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by

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Abstract

In this paper, I derive exact formulas for expected hedging error and transactions costs in option replication for the Black-Scholes economy with exogenously fixed trading points. I derive the formulas using two different volatilities which allow the hedger to use a transactions costs adjusted volatility to determine the hedge portfolio. The expected hedging error is written in an easily recognized form. The four terms in the expectation can be interpreted as terms from Black and Scholes’ (1973) formula with adjusted parameters. This interpretation holds for all future hedging periods even though the expectation is conditional on the stock price at the time of the hedging scheme’s initiation. I also derive an approximation of the expected transactions costs. This approximation has a simple interpretation: for each of the future hedging periods, the approximate expected transactions costs incurred at the end of each hedging period are proportional to the option’s gamma with adjusted parameters, multiplied by the squared expected value of the underlying asset. For the risk neutral economy with no volatility adjustment, I show that present values of the approximate expected transactions costs are identical for each of the future hedging intervals. Moreover, I illustrate that the approximation to the expected transactions costs is accurate except for hedging periods close to the maturity of the contingent claim. Here, the exact expectation tends to be larger than the approximation, even though the expectation is taken only with knowledge of the initial stock price. Finally, I derive an approximation of the variance of the hedging scheme’s cash-flow (the hedging error minus the transactions costs) for each of the future hedging periods. This approximation facilitates evaluation of the tradeoff between cost and variance of the replication strategy.
1 Introduction

Research on the effects of transactions costs and discrete trading on option pricing is based on a sequence of works inspired by Black and Scholes' (1973) seminal results. Several authors recognize that continuous readjustment of the replicating portfolio in the presence of transactions costs is prohibitively expensive (see Gilster and Lee (1984) and Leland (1985)) and therefore not a realistic trading strategy. Instead, the hedging strategy must be implemented discretely and transactions costs must be taken into account. Leland approaches this problem in a continuous-time setting and develops a trading strategy where the sum of all the hedging errors over the contract period including the transactions costs converges to zero. This trading strategy is appealing because the hedge portfolio at each revision point is simply a Black-Scholes hedge with increased volatility. As the revision interval goes to zero, the adjusted volatility goes to infinity. This implies that the cost of the replicating portfolio converges to the price of the underlying asset. Boyle and Vorst (1992) analyze the same problem in a binomial framework. They also conclude that in the limit, the hedging strategy can be approximated by a Black-Scholes strategy with an adjusted volatility. Their suggested adjustment is slightly larger than Leland's. Henrotte (1993) reconciles the two different volatility adjustments by analyzing limits of duplication strategies, where both the time between revision points and the proportional transactions costs go to zero. He confirms both Leland's and Boyle and Vorst's volatility adjustments; Leland's adjustment is appropriate for a time based strategy, Boyle and Vorst's should be used when a percentage move in the underlying asset value triggers the rebalancing trade. Merton (1990) analyzes the transactions costs problem in a discrete two-period model and concludes that the initial holding of the risky asset is larger for the case with transactions costs than for Cox, Ross, and Rubinstein (1979) replication without transactions costs.

All the papers discussed above assume that a trade is placed at each revision point. This assumption is necessary to replicate the call's payoff. Other researchers' works indicate that it is not necessarily optimal to revise the hedge portfolio at each revision point. If the replicating portfolio's payoff function is allowed to dominate the option's payoff function, then the minimum value of the replicating portfolio in a binomial model may be smaller than when exact replication is performed, see Edirisinghe, Naik, and Uppal (1992) and Bensaid, Lesne, Pagès, and Scheinkman (1992). Their results indicate that the holding of the risky asset is larger when transactions costs exist than
for the case without transactions costs. However, Bensaid et al. (1992) show that, in a binomial model where options are settled in-kind, exact replication is the optimal way to manufacture a long portfolio of calls and puts.\(^1\)

In this paper, I analyze the expected hedging error, transactions costs, and variance of an option replication strategy in a Black-Scholes economy where trading is allowed only at exogenously fixed points in time.\(^2\) Figure 1 illustrates the hedger's perspective immediately preceding initiation of the hedging scheme. Before him lies one contract period subdivided into a series of shorter hedging periods. In this economic environment, the hedger is not only interested in the expected hedging error and transactions costs at the end of the first hedging interval (one period ahead), but also the expected hedging errors and transactions costs for all subsequent hedging periods \(m + 1\) periods ahead for \(m = 1, \ldots, M - 1\), where \(M\) is the number of hedging periods in the contract period. To get an apriori measure of the expected cash-flows from the hedging strategy, the expected hedging error minus the expected transactions costs for each of the future hedging periods must be calculated conditionally on the information available to the hedger when the hedging scheme is initiated. Finally, the expected cash-flow for each of the individual periods can be discounted, summed, and added to the initial price of the hedge portfolio providing an aggregate measure of the expected cost of the hedging scheme.\(^3\) A risk-averse hedger is also concerned with the variances of the cash-flows from the replicating strategy. To get an apriori measure, the variance of the cash-flow at each of the rebalancing points must be calculated conditionally on the information set at the initiation of the hedging scheme. The variances of the hedging error minus the transactions costs for each of the hedging periods may be aggregated to a variance for the contract period by summation, because the hedging errors minus the transactions costs are uncorrelated over time.

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\(^1\) Other recent works on move and time based strategies include Shen (1990), Crannan and Swindle (undated), Flesaker and Hughston (1993), Kat (1993), Whalley and Wilmott (1993), and Boyle and Tan (1994). Another sequence of papers analyzes the properties of the intertemporal dynamic programming problem which can be constructed by using an exponential utility function, see Hodges and Neuberger (1989), Davis and Panas (1991), Davis, Panas, and Zariphopolou (1993) and Hodges and Clewlow (1993). Using this approach, the hedge portfolio is rebalanced whenever the option's delta moves outside a non-trading region. Finally, Constantinides (1993) puts very tight bounds on option prices in the presence of transactions costs by deriving reservation purchase prices (reservation write prices) for an investor who buys (sells) a small fraction of an option and subsequently follows a suboptimal trading strategy.

\(^2\) By analyzing the discrete trading strategy directly, we avoid the use of unrealistic limits: an infinite adjusted volatility for Leland's (1986) work and a proportional transactions costs of zero in Henrotte's (1993) analysis.

\(^3\) Determining the discount rate is a non-trivial issue since the presence of transactions costs renders the market dynamically incomplete.
Figure 1: The Hedger's Perspective. The contract period is from $t_0$ to $t_M$. The individual hedging periods are from $t_m$ to $t_{m+1}$, $m = 0, \ldots, M - 1$. The hedger is concerned with the distribution of his cash-flow at the end of each hedging period. The inflow of cash at each rebalancing point is the hedging error minus the transactions costs. Conditional on the initial asset price, the first and second moment of this cash-flow are forward looking measures of the quality of the hedging strategy.
With these two measures in hand, the hedger can now choose the number of hedging periods in the contract period to suit his preferences. In this paper, I provide a menu of tools in closed form that facilitate this choice.

First, I derive exact formulas for the expected hedging error in the replication of a call option one and \( m + 1 \) periods ahead when rebalancing of the replicating portfolio is possible only at a finite number of points in time. Work by Leland (1985) and Boyle and Vorst (1992) suggest that when transactions costs exist, the number of shares in the hedge portfolio should be calculated using an adjusted volatility in Black and Scholes' equation. Therefore, the expressions for the expected hedging errors are calculated using two different volatilities: a stock process volatility and an adjusted hedging volatility. If the hedger agrees with the market on the underlying stock process' volatility, then the hedging volatility is adjusted to offset the effect of transactions costs. The expected hedging error can be calculated for any adjusted volatility but the adjustment given by Leland implies that, in the limit as the length of the hedging periods goes to zero, the sum of the hedging errors inclusive of transactions costs goes to zero with probability one. Furthermore, since the economy is dynamically incomplete, expectations are derived using a mean rate of return of the stock price, which may be different from the riskfree interest rate.

The expected hedging error is written in an easily recognized form. The expectation contains four terms, each of which has an interpretation corresponding to one of the two terms in Black and Scholes' equation, but with adjusted parameters. Rubinstein (1984) was the first to recognize that the expected payoff of an option over a finite holding period equals a Black-Scholes value with adjusted input parameters. The derived one period expected hedging error is a simple application of Rubinstein's result. In this paper, I show that Rubinstein's interpretation holds for any future hedging period, even when the stock price is known only at the beginning of the first hedging period. In addition, it follows directly from the derivation of the expected hedging error that the sum of the expected hedging errors is zero in the risk neutral economy – a result that does not depend on continuous rebalancing.

Second, I derive two approximations for the expected transactions costs. Using Leland's approximation of the period's transactions costs based on the option's gamma, the expected value of the approximate transactions costs is calculated for the rebalancing performed one and \( m + 1 \) periods ahead. From this result it follows directly that the expected transactions costs are the same
for any future period in the risk neutral economy without volatility adjustment.

Third, I derive exact expressions for the expected transactions costs one and \( m + 1 \) periods ahead. The exact and the approximate formulas are compared in numerical examples, and I show that the approximate values are almost equivalent to the exact values for hedging periods far from maturity. However, closer to maturity the approximation error becomes quite substantial. This is a result of the approximation’s failure to take changes in the call’s gamma into account.

Fourth, I illustrate that Leland’s Z-measure of the transactions costs in option replication is a measure of the expected transactions costs in the risk neutral economy. If the stock’s drift rate is different from the riskfree interest rate, Leland’s Z is not necessarily a measure of the expected transaction costs. Yet, it is still the appropriate measure for the difference in the prices caused by transactions costs because the sum of the hedging errors and the transactions costs converges to zero in probability, and this result does not require the stock price’s mean rate of return to be equal to the riskfree interest rate.

Fifth, I derive an approximation of the variance of the replicating strategy. In the presence of transactions costs, the variance of the replicating strategy over one hedging period is defined as the variance of the hedging error minus the transactions costs. Since the hedging errors minus the transactions costs are uncorrelated over time, the variance of the hedging strategy over the whole contract period can easily be obtained by simple summation. Using this result, I quantify the tradeoff between precision and price of the replicating strategy.

Finally, while developing the preceding results, it was necessary to derive formulas for integrals involving normal and bivariate normal density functions. These formulas, some of which I have not been able to find elsewhere, have obvious applications in option pricing, and may also be useful for researchers in other areas of Finance and Economics.

The detailed analysis of the time based strategy is partially motivated by Henrotte’s (1993) work. Henrotte compares the precision of the time based strategy to that of a move based trading strategy where trades are triggered by relative changes in the underlying asset’s value. He finds that the move based strategy does not always provide the better replication in mean-variance space; if very precise replication is required, the time based strategy offers a higher degree of precision at the same cost.\(^4\) Henrotte also shows that the properties of a hedging strategy based on relative

\(^4\)Henrotte (1993) expresses variances of replication strategies in terms of constants multiplied by the expectation
changes in the value of the underlying asset can be analyzed using expectations identical to those of the time base trading strategy. Consequently, when combined with Henrotte's work, this paper's results for the time based strategy can provide traders with efficient tools to analyze the properties of a move based hedging strategy. Finally, traders are often required to close their option books during periods of no trading in the underlying asset or futures on the underlying asset; in other words, for exogenous reasons, books of options written on most underlying assets must be delta neutral at least once a day.

The outline of the paper is as follows: Section 2 presents the underlying assumptions and formulas for the expected hedging errors. In section 3, approximate and exact formulas for expected transactions costs are derived. In both sections, the formulas are illustrated by numerical examples. In section 4, the formulas for the variance of the replicating strategy are derived and the tradeoff between price and precision is quantified. Finally, section 5 concludes the paper.

2 Expected Hedging Error

The hedging error in this section is defined as the difference between the value of the hedge portfolio and the theoretical value of an option after the passage of a fixed period of time. This definition reflects an option writer's point of view. The option price process is assumed to satisfy the Black-Scholes equation. The volatility used in the option price process is called the hedging volatility, and it may or may not be the same as the underlying asset's volatility. The use of an adjusted volatility is motivated by research by Leland (1985) and Boyle and Vorst (1992), who argue that a specific increase in the hedging volatility results in a hedging error that offsets (approximately) the transactions costs that occur when the replicating portfolio is revised. The expected hedging error in any of the future hedging periods is calculated over the distribution of stock prices, which is completely described by the stock process' drift and variance. When the hedging volatility is different from the underlying stock process' volatility, the sum of the hedging errors does not converge to zero as the number of revision points increases to infinity. However,
if Leland's adjusted hedging volatility is used, the sum of the hedging errors and the transactions costs converges to zero in probability as the number of revision points goes to infinity.

The expected hedging error with two volatilities (one to construct the hedge portfolio, and the other in the underlying asset's price process) has another interpretation similar to Rubinstein's (1984) interpretation of the expected payoff of an option. Assume that the hedging volatility is the volatility that prices options in the market at any future point in time. If an individual investor has subjective beliefs about the stock price's drift and volatility, \(^6\) then the expected value of the hedging error can be calculated using this subjective distribution of the stock price. Given this interpretation, the formulas for the expected hedging error can be used to assess the expected hedging error's sensitivity to different estimates of drift and volatility parameters. Therefore, the derived formulas can be used to extend the sensitivity results in Figlewski (1989) to expected values of the hedging error.\(^7\)

### 2.1 Notation and Underlying Assumptions

The basis for this paper is the Black-Scholes economy where two underlying assets exist. Trades can only be executed at a set of pre-specified points in time, \(\{t_0, \ldots, t_m, \ldots, t_M\}\), where \(t_0\) denotes the calendar time when hedging is initiated and \(t_M\) is the calendar time at the option's expiration. The period from \(t_0\) to \(t_M\) is called the contract period. The periods from \(t_m\) to \(t_{m+1}\), \(m = 0, \ldots, M - 1\) are labeled hedging periods. The stock and bond prices at the revision point \(m\) are given by

\[
S_{t_m} = S_{t_0} e^{\left((\mu - \frac{1}{2} \sigma^2)(t_m - t_0) + \sigma w(t_m - t_0)\right)},
\]

(1)

and

\[
R_{t_m} = R_{t_0} e^{\left(r(t_m - t_0)\right)},
\]

(2)

where \(w(\cdot)\) denotes a standard Brownian motion, \(r\) is the riskfree interest rate, and \(\mu\) is the mean rate of return on the risky asset. At each of the revision points the call's value, given the volatility

\(^6\)A discrete time economy where investors with different subjective beliefs about asset volatilities can co-exist is presented by Rubinstein (1976).

\(^7\)Figlewski (1989) uses simulations to assess standard deviations of the hedging errors.
\( \sigma \), is given by the Black-Scholes formula,

\[
C_{tm} = S_{tm} N \left( \frac{\lambda_m + \frac{1}{2} \sigma_m^2}{\sigma_m} \right) - Ke^{(-rT_m)} N \left( \frac{\lambda_m - \frac{1}{2} \sigma_m^2}{\sigma_m} \right),
\]

where \( \sigma_m = \sigma \sqrt{T_m} \), \( \lambda_m = \ln \left( \frac{S_{tm}}{K} \right) + rT_m \), and \( T_m \) denotes the time to maturity of the option contract at revision point \( m \), that is, \( T_m = t_M - t_m \). Note that lowercase \( t_m \) denotes calendar time whereas uppercase \( T_m \) is the time to maturity at revision point \( m \). The number of shares in the hedge portfolio between two revision points \( t_m \) and \( t_{m+1} \) is given by the option’s delta,

\[
\Delta_{tm} = N \left( \frac{\lambda_m + \frac{1}{2} \sigma_m^2}{\sigma_m} \right).
\]

The value of the hedge portfolio’s holding of bonds at the beginning of the period from \( t_m \) to \( t_{m+1} \) is given by

\[
B_{tm} = -Ke^{(-rT_m)} N \left( \frac{\lambda_m - \frac{1}{2} \sigma_m^2}{\sigma_m} \right).
\]

### 2.2 Derivation of the Expected Hedging Error

The expected hedging error is calculated for two different cases. The first expectation is calculated conditionally on the stock price when the hedge is determined. The result is called the one period expected hedging error. The derivation is similar to that of Rubinstein (1984), where the expected payoff of an option over a finite holding period is determined. The second expectation quantifies the expected hedging error in the hedging period from time \( t_m \) to \( t_{m+1} \) when only the value of the stock at time \( t_0 \) is known. The sum of the expected hedging errors for the \( M \) periods (possibly discounted) provides a measure of the expected hedging error over the life of an option contract.

At time \( t_0 \) we sell a call and buy the replicating portfolio. The option’s price and the replicating portfolio’s composition are given by the Black-Scholes formula using an adjusted hedging volatility, \( \bar{\sigma} \). This gives us a net cash flow of zero dollars at time \( t_0 \). At the next rebalancing point, \( t_1 \), the call is bought back and the replicating portfolio is sold. This gives us the period’s hedging error. This hedging process is repeated for all \( t_m \), \( m = 0, \ldots, M - 1 \). The hedging error at a general point in
time \( t_{m+1}, m = 0, \ldots, M - 1 \), is given by

\[
H_{t_{m+1}} = e^{(r(t_{m+1} - t_m))} \overline{B}_{t_m} + \overline{\Delta}_{t_m} S_{t_{m+1}} - \overline{C}_{t_{m+1}},
\]

where terms with bars denote values of (5), (4), and (3) using the hedging volatility, \( \overline{\sigma} \). The one period expected hedging error is defined as the expectation of the above expression over the stock price's distribution at time \( t_{m+1} \) conditional on the share's value at time \( t_m \). When the stock price distribution is lognormal with parameters \( \mu \) and \( \sigma \), this expectation can be calculated using formula (35) from appendix A. This leads to proposition 1.

**Proposition 1** If the call's price and the hedge portfolio's composition are derived from Black and Scholes' formula with an adjusted volatility, \( \overline{\sigma} \), but the stock price over the hedging period is lognormally distributed with parameters \( \mu \) and \( \sigma \), then the one period expected hedging error is given by

\[
E(H_{t_{m+1}} | S_{t_m}) = S_{t_m} e^{(\mu(T_m - T_{m+1}))} \left[ N \left( \frac{\lambda_{m,m}^* + \frac{1}{2} \sigma_{m,m}^*}{\sigma_{m,m}} \right) - N \left( \frac{\lambda_{m,m+1}^* + \frac{1}{2} \sigma_{m,m+1}^*}{\sigma_{m,m+1}} \right) \right] \nonumber \\
- Ke^{-rT_{m+1}} \left[ N \left( \frac{\lambda_{m,m}^* - \frac{1}{2} \sigma_{m,m}^*}{\sigma_{m,m}} \right) - N \left( \frac{\lambda_{m,m+1}^* - \frac{1}{2} \sigma_{m,m+1}^*}{\sigma_{m,m+1}} \right) \right],
\]

where

\[
\sigma_{m,k}^* = \sqrt{\overline{\sigma}^2 T_k + \sigma^2(T_m - T_k)}, \quad k = m, \ldots, M - 1, \\
\lambda_{m,k}^* = \ln \left( \frac{S_{t_m}}{K} \right) + T_k + \mu(T_m - T_k), \quad k = m, \ldots, M - 1.
\]

The first term in the first square bracket represents the delta from the beginning of the period. The delta is multiplied by the expected value of the stock at the end of the period. The first term in the second square bracket multiplied by \(-K\exp(-rT_{m+1})\) represents the bond holding at the beginning of the period plus the accrued interest from \( t_m \) to \( t_{m+1} \). The last two terms represent the negative of the expected value of the call, priced using the adjusted volatility, at the end of the period. As Rubinstein (1984) notes, this expectation is simply the Black-Scholes value of the call at the beginning of the period, where \( E(S_{t_{m+1}}) \) is substituted for \( S_{t_m} \), \( K\exp(r(t_{m+1} - t_m)) \) is substituted for \( K \), and finally \( \sigma_{m,m+1}^* \) is substituted for \( \sigma_{m,m}^* \). Intuitively, the last substitution means that the adjusted variance over the interval from \( t_m \) to \( t_{m+1} \), \( \overline{\sigma}^2(t_{m+1} - t_m) \), has been replaced with
the unadjusted variance, \( \sigma^2(t_{m+1} - t_m) \).

At the beginning of the contract period, the seller of the call may be interested in knowing the size of the expected hedging error over each of the future hedging periods. However, at time \( t_0 \) he does not have the information to apply the result in proposition 1. The relevant expectation is \( E \left[ H_{t_{m+1}} | S_{t_0} \right] \), which can be obtained from (6) by applying the law of iterated expectations and evaluating the expectation using formula (35) in appendix A.

**Proposition 2** If the call’s price and the hedge portfolio’s composition are derived from Black and Scholes’ formula with an adjusted volatility \( \tilde{\sigma} \), but the stock price over the contract period is lognormally distributed with parameters \( \mu \) and \( \sigma \), then the expectation at time \( t_0 \) of the hedging error at time \( t_{m+1}, m = 0, \ldots, M - 1 \), is given by

\[
E(H_{t_{m+1}} | S_{t_0}) = E(E(H_{t_{m+1}} | S_{t_m}) | S_{t_0}) = \left. \right. 
S_{t_0} e^{(\mu(T_0 - T_m) - \frac{1}{2} \sigma^2(Y_m))} - \left. \right. 
- K e^{-r(T_{m+1})} \left[ \left. \right. \right. 
N \left( \frac{\lambda_0, m + \frac{1}{2} \sigma^2(Y_m)}{\sigma(Y_m)} \right) - N \left( \frac{\lambda_0, m + 1 + \frac{1}{2} \sigma^2(Y_{m+1})}{\sigma(Y_{m+1})} \right) \right] .
\]

This expression is very similar to (6). The expected hedging error in the \((m + 1)\)th period given the information at time \( t_0 \) is equal to the one period expected hedging error in the first period where \( S_{t_0} \exp(\mu(T_0 - T_m)) \) is substituted for \( S_{t_0} \), \( K \exp(r(T_0 - T_m)) \) is substituted for \( K \), and the variance over the period \( t_0 \) to \( t_m \) is \( \sigma^2(T_0 - T_m) \) instead of \( \tilde{\sigma}^2(T_0 - T_m) \).

The expected hedging error can be interpreted as four terms similar to the two terms in Black and Scholes’ formula. The first term is equivalent to the first term in Black and Scholes’ equation for the value of a call at time \( t_0 \) where the variance and the expected growth rate of the stock change at the beginning of the hedging period, multiplied by one plus the expected growth of the stock’s price. The second term can be interpreted in the same way with the modification that the variance and the drift shift at the end of the hedging period. The last two terms have similar interpretations.

The expectations in propositions 1 and 2 are calculated using the true probability measure, but the expectations under the risk neutral probability measure are readily obtained; substitute \( r \) for \( \mu \) everywhere in the expectations. This simplification is analyzed in corollary 1.
Corollary 1 In the risk neutral economy, where \( \mu = r \), the present value of the expected future hedging errors equals the difference between two call values at time \( t_0 \), where one is evaluated with a volatility of \( \tilde{\sigma} \), and the other with a volatility of \( \sigma \), that is,

\[
\overline{C}_{t_0} - C_{t_0} = \sum_{m=0}^{M-1} \frac{E(H_{t_{m+1}} | S_{t_0})}{e^{r(T_0 - T_{m+1})}}.
\]

This result follows directly when the sum on the right-hand side of (8) is expanded and terms are canceled. From corollary 1 it is obvious that \( \tilde{\sigma} \) is equal to \( \sigma \) in the risk neutral economy if and only if the sum of the expected hedging errors are equal to zero. Corollary 1 does not depend on the number of revision points over the contract period.

The next corollary addresses the allocation of funds between stocks and bonds in the replicating portfolio.

Corollary 2 The expected value of the replicating portfolio invested in stocks right after rebalancing is given by

\[
E(\Delta_{t_{m+1}} S_{t_{m+1}} | S_{t_0}) = S_{t_0} e^{(\mu(T_0 - T_{m+1}))} N\left(\frac{\lambda_{0,m+1}^* + \frac{1}{2} \sigma_{0,m+1}^*}{\sigma_{0,m+1}^*}\right),
\]

and the expected amount invested in bonds is given by

\[
E(B_{t_{m+1}} | S_{t_0}) = -K e^{(-rT_{m+1})} N\left(\frac{\lambda_{0,m+1}^* - \frac{1}{2} \sigma_{0,m+1}^*}{\sigma_{0,m+1}^*}\right).
\]

Moreover, if \( \mu = r \) and \( \tilde{\sigma} = \sigma \), the present value of the stocks in the replicating portfolio is a martingale, that is,

\[
E\left[\frac{\Delta_{t_{m+1}} S_{t_{m+1}}}{e^{(\mu(T_0 - T_{m+1}))}} | S_{t_0}\right] = \Delta_{t_0} S_{t_0},
\]

and so is the present value of the bond investment,

\[
E\left[\frac{B_{t_{m+1}}}{e^{(r(T_0 - T_{m+1}))}} | S_{t_0}\right] = B_{t_0}.
\]

The martingale property is not a novelty. It follows directly from the work by Harrison and Kreps (1979) and Harrison and Pliska (1981): define two contingent claims where one pays \( S_{t_M} \) at
time \( t_M \) if \( S_{t_M} > K \) and zero otherwise, and the other pays \(-K\) if \( S_{t_M} > K \) and zero otherwise; calculate the discounted risk neutral expectations which, at time \( t \), are given by \( \Delta_t S_t \) and \( B_t \), respectively; since the normalized price processes are martingales in the risk neutral economy, it follows that normalized values of \( \Delta_t S_t \) and \( B_t \) are martingales.

Figure 2 shows the discounted value at time \( t_0 \) of the expected future hedging errors for a call option as a function of the adjusted hedging volatility, \( \tilde{\sigma} \), and the number of hedging periods in the contract period, \( M \). The discounted value of the expected hedging errors is defined as

\[
DV(HE) = \sum_{m=0}^{M-1} \frac{E(H_{t_{m+1}} | S_{t_0})}{e^{r(T_0 - T_{m+1})}}.
\]

(13)

The discounted values of the expected hedging errors are calculated using the following parameters: \( T_0 = 1, \sigma = 0.2, r = 0.06, \mu = 0.16, K = 100, \) and \( S_{t_0} = K \exp(-\mu \times T_0) \), which implies a Black-Scholes value of $3.54. An initial stock price of \( S_{t_0} = K \exp(-\mu \times T_0) \) is chosen such that the option is at-the-money with respect to the expected future stock price. It is clear from figure 2 that it does not take many hedging periods to get an expected hedging error close to zero when the hedging volatility is the same as the underlying asset’s volatility (20% per year). One period is insufficient, but when 10 or more hedging periods are used, the discounted values of the expected future hedging errors are very close to zero. When the hedging volatility is larger than the underlying asset’s volatility, the discounted value of the future hedging errors is positive\(^4\) (for \( \mu \) close to \( r \) and/or a large number of hedging periods). This is exactly the property Leland uses to construct his transactions costs adjusted volatility. Using this adjusted volatility, the flow of funds from the hedging errors offsets the incurred transactions costs when the lengths of the hedging periods go to zero. On the other hand, if the adjusted volatility is less than the underlying asset’s volatility, an inflow of funds is required to maintain the Black-Scholes hedge.

Figure 3 uses the same base values as figure 2, but now \( \tilde{\sigma} \) is kept constant at 0.2 while the asset’s drift rate, \( \mu \), is varied from 0.06 to 0.2. Figure 3 illustrates the result of corollary 1. If the stock’s drift rate is equal to the riskfree interest rate, the discounted value of the expected hedging errors is equal to zero no matter how many hedging periods we have in the contract period. The

\(^{4}\)Of course, when the hedging volatility is larger than the underlying asset’s volatility, more funds must be allocated to the hedge portfolio initially than if the true volatility is used.
Figure 2: Discounted Value of Expected Future Hedging Errors, $DV(HE)$, as a Function of Adjusted Volatility and Number of Hedging Periods. $DV(HE) = \sum_{m=0}^{M-1} \frac{E(H_{m+1} | S_{t_0})}{e^{r(T_0 - t_{m+1})}}$. Figure 2 shows the discounted value at time $t_0$ of the expected future hedging errors for a call option as a function of the adjusted hedging volatility, $\tilde{\sigma}$, and the number of hedging periods in the contract period, $M$. The discounted values of the expected hedging errors are calculated using the following parameters: $T_0 = 1$, $\sigma = 0.2$, $r = 0.06$, $\mu = 0.16$, $K = 100$, and $S_{t_0} = K \exp(-\mu \cdot T_0)$. 

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Figure 3: Discounted Value of the Expected Future Hedging Errors, \( DV(HE) \), as a Function of Underlying Asset's Drift Rate, \( \mu \), and Number of Hedging Periods, \( M \). 

\[
DV(HE) = \sum_{m=0}^{M-1} \frac{E(H_{t+m+1}|S_{t_0})}{e^{r(t_0-t_{m+1})}}
\]

Figure 3 shows the discounted value at time \( t_0 \) of the expected future hedging errors for a call option as a function of the underlying asset's drift rate, \( \mu \), and the number of hedging periods in the contract period, \( M \). The discounted values of the expected hedging errors are calculated using the following parameters: \( T_0 = 1 \), \( \sigma = \bar{\sigma} = 0.2 \), \( r = 0.06 \), \( K = 100 \), and \( S_{t_0} = K \exp(-0.16 \times T_0) \).
figure also illustrates that the expected hedging error converges to zero very quickly as the number of hedging periods increases when $\sigma = \bar{\sigma}$. For 50 hedging periods, the discounted value of the expected hedging errors is measured in pennies even when the excess rate of return on the risky asset is 14%.

3 Expected Transactions Costs

In this section, I derive the expected value of the transactions costs that occur at each of the future rebalancing points. The transactions costs are assumed to be a constant proportion of the value of the traded stocks.

Two different sets of expressions are calculated. First, I derive an approximation of the expected transactions costs at the end of the first and the $(m+1)$th hedging period. Thereafter, I calculate exact expressions. When transactions costs exist, the use of two different volatilities is motivated directly by Leland's work. Leland shows that if the hedging volatility is adjusted by $\bar{\sigma}^2 = \sigma^2[1 + \sqrt{(2/\pi)}k/(\sigma\sqrt{\Delta t})]$, where $\Delta t = (t_{m+1} - t_m)$, then the hedging error, as defined in the previous section, minus the transactions costs converges to zero in probability as the number of revision points goes to infinity.

3.1 Approximation of the Expected Transactions Costs

Leland (1985) shows, using a first order Taylor approximation, that the expected transaction costs one period ahead can be approximated by

$$E(\text{ATC}_{t_{m+1}} | S_{t_m}) = \frac{1}{2}k\bar{\Gamma}_{t_m} S_{t_m}^2 E_{t_m} \left| \frac{S_{t_{m+1}} - S_{t_m}}{S_{t_m}} \right|,$$  \hspace{1cm} (14)

where

$$\bar{\Gamma}_{t_m} = \frac{1}{S_{t_m} \sigma_{m,m}^* \sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{S_{t_{m+1}} - S_{t_m}}{\sigma_{m,m}^*}\right)^2},$$  \hspace{1cm} (15)

and $k$ denotes the round trip transactions costs on one share of stock. The bar in $\bar{\Gamma}_{t_m}$ denotes the use of an adjusted volatility. Using formula (34) in appendix A, the expectation in equation (14) is
calculated as\(^9\)

\[
V(\sigma, \mu, \Delta t) \equiv E_{t_m} \left| \frac{S_{t_m+1} - S_{t_m}}{S_{t_m}} \right| =
\]

\[
e^{(\mu \Delta t)} - 1 + 2 \left[ N \left( \frac{\left( \frac{1}{2} \sigma^2 - \mu \right) \Delta t}{\sigma \sqrt{\Delta t}} \right) - e^{(\mu \Delta t)} N \left( \frac{-\frac{1}{2} \sigma^2 - \mu}{\sigma \sqrt{\Delta t}} \right) \right],
\]

(16)

and proposition 3 follows.

**Proposition 3** Under this paper's assumptions about price processes, hedging policy, and transactions costs, the expected approximate transactions costs at the end of the hedging period from time \(t_m\) to time \(t_{m+1}\) conditional on the stock price at the beginning of the hedging period, \(S_{t_m}\), are given by

\[
E(\text{ATC}_{t_{m+1}} | S_{t_m}) = \frac{1}{2} k \Gamma_{t_m} S_{t_m}^2 V(\sigma, \mu, \Delta t).
\]

(17)

An approximation of the expected transactions costs at any future revision point conditional on the stock price at time \(t_0\) can also be calculated as

\[
E(\text{ATC}_{t_{m+1}} | S_{t_0}) = \frac{1}{2} k E_{t_0} \left[ \Gamma_{t_m} S_{t_m}^2 \left| \frac{S_{t_{m+1}} - S_{t_m}}{S_{t_m}} \right| \right] = \frac{1}{2} k E_{t_0} \left[ \Gamma_{t_m} S_{t_m}^2 \right] E_{t_m} \left[ \frac{S_{t_{m+1}} - S_{t_m}}{S_{t_m}} \right],
\]

because \(\Gamma_{t_m} S_{t_m}^2\) and \((S_{t_{m+1}} - S_{t_m})/S_{t_m}\) are independent. Computing the expectation of \(\Gamma_{t_m} S_{t_m}^2\) conditional on the information at time \(t_0\) leads to proposition 4.

**Proposition 4** Under this paper's assumptions about price processes, hedging policy, and transactions costs, the expected approximate transactions costs at the end of the hedging period from time \(t_m\) to time \(t_{m+1}\) conditional on the stock price at the beginning of the contract period, \(S_{t_0}\), are given by

\[
E(\text{ATC}_{t_{m+1}} | S_{t_0}) = \frac{1}{2} k \Gamma_{0,t_m}^* \left( S_{t_0} e^{\mu(T_0-T_m)} \right)^2 V(\sigma, \mu, \Delta t),
\]

(18)

\(^9V(\sigma, \mu, \Delta t)\) can also be approximated by \(\sqrt{2/\pi \sigma \sqrt{\Delta t}}\). This approximation is obtained by assuming that \((S_{t_{m+1}} - S_{t_m})/S_{t_m}\) is normally distributed, see Leland (1986).
where

\[
\Gamma_{0,m} = \frac{1}{S_{t_0} e^{\left(\mu(T_0 - T_m)\right)} \sigma_{0,m} \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\bar{T}_{0,m} + \sigma_{0,m} \bar{T}_{0,m}}{\sigma_{0,m}}\right)^2}.
\]

(19)

Here, the similarity between the expected transactions costs one and \(m + 1\) periods ahead is striking. The expected approximate transactions costs paid at time \(t_{m+1}\) can be obtained from the formula for the expected approximate transactions costs at time \(t_1\) simply by replacing \(S_{t_0}\) with \(S_{t_0} e^{\left(\mu(T_0 - T_m)\right)}\), \(K\) with \(K e^{\left(\mu(T_0 - T_m)\right)}\), and finally replacing the adjusted variance from \(t_0\) to \(t_m\), \(\bar{\sigma}^2(T_0 - T_m)\), with the unadjusted variance, \(\sigma^2(T_0 - T_m)\). In effect, this proposition shows that the expected approximate transactions costs in each of the future periods have the same structure, and are easily calculated as a constant times an adjusted gamma multiplied by the squared expected stock price at the beginning of the hedging period.\(^{10}\)

\(^{10}\)If a trader follows a move based strategy, that is, he rebalances the option hedge whenever the value of the underlying asset has changed by more than \(\delta + 100\) percent since the hedge was last rebalanced, then the expected costs of transacting over the period \(dt\), starting at \(t_m\), can be approximated by

\[
\frac{k}{\delta} \frac{1}{2} \sigma^2 \Gamma_{0,m} (S_{t_0} e^{\left(\mu(T_0 - T_m)\right)})^2 dt,
\]

when \(\delta, k,\) and \(dt\) are small. This result is obtained by using the expected value of \(\bar{T}_t S_t^2\) derived in this paper and the approximation of the expected transactions costs for a move based hedging strategy derived and expressed by Henrotte (1993) as

\[
\frac{k}{\delta} \frac{1}{2} \sigma^2 E \left[ \int_{t_1}^{t} \bar{T}_t S_t^2 dt \right].
\]

For the move based trading strategy, the appropriate adjustment to the volatility to account for transactions costs is \(\bar{\sigma} = \sigma \sqrt{1 + k/\delta} \). If \(\bar{\sigma} = \sigma\) and \(\mu = r\), the present value of the transactions costs flow over the interval \(\Delta t\) simplifies to

\[
\frac{k}{\delta} \frac{1}{2} \sigma^2 \Gamma_{t_0} S_{t_0}^2 dt.
\]

In other words, the expected value of the transactions costs for the contract period from \(t_0\) to \(t_M\) can be approximated by

\[
\frac{k}{\delta} \frac{1}{2} \sigma^2 \Gamma_{t_0} S_{t_0}^2 T_0.
\]

For the time based strategy, a similar approximation can be derived as

\[
k \sqrt{\frac{2}{\pi}} \frac{1}{\sigma \sqrt{\Delta t}} \frac{1}{2} \sigma^2 \Gamma_{t_0} S_{t_0}^2 T_0.
\]

This expression is obtained by approximating \(V(\sigma, \mu, \Delta t)\) by \(\sqrt{\frac{2}{\pi}} \sigma \sqrt{\Delta t}\).
A corollary follows directly from proposition 4, when $\mu = r$, and $\bar{\sigma} = \sigma$. In this case the normalized value of $\Gamma_t S_t^2$ is a martingale.

**Corollary 3** If the future hedging periods are of the same length, the economy is risk neutral, and $\bar{\sigma} = \sigma$, then the present values of the expected approximate transactions costs are the same for each of the future hedging periods, that is,

$$\frac{E(\text{ATC}_{t_m} | S_{t_0})}{e^{r(T_0 - T_m)}} = \frac{E(\text{ATC}_{t_n} | S_{t_0})}{e^{r(T_0 - T_n)}}, \text{ for } n, m \in \{1, \ldots, M\}. \quad (20)$$

This result is a direct consequence of the martingale property of the normalized value of $\Gamma_t S_t^2$. If the option is at-the-money in a maximum gamma sense, that is, $S_{t_m} = K \exp(-rT_m + 0.5\sigma^2 T_m)$, but far from maturity, $\Gamma_{t_m} S_{t_m}^2$ is relatively small. The expectation of $\Gamma_{t_n} S_{t_n}^2$, $n > m$, is affected by two things. The gamma increases as time to maturity decreases if the stock price stays at-the-money. However, the distribution of the stock price over which the expectation at time $t_0$ is taken becomes more disperse. These two effects offset each other, which, in this case, implies that the present value of $\Gamma_{t_m} S_{t_m}^2$ is a martingale.

Proposition 3 can be used to approximate the current flow of transactions costs when a portfolio of options is hedged. $\Gamma_{t_0} S_{t_0}^2$ for the individual options can be summed to a dollar gamma for the portfolio. If we take the absolute value\(^\text{11}\) of the portfolio's dollar gamma and multiply it by $0.5kV(\sigma, \mu, \Delta t)$,\(^\text{12}\) we obtain a measure of the expected costs of rebalancing the portfolio at time $t_1$. Of course, this measure is only valid if the portfolio’s gamma does not change significantly between time $t_0$ and $t_1$.

### 3.2 Exact Value of the Expected Transactions Costs

We now explore the exact values of the expected transactions costs. First, consider the expected transactions costs one period ahead, that is, from time $t_m$ to $t_{m+1}$. We are interested in the absolute value of the changes in the delta over the hedging period multiplied by the stock price at the end of

\(^{11}\)It follows that there are large economies of scale (with respect to transactions costs) from hedging large books of options that contain options with both convex and concave payoffs.

\(^{12}\)Here, we ignore an asymmetry which exists in most markets, that is, transactions costs from hedging a short position are often larger than that those from hedging a long position. This asymmetry comes from the trader's opportunity to implicitly make the market in the underlying asset when a positive gamma position is being hedged.
the period. This gives us the dollar amount traded at time $t_{m+1}$ to adjust the hedge. Multiply by $0.5k$ to obtain the realized transactions costs. Taking the expectation of this expression conditional on the value of the stock at time $t_m$ gives us the formula for the expected transactions costs one period ahead in time. This expectation is evaluated using formulas (33), (34), and (36) from appendix A, and leads to proposition 5.

**Proposition 5** Under this paper’s assumptions about price processes, hedging policy, and transactions costs, the expected transactions costs at the end of the hedging period from time $t_m$ to time $t_{m+1}$ conditional on the stock price at the beginning of the hedging period, $S_{t_m}$, are given by

$$E(TC_{t_{m+1}}|S_{t_m}) = \frac{1}{2} k E_{t_m}[(\Delta_{t_{m+1}} - \Delta_{t_m})S_{t_{m+1}}] =$$

$$\frac{1}{2} k S_{t_m} e^{(\mu \Delta t)} \left( [2 N(a_m) - 1] N(c_m) + N(b_m) - 2 N(a_m, b_m, \rho_m) \right) =$$

$$\frac{1}{2} k S_{t_m} e^{(\mu \Delta t)} \left( 2 [N(a_m, c_m, 0) - N(a_m, b_m, \rho_m)] + [N(b_m) - N(c_m)] \right),$$

where

$$a_m = \frac{z_m^* - (\mu + \frac{1}{2} \sigma^2) \Delta t}{\sigma \sqrt{\Delta t}},$$

$$b_m = \frac{\lambda_{m,m+1} + \frac{1}{2} \sigma_{m,m+1}^2}{\sigma_{m,m+1}^*},$$

$$c_m = \frac{\lambda_{m,m}^* + \frac{1}{2} \sigma_{m,m}^2}{\sigma_{m,m}^*},$$

$$z_m^* = \ln \left( \frac{S_{t_m}}{K} \right) \left( \frac{\sqrt{T_{m+1}}}{\sqrt{T_m}} - 1 \right) + \left( r + \frac{1}{2} \sigma^2 \right) \left( \sqrt{T_m T_{m+1}} - T_{m+1} \right),$$

$$\rho_m = -\frac{\sigma \sqrt{\Delta t}}{\sigma_{m,m+1}^*}.$$

As in the previous section, the expected transactions costs for the $(m + 1)$th period given the stock price at time $t_0$ can be determined by applying the law of iterated expectations as well as formulas (37) and (38) from appendix A.

**Proposition 6** Under this paper’s assumptions about price processes, hedging policy, and transactions costs, the expected transactions costs at the end of the hedging period from time $t_m$ to time
$t_{m+1}$ conditional on the stock price at the beginning of the contract period, $S_{t_0}$, are given by

$$E(TC_{t_{m+1}} | S_{t_0}) = E(E(TC_{t_{m+1}} | S_{t_m}) | S_{t_0}) = \frac{1}{k} S_{t_0} e^{\left(\sigma(T_0 - T_{m+1})\right)} \left(2 \left[ N\left(d_m, f_m, \rho_1\right) - N\left(d_m, \epsilon_m, \rho_2\right)\right] + \left[ N\left(\epsilon_m\right) - N\left(f_m\right)\right]\right),$$

where

$$d_m = \frac{\sqrt{T_{m+1}}(\lambda_{0,m}^* + \frac12 \sigma_{0,m}^2) - \left(\lambda_{0,m+1}^* + \frac12 \sigma_{0,m+1}^2\right)}{\sigma_{m}},$$

$$e_m = \frac{\lambda_{m+1}^* + \frac12 \sigma_{0,m+1}^2}{\sigma_{0,m+1}},$$

$$f_m = \frac{\lambda_{0,m}^* + \frac12 \sigma_{0,m}^2}{\sigma_{0,m}},$$

$$\rho_1 = \frac{\sigma^2(T_0 - T_m) \tau_m}{\sigma_{0,m}^*},$$

$$\rho_2 = \frac{\sigma^2((T_0 - T_m) \tau_m - \Delta t)}{\sigma_{0,m+1}^*},$$

$$\delta_m = \sigma \sqrt{\Delta t + (T_0 - T_m)^2},$$

$$\tau_m = \frac{\sqrt{T_{m+1}}}{\sqrt{T_m}} - 1.$$

Clearly, the formulas for the expected transactions costs are not as intuitively appealing as the approximate formulas for the expected transactions costs. In addition, they involve the evaluation of bivariate normal distribution functions. However, note that the structures of the expected transactions costs are the same for the expected transactions costs one and $m + 1$ periods ahead.

Figure 4 shows the expected hedging errors, the approximation of the expected transactions costs, and finally, the exact values of the expected transactions costs in 52 weekly hedging periods in a one year contract. The values in the figure are calculated under the same assumptions as figure 2, except that the adjusted volatility $\tilde{\sigma}$ is given by Leland's (1985) adjusted volatility and weekly rebalancing is assumed. When Leland's adjusted volatility is used, the sum of the hedging errors minus the transactions costs converges to zero in probability. As anticipated, the expected hedging errors and the expected transactions costs are almost the same for each of the individual hedging periods. However, we notice that the approximation to the expected transactions costs deviates
Figure 4: Expected Weekly Hedging Errors and Transactions Costs. Figure 4 shows the expected hedging errors, $E(\text{HE})$, the approximation of the expected transactions costs, $E(\text{ATC})$, and finally the exact values for the expected transactions costs, $E(\text{TC})$, for 52 weekly hedging periods in a one year contract hedging a call option. The weekly expected hedging errors and transactions costs are calculated using the following parameters: $T_0 = 1$, $\sigma = 0.2$, $\Delta t = 1/52$, $k = 0.01$, $\hat{\sigma}^2 = \sigma^2[1 + \sqrt{(2/\pi)}k/(\sigma \sqrt{\Delta t})]$ (Leland’s adjustment), $r = 0.06$, $\mu = 0.16$, $K = 100$, and $S_{t_0} = K \times \exp(-\mu \times T_0)$. 
significantly from the exact values in the final weeks of the contract period. For earlier hedging periods, the approximate formula for the transactions costs is very precise.

Figure 5 shows the expected hedging errors and the expected transactions costs under the same assumptions as those for figure 4, but calculated by Monte Carlo simulations. This figure is included to verify the derived formulas. Figure 6 plots the discounted values of the expected future hedging errors, the expected value of the approximate future transactions costs, and the expected value of the future transactions costs as a function of the stock’s drift rate when the hedging volatility is adjusted using Leland’s correction. The input parameter values are the same as in figure 4 and 5. The stock price at $t_0$ is kept constant at $100 \times \exp(-0.16 \times T_0)$ when $\mu$ is varied. Figure 6 illustrates that the discounted value of the expected transactions costs is a function of the stock’s drift rate when the hedge portfolio is constructed using an adjusted volatility (Leland’s adjustment). Note that the curve for the discounted value of the approximate transactions costs and the discounted value of the hedging errors are almost the same when $\mu$ is close to $r$. This illustrates that Leland’s volatility adjustment is based on the approximation and not the exact size of the transactions costs. Of course, this difference vanishes in the limit as $\Delta t \to 0$.

4 Approximation of the Variance of the Replication Strategy

In the previous sections, I derived expressions for expected hedging errors and transactions costs when the hedge portfolio is formed using an adjusted hedging volatility. When this volatility is adjusted as devised by Leland (1985), the sum of hedging errors minus transactions costs converges to zero in probability. Since this limiting strategy is prohibitively expensive and convergence to the limit is slow (of order $\sqrt{\Delta t}$), for practical applications it is necessary to quantify the variance of the replicating strategy as the length of the hedging periods converges to zero. The results in this section enable us to analyze the variance of the hedging strategy as a function of trading intensity and the input variables in Black and Scholes’ equation. The variance of the replicating strategy over one hedging period is defined as the variance of the hedging error minus the transactions costs at the end of that particular period. A natural measure of the variance of the hedging strategy over the entire contract period is the variance of the sum of the discounted hedging errors minus the transactions costs. This quantity is easily obtained from the one period variances because the
Figure 5: Expected Weekly Hedging Errors and Transactions Costs, Monte Carlo Results, 500,000 Simulations. Figure 5 shows estimates of the expected hedging errors, E(HE), and the expected transactions costs, E(TC), obtained by simulation for 52 weekly hedging periods in a one year contract hedging a call option. The weekly expected hedging errors and transactions costs are calculated using the following parameters: $T_0 = 1$, $\sigma = 0.2$, $\Delta t = 1/52$, $k = 0.01$, $\bar{\sigma}^2 = \sigma^2 [1 + \sqrt{(2/\pi)k/(\sigma \sqrt{\Delta t})}]$ (Leland’s adjustment), $r = 0.06$, $\mu = 0.16$, $K = 100$, and $S_{t_0} = K \times \exp(-\mu \times T_0)$. 
Figure 6: Discounted Value of Expected Future Hedging Errors and Transactions Costs as a Function of Underlying Asset's Drift Rate. Figure 6 shows the discounted value of all future expected hedging errors, \( DV(HE) \), expected approximate transactions costs, \( DV(ATC) \), and finally the discounted values of all future expected transactions costs, \( DV(TC) \), as a function of the underlying asset's drift rate. 52 hedging periods in a one year contract are assumed. The weekly expected hedging errors and transactions costs are calculated using the following parameters: \( T_0 = 1, \sigma = 0.2, \Delta t = 1/52, k = 0.01, \tilde{\sigma}^2 = \sigma^2 [1 + \sqrt{(2/\pi)k/(\sigma\sqrt{\Delta t})}] \) (Leland's adjustment), \( r = 0.06, K = 100, \) and \( S_{t_0} = K \times \exp(-0.16 \times T_0) \).
hedging errors minus transactions costs for different hedging periods are uncorrelated. When the results in this section are combined with the results in previous sections, a mean-variance frontier can be constructed as the length of the hedging period is varied.\textsuperscript{13} This allows the hedger to pick a trading intensity which suits his preferences.\textsuperscript{14}

An exact expression for the variance of the replicating strategy can be calculated using formulas in appendix A. Preliminary calculations show that the exact formulas are likely to span several pages, so only an approximation is derived. In particular, the hedging error over any future hedging period, conditional on the stock price at the beginning of the contract period, can be approximated by

$$AHE_{t_{m+1}} = \frac{1}{2} \hat{\Gamma}_{t_m} \hat{S}_{t_m}^2 \left( \tilde{\sigma}^2 \Delta t - \left( \frac{S_{t_{m+1}} - S_{t_m}}{S_{t_m}} \right)^2 \right).$$

Taking expectations conditional on the stock price at the beginning of the contract period in conjunction with a Taylor approximation yields

$$E(AHE_{t_{m+1}} | S_{t_0}) = \frac{1}{2} \Gamma_{0,m}^* \left( S_{t_0} e^{\mu(T_0 - T_m)} \right)^2 (\tilde{\sigma}^2 - \sigma^2) \Delta t.$$  \hspace{1cm} (22)

In addition, the expected approximate transactions costs can be simplified further by ignoring higher order terms and approximating $V(\sigma, \mu, \Delta t)$ by $\sqrt{\frac{2}{\pi}} \sigma \sqrt{\Delta t}$. This leads to a simple expression for the expected approximate transactions costs.

$$E(ATCAP_{t_{m+1}} | S_{t_0}) = \frac{1}{2} k^* \Gamma_{0,m}^* \left( S_{t_0} e^{\mu(T_0 - T_m)} \right)^2 \sqrt{\frac{2}{\pi}} \sigma \sqrt{\Delta t}.$$ \hspace{1cm} (23)

Subtracting these two approximations from each other yields

$$E[AHETC_{t_{m+1}} | S_{t_0}] = E(AHE_{t_{m+1}} - ATCAP_{t_{m+1}})$$

\textsuperscript{13}Using the results in footnotes 10 and 15, the corresponding tradeoff between cost and variance of a move based trading strategy can be constructed.

\textsuperscript{14}I implicitly assume that the hedger has quadratic preferences.
\[ = \frac{1}{2} \Gamma_{\sigma,m} (S_{t_0} e^{\mu(T_0 - T_m)})^2 \left[ (\bar{\sigma}^2 - \sigma^2) \Delta t - k \sqrt{\frac{2}{\pi}} \sigma \sqrt{\Delta t} \right]. \] (24)

This expression demonstrates that the adjusted volatility, which equates the expected value of the approximate hedging error and the expected approximate transactions costs, is the \( \bar{\sigma} \) which solves

\[ (\bar{\sigma}^2 - \sigma^2) \Delta t - k \sqrt{\frac{2}{\pi}} \sigma \sqrt{\Delta t} = 0. \] (25)

The result is Leland's volatility adjustment.

After these preliminary results we derive the variance of the hedging policy over one period, given that we know the value of the underlying asset at the beginning of the hedging period. We define the approximate variance of the cash-flows from the hedging scheme one period ahead as the variance of \( \text{AHETC}_{t_{m+1}} \) conditional on \( S_{t_m} \). Evaluation of this variance is simplified by the fact that \( \Gamma_{t_m} S_{t_m}^2 \) is in the information set at the beginning of the hedging period. Therefore,

\[ \text{VAR}(\text{AHETC}_{t_{m+1}} | S_{t_m}) = \frac{1}{4} (\Gamma_{t_m} S_{t_m}^2)^2 \left[ E_{t_m} \left[ \bar{\sigma}^2 \Delta t - \left( \frac{S_{t_{m+1}} - S_{t_m}}{S_{t_m}} \right)^2 - k \left| \frac{S_{t_{m+1}} - S_{t_m}}{S_{t_m}} \right| \right] \right] - \left[ (\bar{\sigma}^2 - \sigma^2) \Delta t - k \sqrt{\frac{2}{\pi}} \sigma \sqrt{\Delta t} \right]^2. \] (26)

Using numerous approximations and moment generating functions, evaluation of this expectation leads to the following proposition.

**Proposition 7** Under this paper's assumptions about price processes, hedging policy, and transactions costs, the approximate one period variance of the hedging policy at the end of the hedging period from time \( t_m \) to time \( t_{m+1} \) conditional on the stock price at the beginning of the hedging period, \( S_{t_m} \), is given by

\[ \text{VAR}(\text{AHETC}_{t_{m+1}} | S_{t_m}) = \frac{1}{4} (\Gamma_{t_m} S_{t_m}^2)^2 \left( k^2 \left( 1 - \frac{2}{\pi} \right) \sigma^2 \Delta t + 2k \sqrt{\frac{2}{\pi}} \sigma^3 (\Delta t)^{3/2} + 2 \sigma^4 (\Delta t)^2 \right). \] (27)
Proposition 7 requires knowledge of the stock price at the beginning of the hedging period. Only the stock price at the beginning of the first hedging period is known when the hedging scheme is initiated. For future hedging periods, the expected value of \( (\bar{T}_m S^2_{t_m})^2 \) conditional on the stock price at time \( t_0 \) must be determined. Define

\[
\sigma^*_{m,k} = \sqrt{2\sigma^2(T_m - T_k) + \sigma^2 T_k},
\]  

then

\[
E((\bar{T}_m S^2_{t_m})^2 | S_{t_0}) = \Psi_{0,m} = \frac{(S_{t_0}e^{\mu(T_0-T_m)})^2}{2\pi \sigma T_m \sigma_{0,m} \sigma^*_{m,k}} e^{-\frac{(\frac{1}{2} \sigma_{0,m}^2 + \frac{1}{4} \sigma^*_{m,k}^2)^2 - 2\sigma^2(T_0-T_m)\sigma_{0,m}}{\sigma^*_{m,k}^2}}.
\]

(29)

Using the law of iterated expectations, the independence of \((S_{t_{m+1}} - S_{t_m})/S_{t_m}\) and \(S_{t_m}\), and the above expectation leads to the next proposition.\(^{15}\)

**Proposition 8** Under this paper's assumptions about price processes, hedging policy, and transactions costs, the approximate one period variance of the hedging policy at the end of the hedging period from time \( t_m \) to time \( t_{m+1} \) conditional on the stock price at the beginning of the contract period, \( S_{t_0} \), is given by

\[
VAR(\text{AHETC}_{t_{m+1}}|S_{t_0}) =
\]

\(^{15}\)If the trader follows a move based hedging strategy with \( \delta = \sigma \sqrt{1+k/\delta} \), where \( \delta \) represents the relative change in the underlying asset which triggers rebalancing, Henrotte (1993) shows that the variance of the hedging strategy can be approximated by

\[
\frac{\sigma^2}{6} (\delta + k) E \left[ \int_{t_0}^{t_m} (\bar{T}_t S^2_t)^2 dt \right].
\]

(30)

Applying Fubini's theorem and using the expectation of \((\bar{T}_t S^2_t)^2\) derived in this paper, it follows that the variance of the move based replicating strategy can be approximated by

\[
\frac{\sigma^2}{6} (\delta + k) \sum_{n=0}^{N-1} \Psi_{0,n} dt,
\]

(31)

when \( \delta, k, \) and \( dt = t_{n+1} - t_n \) are small. Note that in expression (31) \( n = 0, \ldots, N-1 \), denotes a discretization of time which is sufficiently fine to yield satisfactory convergence of (31) to (30). The corresponding expression for the time based strategy is

\[
\frac{1}{4} \left( k^2 \left( 1 - \frac{2}{\pi} \right) \sigma^2 + 2k \frac{2}{\pi} \sigma^3 (\Delta t)^{1/2} + 2\sigma^4 (\Delta t) \right) \sum_{m=0}^{M-1} \Psi_{0,m} \Delta t.
\]
\[ \frac{1}{4} \Psi_{0,n} \left( k^2 \left( 1 - \frac{2}{\pi} \right) \sigma^2 \Delta t + 2k \sqrt{\frac{2}{\pi} \sigma^3 (\Delta t)^{3/2} + 2 \sigma^4 (\Delta t)^{2}} \right) . \]

Figures 7 through 9 illustrate the results in this section. Figure 7 plots the variance of the cash-flow at each rebalancing point as a function of hedging period for our standard hedging contract with 52 hedging weeks in a one year contract. Figure 7 shows that the hedging scheme is less precise closer to the expiration date of the option. Keep in mind that the variance is a forward looking measure calculated conditionally on the stock price at the initiation of the one year contract. Therefore, the observed pattern is not simply the result of an increase in the option’s gamma as the time to maturity goes to zero when the option is at-the-money. Figure 8 adds another dimension to figure 7. Figure 8 plots the variance of the cash-flow at the end of each hedging period as a function of the striking price and hedging week. As expected, we note that replication of options deeply in- or out-of-the-money can be performed precisely.

Finally, figure 9 illustrates the tradeoff between price of the option and the precision of the replicating strategy. In this figure, the standard deviation of the replicating strategy is defined as

\[ \text{STD} = \sqrt{\sum_{m=0}^{M-1} \text{VAR} \left( \frac{\text{AHETC}_{t,m+1}}{e^{r(T_0-T_{m+1})}} \right) S_{t_0}} . \tag{32} \]

The coordinates in the figure are calculated using 10 to 200 hedging periods in the analyzed one year contract. As the number of hedging periods specified in the contract increases, the standard deviation decreases while the price of the replicating portfolio increases. This is caused by an increasing volatility adjustment as the number of hedging periods increases. A proportional transactions cost of 1% is realistic when hedging is performed using the underlying asset (equity). However, when liquid futures are used for hedging purposes, round trip transactions costs can easily be reduced by more than one order of magnitude. Consequently, when an active futures market exists, a given level of precision can be obtained at significantly lower costs than figure 9 indicates.
Figure 7: Variance of Cash-flow from Replicating Strategy as a Function of Hedging Period. Figure 7 illustrates the variance of the hedging error minus transactions costs, VAR(AHETC), for each of the 52 weekly hedging periods in a one year contract hedging a call option. The variance of the weekly cash-flow is calculated conditional on the stock price at the beginning of the hedging contract using the following parameters: $T_0 = 1$, $\sigma = 0.2$, $\Delta t = 1/52$, $k = 0.01$, $\tilde{\sigma}^2 = \sigma^2[1 + \sqrt{(2/\pi)k/(\sigma\sqrt{\Delta t})}]$ (Leland’s adjustment), $r = 0.06$, $\mu = 0.16$, $K = 100$, and $S_{t_0} = K \exp(-\mu * T_0)$. 
Figure 8: Variance of Cash-flow from the Replicating Strategy as a Function of Hedging Period and Striking Price. Figure 8 shows the variance of the hedging error minus transactions costs, VAR(AHETC), as a function of hedging week and strike price for a one year contract hedging a call option. The strike price is varied from 60 to 140. The variance of the weekly cash-flow is calculated conditional on the stock price at the beginning of the hedging contract using the following parameters: $T_0 = 1$, $\sigma = 0.2$, $\Delta t = 1/52$, $k = 0.01$, $\sigma^2 = \sigma^2[1 + \sqrt{(2/\pi)}k/(\sigma\sqrt{\Delta t})]$ (Leland's adjustment), $r = 0.06$, $\mu = 0.16$, and $S_{t_0} = 100 \exp(-\mu \cdot T_0)$.
Figure 9: Price Plotted Against Standard Deviation of Replicating Strategy. Figure 9 shows an aggregate measure of the standard deviation of the cash flows from the hedging contract,

\[
\text{STD} = \sqrt{\sum_{m=0}^{M-1} \text{VAR}\left( \frac{A_{\text{HEDG}} t_{m+1}}{e^{r(T_0-T_m)}} \mid S_{t_0} \right)},
\]

plotted against the price of a one year contract hedging a call option. \(\Delta t\) is varied from 1/10 (lower right corner) down to 1/200 (upper left corner) of a year. The standard deviation is calculated conditional on the stock price at the beginning of the hedging contract using the following parameters: \(T_0 = 1, \sigma = 0.2, k = 0.01, \sigma^2 = \sigma^2[1 + \sqrt{(2/\pi)k/(\sigma \sqrt{\Delta t})}]\) (Leland’s adjustment), \(r = 0.06, \mu = 0.16, K = 100, \) and \(S_{t_0} = 100 \times \exp(-\mu \times T_0).\) The option’s value ignoring transactions costs is 3.53. The expected transactions costs equals total costs minus 3.53.
5 Conclusion

This paper presents formulas for expected hedging errors, transactions costs, and variances in option replication when trades are placed at exogenously fixed points in time. These formulas are expressed as functions of two different volatilities: the asset volatility and the hedging volatility. Expectations and variances are derived using the asset’s true drift rate and volatility. This paper’s results can be used to quantify the tradeoff between expected costs and variance of an option hedging strategy when transactions costs exist. Previously derived analytical results for the Black-Scholes economy relied on unrealistic limits in which either the volatility adjustment is infinite (Leland (1985)) or the transactions costs go to zero as the lengths of the hedging periods go to zero (Henrotte (1993)). In this paper, analytical results are derived directly for the non-limiting case. The main body of the paper is used to analyze a time based trading strategy. However, footnotes 10 and 15 show how the developed results can be used to quantify the corresponding tradeoff between expected cost and variance of a move based trading strategy where a rebalancing trade is triggered by a small relative change in the underlying asset’s price. This paper’s analytical results significantly reduce the computational time required to quantify a given trading strategy’s properties. Henrotte (1993) uses Monte Carlo simulations to evaluate the required expectations of some important time integrals. However, when one interchanges the order of integration and evaluates the expectation analytically using the results of this paper, only a single time integral must be computed numerically.
A Useful Integrals

This appendix lists a series of integrals which assist in the derivation of the closed form expressions for option hedges when transactions costs are present. The derivations of the expressions are not performed, but an outline of the proof is given for each of the formulas. In the following list of formulas, \( N(\cdot) \) and \( N(\cdot, \cdot, \cdot) \) denote the univariate and bivariate normal distribution function, respectively.

\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}(Ax^2+By+C)} \, dx = \frac{\sqrt{2\pi}}{\sqrt{A}} e^{-\frac{1}{2} \left( C - \frac{E^2}{4A} \right)}
\]

(33)

\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}(Ax^2+By+C)} \, dx = \frac{\sqrt{2\pi}}{\sqrt{A}} e^{-\frac{1}{2} \left( C - \frac{E^2}{4A} \right)} N \left( \sqrt{A}x + \frac{B}{2\sqrt{A}} \right)
\]

(34)

\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}(Ax^2+By+C)} N(Dx + E) \, dx = \frac{\sqrt{2\pi}}{\sqrt{A}} e^{-\frac{1}{2} \left( C - \frac{E^2}{4A} \right)} N \left( \sqrt{\frac{A}{A+D^2}}(E - \frac{BD}{2A}) \right)
\]

(35)

\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}(Ax^2+By+C)} N(Dx + E) \, dx = \frac{\sqrt{2\pi}}{\sqrt{A}} e^{-\frac{1}{2} \left( C - \frac{E^2}{4A} \right)} N \left( \sqrt{\frac{A}{A+D^2}}(E - \frac{BD}{2A}), -\frac{D}{\sqrt{A+D^2}} \right)
\]

(36)

\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}(Ax^2+By+C)} N(Dx + E, Fx + G, \rho) \, dx = \frac{\sqrt{2\pi}}{\sqrt{A}} e^{-\frac{1}{2} \left( C - \frac{E^2}{4A} \right)} N \left( \sqrt{\frac{A}{A+D^2}}(E - \frac{BD}{2A}), \sqrt{\frac{A}{A+F^2}}(G - \frac{BF}{2A}), \frac{DF+Ar}{\sqrt{(A+D^2)(A+F^2)}} \right)
\]

(37)

In the case of zero correlation the above equation becomes

35
\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}((Ax^2+Bx+C)N(Dx+E)N(Fx+G)dx = \\
\frac{\sqrt{2\pi}}{\sqrt{A}} e^{-\frac{1}{2}(C-\frac{B^2}{4A})} \left( \frac{\sqrt{A}}{A+D^2}(E-\frac{BD}{2A}), \frac{\sqrt{A}}{A+F^2}(G-\frac{BF}{2A}), \frac{DF}{\sqrt{(A+D^2)(A+F^2)}} \right)
\]

The derivation of equation (33) and (34) involves a simple completion of the square. The derivation of equation (35) requires the use of the following trick. Define

\[
f(\epsilon) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}((Ax^2+Bx+C)N(Dx+\epsilon)dx,
\]

where

\[
N(\bar{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\bar{y}} \exp\left(-\frac{x^2}{2}\right) dx.
\]

Use the fact that

\[
f(E) = f(-\infty) + \int_{-\infty}^{E} f'(\epsilon)d\epsilon = \int_{-\infty}^{E} f'(\epsilon)d\epsilon.
\]

Calculate

\[
f'(\epsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((Ax^2+Bx+C)e^{-\frac{1}{2}((Dx+\epsilon)^2)dx,}
\]

and use formula (33) to get

\[
f'(\epsilon) = \frac{1}{\sqrt{A+D^2}} e^{-\frac{1}{2}\left(C+\epsilon^2-\frac{(B+2DE)^2}{4(A+D^2)}\right).
\]

Finally, we integrate over \(\epsilon\) which gives us the equation (35).

To derive equation (36) we have to complete the square for a bivariate normal distribution. First define

\[
f(\bar{x}, \epsilon) = \int_{-\infty}^{\bar{x}} e^{-\frac{1}{2}((Ax^2+Bx+C)N(Dx+\epsilon)dx.
\]

Differentiate with respect to \(\epsilon\), simplify, and integrate back over \(\epsilon\) up to \(E\). This gives us

\[
f(\bar{x}, E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\bar{x}} \int_{-\infty}^{E} e^{-\frac{1}{2}((A+D^2)x^2+c^2+2Dxz+Bz+C)d\epsilon dx.
\]

Now the problem is to rewrite this expression as a constant times a bivariate normal distribution.
This is a fairly cumbersome process and it will not be repeated here.

Equation (38) is derived using the same technique. The only difference is that we now have to differentiate with respect to two variables. Define

\[
 f(\epsilon, \gamma) = \int_{-\infty}^{\infty} N(Dx + \epsilon, Fx + \gamma, \rho) e^{-\frac{1}{2}(Ax^2 + Bx + C)} dx
\]

\[
 = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left( \frac{(Fz + \gamma)^2 + (Dz + \epsilon)^2 - 2\rho Fz + \gamma Dz + \epsilon}{1 - \rho^2} \right)} e^{-\frac{1}{2}(Ax^2 + Bz + C)} dz dy dx
\]

Use the fact that

\[
 f(E, G) = \int_{-\infty}^{E} \int_{-\infty}^{G} f_{12}(\epsilon, \gamma) d\epsilon d\gamma,
\]

where

\[
 f_{12}(\epsilon, \gamma) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left( \frac{(Fz + \gamma)^2 + (Dz + \epsilon)^2 - 2\rho Fz + \gamma Dz + \epsilon}{1 - \rho^2} \right)} e^{-\frac{1}{2}(Ax^2 + Bx + C)} dx.
\]

Use formula (34) to integrate over \( x \). Thereafter the solution is found by completing the square of the bivariate normal distribution function.

Formula (37) is a special case of formula (38).
References


