen by the firm. Now let $V$ drop from $100 to $90. Using (6) and (10) to compute the current values of debt and equity gives:

$$D = $91.79 \quad E = $23.14.$$ 

The firm’s total value can now be increased by reducing debt. The firm should cut its coupon by 10% to $5.85, since $C^*$ is proportional to $V$ which has fallen from 100 to 90. This would increase the total firm value from $114.95 to $115.60.

But consider the firm repurchasing 10% of its debt, to achieve the new optimal leverage. The coupon is reduced from $6.50 to $5.85 and $V_b$ falls by 10% to $47.52. The firm must pay (at least) $9.18 to retire 10% of the bonds whose value is $91.79 prior to repurchase. It will raise this amount by issuing additional stock worth $9.18. Again using (6) and (10) to compute debt and equity values with the lower coupon gives

$$D = $86.65 \quad E = $28.95.$$ 

Debtholders are clearly better off, having received payments of $9.18 to retire 10% of their holdings, plus retaining holdings worth $86.65. But the original equity holders have had their stock diluted: $9.18 of stock—the amount raised to pay the debtholders—now belongs to new shareholders, leaving the original shareholders with stock worth $28.95 - $9.18 = $19.77. This is less than the $23.14 value of their shares prior to repurchase. Although the total value of the firm would be increased by the restructuring, equity holders cannot benefit from the repurchase, and will block such refinancing.

---

52 Note debt becomes more valuable per unit, as the coupon is reduced. We are assuming here that the entire amount of repurchase can be effected at the lowest (i.e. initial) price. Any higher price would magnify the losses to equity holders.
The example shows that straightforward restructuring through debt repurchases or sales will not be possible although such changes could increase total firm value. To capture such potential increases, changes in the terms of the debtholders' securities will be required. These types of restructurings will be labelled debt renegotiation. In our example, renegotiation to replace current debt with convertible debt could be used to achieve the optimal coupon level. By agreeing to exchange the current debt for debt with coupon $5.85 (worth $86.65), plus (say) a convertibility privilege into stock worth $5.50, the value of the new debt will be $92.15. This exceeds the $91.79 value of the current debt paying a $6.50 coupon, so bondholders will benefit. And stockholders will also benefit, by the rise in the equity value of $5.81 ($28.95 - $23.14) less the $5.50 value of the convertibility option given bondholders.

Renegotiation of unprotected debt is particularly simple when bankruptcy is imminent (V is close to V_D), and C > C_{max}(V). In this case, a small reduction in the coupon will increase the value of both debt and equity—with no further compensation to bondholders (such as the convertibility privilege) being required. However, additional considerations (or "side payments") to bondholders still may be required to reduce the coupon all the way to the optimal level C*(V).53

X. CONCLUSION

By assuming a debt structure with time-independent payouts, we have been able to develop closed form solutions for the value of debt and for optimal capital structure. This permits a detailed analysis of the behavior of bond prices and optimal debt-equity ratios as firm value, risk, taxes, interest rates, bond covenants, payout rates, and bankruptcy costs change.

53 The firm may be able to reduce its coupon payment to C*(V) with no side payments, if the value of debt D*(V) at the optimal coupon is greater than the value of debt D(V) when the renegotiation begins. This assumes stockholders can credibly make a "take it or leave it" offer to bondholders. Note that the firm may wait until the brink of bankruptcy before renegotiating, since this will minimize the ratio D(V)/D*(V).
The analysis examines two types of bonds: those which are protected by a positive net worth covenant, and those which are not. The distinction is critical in determining when bankruptcy is triggered, which in turn is critical in determining bond values and optimal leverage. We argued that long term debt rarely has positive net worth covenants, while short term financing may require positive net worth to be continuously rolled over.

Our results indicate that protected debt values and unprotected "investment grade" debt values behave very much as expected. Unprotected "junk" bonds exhibit quite different behavior. For example, an increase in firm risk will increase debt value, as will a decrease in the coupon. Such behavior is not exhibited by protected "junk" bonds.

Two curious aspects of optimal leverage are observed. First, a rise in the riskfree interest rate (increasing the cost of debt financing) leads to a greater optimal debt level. Higher interest rates generate greater tax benefits, which in turn dictates more debt despite its higher cost. Second, the optimal debt for firms with higher bankruptcy costs may carry a lower interest rate than for firms with lower bankruptcy costs. This is because firms will choose significantly lower optimal leverage when bankruptcy costs are substantial, making debt less risky. This result does not hold for protected debt: higher bankruptcy costs imply higher interest rates at the optimal leverage.

Optimal leverage, yield ratios, and equity risk are well within historical norms for protected debt. But optimal leverage seems high (and/or yield ratios seem low) for unprotected debt. Variants of the basic assumptions, discussed in Section VII, seem needed for unprotected debt to fall within historical norms. The most important modification is dropping the requirement that payouts to bondholders be externally financed.

Issuing debt without protective net-worth covenants yields greater tax benefits, and would seem to dominate issuing protected debt. However, this conclusion may be reversed if firms have the ability to increase the riskiness of their activities through "asset substitution." As
is widely recognized, increasing risk will transfer value from bondholders to stockholders when debt is unprotected, leading cautious bondholders to demand higher interest rates even when the firm currently has low risk. But such costs typically are not incurred when firms issue protected debt: stockholders will not gain by increasing firm risk when debt is protected by a positive net worth covenant, and bondholders will not need to demand higher interest rates in anticipation of riskier firm activities. Protected debt may be the preferred form of financing in these situations, despite having lesser potential tax benefits.

Our results offer some preliminary insights on debt repurchases and on debt renegotiations. The former cannot be used to adjust leverage continuously to its optimal level: bondholders will block further debt issuance, and shareholders will block (marginal) debt reductions. Debt renegotiation can achieve simultaneous increases in debt and equity value. But the costly nature of renegotiation suggests it would be uneconomic to do so continuously (see Fischer, Heinkel, and Zechner [1989]). Our analysis suggests that it may be optimal for shareholders to wait until the brink of bankruptcy before renegotiating. When bankruptcy is neared, a reduction in coupon payments to the optimal level may benefit both stockholders and bondholders, without additional side payments.

While we have not emphasized equity values, our analysis also provides some interesting insights. Equity return volatility will be stochastic, changing with the level of firm asset value \( V \). This (and the possibility of bankruptcy) will have important ramifications for option pricing.\(^{54}\)

The model can be extended in several further dimensions. Multiple classes of long-term debt can be analyzed, recognizing that payments to the various classes of debtholders when bankruptcy occurs are determined by seniority rules. More difficult extensions will involve finite-lived debt, dynamic restructuring, and a stochastic term structure of riskfree interest rates.

\(^{54}\) Preliminary work has been done by Klaus Toft on this subject.
APPENDIX A

We assume in this case that instantaneous tax benefits = 0 whenever $V \leq V_T$. There are no carryforwards.

Differential equation (4) with $C = 0$ has solution:

$$TB(V) = A_1 V + A_2 V^{-X}, \quad V_B < V \leq V_T.$$  

Differential equation (4) with instantaneous tax benefit $\tau C$ realized has solution:

$$TB(V) = (\tau C/r) + B_2 V^{-X}, \quad V \geq V_T.$$  

$TB(V)$ must satisfy:

$$TB(V_B) = 0: \quad A_1 V_B + A_2[V_B]^{-X} = 0 \quad (1)$$

$$TB(V_T) = TB(V_T): \quad A_1 V_T + A_2 V_T^{-X} = (\tau C/r) + B_2 V_T^{-X} \quad (2)$$

$$TB'(V_T) = TB'(V_T): \quad A_1 - XA_2 V_T^{-X-1} = -XB_2 V_T^{-X-1} \quad (3)$$

Solutions:

$$A_1 = (\tau C/r)(X/(X+1))(1/V_T)$$

$$A_2 = -(\tau C/r)(X/(X+1))(V_B^{X+1}/V_T)$$

$$B_2 = -(\tau C/r)(X/(X+1))(1/V_T)(V_B^{X+1} + (1/X)V_T^{X+1})$$

To find $V_B$, we again set $dE/dV|_{V=V_B} = 0$. Substituting in expression for tax benefits for $V \leq V_T$ gives

$$E = v - D = V + A_1 V + A_2 V^{-X} - \alpha V_B(V/V_B)^{-X} - C/r - [(1-\alpha) V_B - (C/r)](V/V_B)^{-X}$$

$$dE/dV = 1 + A_1 - XA_2 V_B^{-X-1} + \alpha X(V/V_B)^{-X-1} + X[(1-\alpha) V_B - (C/r)](V/V_B)^{-X-1}(1/V_B) = 0$$

evaluating at $V = V_B$:

$$1 + A_1 - XA_2 V_B^{-X-1} + X - (C/r)(X/V_B) = 0.$$  

Substituting for $A_1$ and $A_2$ gives

$$V_B = CV_T X/[rV_T(1+X) + \tau CX].$$

$D$ can be computed from (6); $v = V + (\tau C/r) + B_2 V^{-X} - \alpha V_B(V/V_B)^{-X}$ when $V > V_T$.

Note we can rewrite the expression for $V_B$ as

$$V_B = V_B V_T/(1-\tau V_T + \tau V_B) > V_B \quad \text{(since } V_T > V_B), \quad \text{where } V_B \text{satisfies (11).}$$
APPENDIX B

Since \( E = v - D \),

\[
[(\partial E/\partial C) + (D/C)]dC = \{(\partial v/\partial C) - (\partial D/\partial C) \cdot (D/C)\}dC.
\]

Define \( V^* \) as the firm asset value at which the current coupon would be optimal for protected debt, i.e. for which \( \partial v(V^*)/\partial C = 0 \). Recalling (28), it follows that (29) will be strictly positive when \( \partial v/\partial C = 0 \). Continuity implies that there exists a neighborhood of values \( V \) around the value \( V^* \), for which (29) is strictly positive. For all \( V < V^* \) in this neighborhood, firm value \( v \) would be increased by lowering the coupon since \( v \) is strictly concave in \( C \). But because (29) is positive, current stockholders' equity value will fall when \( v dC < 0 \), and shareholders will resist reducing the coupon to its optimal level.
REFERENCES


## Table A

**Comparative Statics of Financial Variables:**

**Unprotected Debt**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Homogeneity (1)</th>
<th>Shape (2)</th>
<th>Limit As</th>
<th>Sign of Change in Instrument for an Increase in:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$V - \infty$ (3)</td>
<td>$V - V_h$ (4)</td>
</tr>
<tr>
<td>a. Debt $D$</td>
<td>Degree 1 in V, C</td>
<td>Concave in V, C</td>
<td>$C/r$</td>
<td>$\frac{C(1-\alpha)(1-\tau)}{(r+.5\sigma^2)}$</td>
</tr>
<tr>
<td>b. Interest Rate $R$</td>
<td>Degree 0 in V, C</td>
<td>Convex in V/C</td>
<td>$r$</td>
<td>$\frac{(r+.5\sigma^2)}{(1-\alpha)(1-\tau)}$</td>
</tr>
<tr>
<td>c. Total Value $x$</td>
<td>Degree 1 in V, C</td>
<td>Concave in V, C</td>
<td>$V+\tau C/r$</td>
<td>$\frac{C(1-\alpha)(1-\tau)}{(r+.5\sigma^2)}$</td>
</tr>
<tr>
<td>d. Equity Value $E$</td>
<td>Degree 1 in V, C</td>
<td>Convex in V, C</td>
<td>$V-(1-\tau)C/r$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

* Sign reversal as $V - V_h$ only if $\alpha$ and/or $\tau > 0$.
TABLE B
COMPARATIVE STATICS OF FINANCIAL VARIABLES
AT OPTIMAL LEVERAGE RATIO

<table>
<thead>
<tr>
<th>Variable</th>
<th>Shape</th>
<th>( \sigma^2 ) (2)</th>
<th>( r ) (3)</th>
<th>( g ) (4)</th>
<th>( \ell ) (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Optimal Coupon ( C^* )</td>
<td>Linear in ( V )</td>
<td>( &lt; 0, \ \sigma^2 ) small ( &gt; 0, \ \sigma^2 ) large</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
</tr>
<tr>
<td>B. Optimal Debt ( D^* )</td>
<td>Linear in ( V )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
</tr>
<tr>
<td>C. Optimal Leverage ( L^* )</td>
<td>Invariant to ( V )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
</tr>
<tr>
<td>D. Interest Rate ( R^* )</td>
<td>Invariant to ( V )</td>
<td>( &gt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
</tr>
<tr>
<td>E. Total Value ( y^* )</td>
<td>Linear in ( V )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
</tr>
<tr>
<td>F. Equity Value ( E^* )</td>
<td>Linear in ( V )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0^{**} )</td>
</tr>
<tr>
<td>G. Bankruptcy Level ( V_g^* )</td>
<td>Linear in ( V )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( &lt; 0 )</td>
<td>( &gt; 0^{**} )</td>
</tr>
</tbody>
</table>

** No effect if \( \alpha = 0 \).
### TABLE C
COMPARATIVE STATICS OF FINANCIAL VARIABLES
AT OPTIMAL LEVERAGE RATIO: PROTECTED DEBT

<table>
<thead>
<tr>
<th>Variable</th>
<th>Shape</th>
<th>$\sigma^2$ (2)</th>
<th>$r$ (3)</th>
<th>$\alpha$ (4)</th>
<th>$\tau$ (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Optimal Coupon $C^*$</td>
<td>Linear in $V$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$  (= 0 \text{ if } \alpha = 0)</td>
</tr>
<tr>
<td>B. Optimal Debt $D^*$</td>
<td>Linear in $V$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$    (= 0 \text{ if } \alpha = 0)</td>
</tr>
<tr>
<td>C. Optimal Leverage $L^*$</td>
<td>Invariant to $V$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$ (&gt; 0 \text{ if } \alpha &gt; 0 &amp; \sigma^2 &gt; 0)</td>
</tr>
<tr>
<td>D. Interest Rate $R^*$</td>
<td>Invariant to $V$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$    (= 0 \text{ if } \alpha = 0)</td>
</tr>
<tr>
<td>E. Total Value $z^*$</td>
<td>Linear in $V$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>F. Equity Value $E^*$</td>
<td>Linear in $V$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0 \text{ if } \alpha = 0$</td>
</tr>
<tr>
<td>G. Bankruptcy Level $V_g^*$</td>
<td>Linear in $V$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0 \text{ if } \alpha &gt; 0 \text{ or } \alpha = 0$</td>
</tr>
</tbody>
</table>

Shading represents different behavior than unprotected debt.
FIGURE 1: UNPROTECTED DEBT VALUE

\[ r = 6\% \quad \text{Alpha} = 50\% \quad \text{Tax Rate} = 35\% \]

Debt Value \([D]\)

$0 \quad $2 \quad $4 \quad $6 \quad $8 \quad $10 \quad $12 \quad $14$

$0 \quad $20 \quad $40 \quad $60 \quad $80 \quad $100 \quad $120 \quad $140$

Coupon \([C]\)

Volatility
- 15%
- 20%
- 25%
FIGURE 2: UNPROTECTED DEBT VALUE

\[ r = 6\% \quad \text{Volatility} = 20\% \quad \text{Tax Rate} = 35\% \]

\[ \text{Debt Value} [D] \]

\[ \text{Coupon} [C] \]

\[ \text{Volatility} \quad 100\% \quad 50\% \quad 0\% \]
FIGURE 3: UNPROTECTED DEBT VALUE

$r = 6\%$  Alpha = 50$\%$  Tax Rate = 35$\%$

Volatility

$15\%$  $20\%$  $25\%$

Leverage

$D/V$

Debt Value $[D]$
FIGURE 4: YIELD SPREADS, UNPROTECTED DEBT

$r = 6\%$  \hspace{1em}  $\text{Alpha} = 50\%$  \hspace{1em}  $\text{Tax Rate} = 35\%$

![Graph of Yield Spreads]

Yield Spread [bps]

Coupon [C]
FIGURE 5: YIELD SPREADS, UNPROTECTED DEBT

\[ r = 6\% \quad \text{Alpha} = 50\% \quad \text{Tax Rate} = 35\% \]
FIGURE 6: FIRM VALUE (UNPROTECTED DEBT)

$r = 6\%$  
Alpha = 50\%  
Tax Rate = 35\%

Volatility

$15\%$  
$20\%$  
$25\%$

Firm Value

$[V]$

Coupon

$\$140$  
$\$120$  
$\$100$  
$\$80$  
$\$60$  
$\$40$  
$\$20$  
$\$0$  
$\$12$  
$\$14$  
$\$16$
FIGURE 7: FIRM VALUE (UNPROTECTED DEBT)

\[ r = 6\% \quad \text{Alpha} = 50\% \quad \text{Tax Rate} = 35\% \]
FIGURE 8: OPTIMAL COUPON (UNPROTECTED DEBT)

$r = 6\%$  Tax Rate = 35\%

Volatility

Bankruptcy Cost (Alpha)

Coupon [C]
FIGURE 10: PROTECTED DEBT VALUE

$r = 6\%$  \hspace{1cm} \text{Alpha} = 50\%$  \hspace{1cm} \text{Tax Rate 35\%}$

![Graph showing the relationship between Debt Value and Coupon for different volatility levels.](image)
FIGURE 11: PROTECTED DEBT VALUE

\[ r = 6\% \quad \text{Alpha} = 50\% \quad \text{Tax Rate} = 35\% \]
FIGURE 12: YIELD SPREADS, PROTECTED DEBT

\[ r = 6\% \quad \text{Alpha} = 50\% \quad \text{Tax Rate} = 35\% \]
FIGURE 13: FIRM VALUE (PROTECTED DEBT)

$r = 6\%$  \hspace{1cm} \text{Alpha} = 50\% \hspace{1cm} \text{Tax Rate} = 35\%$

![Graph showing firm value vs. leverage for different volatility levels (25%, 20%, 15%) at a constant interest rate of 6% and an alpha of 50%.]