Options on Leveraged Equity with Default Risk

by

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Abstract

In this paper, I derive option pricing formulas for call and put options written on leveraged equity in an economy with corporate taxes and bankruptcy costs. The firm can be forced into bankruptcy by breaching a net-worth covenant, or it may declare bankruptcy when it is optimal for equityholders to do so. Consequently, option values and sensitivities depend on structural variables such as the corporate tax rate, the firm's coupon payments, and the firm value at which bankruptcy is declared. The derived formulas for calls and puts on equity with default risk simplify to Black-Scholes type formulas for down-and-out barrier options if bankruptcy is declared as soon as the value of the firm's assets equals the after-tax value of the promised coupon payments on the debt. If the capital structure contains no debt, the pricing results simplify to Black-Scholes formulas for call and put options. The model developed in this paper relates implied Black-Scholes volatility for equity options to structural characteristics such as leverage and the debt's protective covenants. Options priced by the proposed model are characterized by Black-Scholes implied volatilities which are decreasing in striking price. Moreover, equity options on firms with protected debt have more pronounced volatility skews than options on firms with unprotected debt. Finally, I show how to evaluate the term structure of default spreads for corporate interest-only strips.
1 Introduction

The direct link between financing decisions at the firm level and the pricing of derivative securities has largely been ignored in academic finance literature. Most existing finance literature treats the pricing of securities in the firm’s capital structure and the pricing of options on these securities as separate research areas. Generally, securities in the capital structure are priced based on assumptions about the value process of the firm’s assets; hedging arguments are invoked and the securities in the capital structure are priced as derivative securities.\(^1\) In equity option pricing theory, the underlying asset’s price is assumed to follow a stochastic process; based on this particular process a pricing formula is derived.\(^2\) In other words, different primitive assumptions are made for different purposes, and result in inconsistent pricing of securities. For example, if the asset value process for a leveraged firm is assumed to follow a geometric Brownian motion, then the possibility of bankruptcy implies that the equity process cannot follow a geometric Brownian motion. Stochastic price processes should be derived from primitive assumptions that create a set of consistent price processes for all securities related to the same underlying real assets, thus eliminating potential arbitrage opportunities between different classes of securities which may arise when inconsistent models are used. In addition, if the values of securities in the capital structure depend on leverage, then the firm’s financing decisions automatically affect the values of derivative securities written on securities in the firm’s capital structure.

In this paper, I derive closed form option prices consistent with the firm’s overall capital structure. The model for the underlying capital structure, developed by Leland (1994), implies that the equity’s volatility is a function of asset values. Consequently, this dependence is built into this paper’s option pricing formulas. The endogenous nature of the equity price process distinguishes this paper from most existing literature; the price process is not assumed – it is derived from primitive assumptions about the capital structure and no arbitrage opportunities.\(^3\)

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\(^3\)Previous models in the literature where volatility varies as a function of equity values include Rubinstein’s (1983) displaced diffusion and Cox and Ross’ (1976) constant elasticity of variance option pricing formula.
The option pricing problem solved in this paper is essentially a compound option problem with a knock-out boundary, where the value of equity is the underlying option. Geske (1979) models option prices as compound options, where the equity itself is modeled as an option on the firm’s assets. Geske uses Black and Scholes’ (1973) and Merton’s (1974) model of a capital structure with zero coupon debt with finite maturity. Geske’s model also leads to option price processes which are consistent with derived equity and debt price processes. However, Geske does not analyze a perpetual capital structure; after the debt’s maturity date, the company is fully equity financed. Also neglected is the presence of taxes and bankruptcy costs which may affect the total value of the firm’s securities.

In this paper, the firm’s capital structure is assumed to consist of equity and perpetual debt with constant continuous coupon payments. Perpetual debt in the capital structure reflects the assumption that the firm’s capital structure does not change abruptly over time, and is a logical counterpart to a capital structure with one zero coupon bond and equity. Pricing of securities in the capital structure simplifies significantly when this structure is used and Leland (1994) obtains closed form solutions for debt and equity values as functions of the riskfree interest rate, payout rate, volatility of underlying assets, bankruptcy trigger, corporate tax rate, coupon rate, value of underlying assets, and proportional bankruptcy costs. Depending on the indenture provisions, bankruptcy can be declared by equityholders when it is optimal to do so, or the company can be forced into bankruptcy by a net-worth covenant. The value of the underlying assets at which bankruptcy will occur is relevant to option pricing; the mechanism behind bankruptcy does not affect option values. However, it does affect some of the option’s sensitivities. Since the equity value is a function of the corporate tax rate, coupon payments, interest rate, payout rate, volatility, bankruptcy trigger, and underlying asset values, option prices are also modeled as functions of these fundamental variables.

The derived formulas for calls and puts on equity with default risk simplify to Black-Scholes type formulas for down-and-out options if bankruptcy is declared as soon as the value of the underlying assets equals the after-tax value of the promised payments on the debt. When the capital structure contains no debt, the pricing results simplify to Black-Scholes formulas for calls and puts.

I compare the developed model to the Black-Scholes model by mapping implied Black-Scholes

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4In Geske (1977), coupon paying debt is analyzed as a compound option.
volatility structures which result from prices generated by the proposed model. For a highly leveraged firm, the implied volatility decreases in striking price. This pattern is less pronounced for a moderately leveraged firm than for a highly leveraged firm. Moreover, equity options for firms with protected debt in the capital structure tend to have more pronounced volatility skews than options on firms with unprotected debt. This difference is less significant for firms with low leverage.

Finally, I illustrate how the derived call option pricing formula can be used to analyze the term structure of default risk for corporate interest-only securities when the payout rate on the firm’s underlying assets is zero. Given a positive payout rate, the value of the corporate interest-only security can be derived directly.

A disadvantage of the model developed in this paper is that the bankruptcy trigger, bankruptcy costs, volatility of underlying assets, and underlying asset value are not directly observable. Yet the model is still testable because it restricts the co-movements of derivative securities, debt, and equity values. The unobservable variables may be estimated from a joint system of equations derived from prices of all securities which depend upon the value of the firm’s underlying assets. Such estimates may eventually be used to assess the cross-sectional differences in bankruptcy costs, effective corporate tax rates, and volatility of underlying assets. Furthermore, the model can be tested simply by comparing predicted cross-sectional pricing biases with observed biases.

The remainder of the paper is organized as follows: Section 2 reviews the foundation for the paper, the underlying capital structure theory. In section 3, the pricing formulas for calls and puts on equity with default risk are developed, the pricing results are analyzed, and the formulas are simplified for certain special cases. Finally, section 4 concludes the paper.

2 The Capital Structure Theory

In this section, I briefly review the capital structure theory developed by Leland (1994). Leland’s model is the foundation for the equity option pricing formula derived in this paper. The value of the firm's assets is assumed to follow an Ito-process

\[ dV = (\eta(V, t) - \delta)V dt + \sigma V dw(t), \] 

(1)
where $dw(t)$ is the increment of a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, $\eta(V, t)$ is the instantaneous drift rate, $\delta$ is the proportional payout rate, and $\sigma$ is the volatility. The value of the firm’s assets is assumed to be independent of the capital structure.\(^5\)

The firm has issued debt which pays the bondholders a continuous stream of coupon payments $C^\prime$ per time interval. The bond’s covenants allow for liquidation of assets at a rate $\delta$ of the total value of the firm’s assets.\(^6\) The payout $\delta V dt$ can be interpreted as the after-tax cash-flow available to both bond and equityholders. The after-tax value of the coupon payments $(1 - \tau)C dt$ is taken from this cash-flow and the remaining cash-flow is the net-payout, or dividends, received by the equityholders. Using this interpretation, the payout rate $\delta$ can be determined such that the after-tax net payout to the equityholders, $(\delta V - (1 - \tau)C) dt$, corresponds to an exogenously given dividend payment over the time interval $dt$. The quantity $(\delta V - (1 - \tau)C) dt$ may be negative. In this case, new equity must be issued to finance the after-tax coupon payments. If $(\delta V - (1 - \tau)C) dt$ is positive, then the cash-flow from liquidating assets is more than sufficient to cover the after-tax coupon payments; equityholders receive dividend payments.

The net-payout, $(\delta V - (1 - \tau)C) dt$, only corresponds to the dividend payments at the initial value of $V$. However, it is realistic to assume that the dividend payments will increase if the value of the firm increases. Conversely, if the value of the firm decreases, the dividend payments may be cut and the firm may eventually decide to raise new equity. The assumption used in this paper implies that the net payout increases (decreases) relatively faster than the value of the underlying assets when the firm value increases (decreases).

No arbitrage opportunities implies that the value of any security with payoff's contingent on the value of the firm’s assets must satisfy the fundamental partial differential equation (PDE) for derivative securities,

$$\frac{1}{2}\sigma^2 V^2 F_{VV}(V, t) + (r - \delta)V F_V(V, t) + F_t(V, t) + \kappa = r F(V, t),$$

where $\kappa$ is the constant continuous payout received by the owner of the security in question,

\(^5\)This is in tradition with the classic capital structure literature pioneered by Modigliani and Miller (1958), Merton (1974), and Brennan and Schwartz (1978a).

\(^6\)Merton (1974) analyzes perpetual debt under the assumption that the firm liquidates assets to pay a continuous stream of coupon payments.
\( \tau \) is the continuously compounded interest rate, and \( F(V, t) \) is the security value process, and subscripts denote partial derivatives. Because we only consider time independent securities, the PDE simplifies to an ordinary differential equation with known general solution. Finally, boundary conditions lead to closed form solutions for debt, equity, bankruptcy costs, and the value of the tax shield. The time independent equity value is

\[
E = V - A + BV^{-x},
\]

(2)

where

\[
A = \frac{(1 - \tau)C}{r},
\]

\[
B = \left[ \frac{(1 - \tau)C}{r} - V_0 \right] V_0^x,
\]

\[
x = \frac{m + \mu}{\sigma^2},
\]

and

\[
m = \sqrt{\mu^2 + 2r\sigma^2},
\]

(3)

\[
\mu = r - \delta - \frac{1}{2}\sigma^2.
\]

(4)

The corporate tax rate is given by \( \tau \), and \( V_0 \) is the value of the underlying assets at which equityholders are either forced or decide to declare bankruptcy. \( A \) can be interpreted as the present value of all future promised after-tax coupon payments. If the payout rate \( \delta \) is zero, \( x \) simplifies to \( 2r/\sigma^2 \).

Debt values are given by

\[
D = \frac{C}{r} + \left[ (1 - \alpha)V_0 - \frac{C}{r} \right] \left[ \frac{V}{V_0} \right]^{-x},
\]

(5)

where \( \alpha \) is the proportional bankruptcy costs. The bankruptcy trigger may be determined endogenously if it is within the equityholder’s discretion to declare bankruptcy. The asset value at which bankruptcy is optimal is derived by Leland (1994) as

\[
V_0 = \frac{(1 - \tau)Cx}{r(x + 1)}.
\]

(6)
The above model of equity values implies that the volatility of equity is time independent, yet highly dependent on leverage; as the value of the firm's underlying assets falls to a relatively low level, the return on equity becomes highly volatile. For high asset values, the equity volatility does not change significantly with changes in equity values.

Equity values can be written slightly differently as

\[ E = V - \frac{(1 - \tau)C}{r} (1 - p_b) - V_b p_b, \]  

where

\[ p_b = \left( \frac{V}{V_b} \right)^{-x}. \]  

The term \( p_b \) can be interpreted as the present value of a security that pays one dollar the moment the firm defaults. Consequently, \( (1 - p_b) \) is the present value of a security that pays one dollar now, with the added obligation to pay the dollar back as soon as the firm defaults. Writing the value of equity in this form illustrates that the equity value consists of three terms: (i) the value of the firm's asset, (ii) minus the after-tax value of a riskfree perpetual coupon stream times \( (1 - p_b) \), and (iii) minus \( V_b \) times the bankruptcy state price, \( p_b \).\(^7\) Likewise, the value of debt can be written as

\[ D = \frac{C}{r} (1 - p_b) + (1 - \alpha) V_b p_b. \]  

Options can be written on any traded security in the capital structure. Equity options are natural derivative securities to consider, but options on debt can also be written in this environment; they would represent options on default risk. Since interest rate risk is ignored, I only consider the evaluation of equity options.

\(^7\)Alternatively, \( p_b \) could have been calculated as \( p_b = \int_0^\infty f(t)e^{-r't} dt \), where \( f(t) \) represents the first passage time density for \( V \) hitting \( V_b \), where \( V \) follows a geometric Brownian motion with drift \( r - \delta \) and volatility \( \sigma \).
Figure 1: Equity Values as Functions of Underlying Asset Value for Three Different Bankruptcy Triggers: $V_b = C/r = 100$, $V_b = (1 - \tau)C/r = 75$, and $V_b = (1 - \tau)Cx/(r(x + 1)) = 43.8$ (optimal bankruptcy). The following input values are used for all three functions: $C = 5$, $\tau = 0.05$, $\sigma = 0.2$, $\tau = 0.25$, and $\delta = (1 - \tau)C/100 = 0.0375$. 
Before the option pricing formulas are derived, it is useful to examine equity values as functions of the value of the firm’s assets. The function given in (2) can be strictly concave, linear, or strictly convex. Figure 1 plots equity values as functions of the value of the firm’s assets in an example with \( C = 5, \ r = 0.05, \ \tau = 0.25, \ \delta = (1 - \tau)C/100 = 0.0375, \) and \( \sigma = 0.2 \) for three different values of the bankruptcy trigger. The function labeled \( V_b = 100 \) shows equity values for a firm where debt is subject to a net-worth covenant that triggers bankruptcy if the firm’s asset value falls below the present value of the promised payments, \( V_b = C/r. \) This covenant implies that the equity value function is concave in the asset value. An option interpretation can explain the concavity of the equity value as a function of asset value. The term \( V - A \) in the equity value function (2) represents the value of the firm’s unleveraged assets, less the after-tax value of the promised coupon payments, which is linear in \( V. \) The term \( BV^{-x} = (A - V_b)p_b \) represents the value of a perpetual cash-at-hit option where the owner pays the value of the firm while being released from the obligation to pay the future coupon stream if the knock-out barrier is reached. When \( A < V_b \) the equity owner is short a perpetual cash-at-hit option with a cash payment of \( V_b - A. \) Since the cash-at-hit option value is a convex function of the firm’s assets, the equity value is a concave function of asset value when \( A < V_b. \) The function labeled \( V_b = 75 \) shows equity value as a function of asset value where bankruptcy is triggered immediately if the asset value reaches \( (1 - \tau)C/r. \) In this case, equity value is a linear function of asset value. As shown below, option pricing formulas for this scenario simplify substantially. Finally, the function labeled \( V_b = 43.8 \) represents equity value as a function of asset value when bankruptcy is declared by equityholders at the optimal asset value, \( V_b = (1 - \tau)Cx/(r(x + 1)). \) This graph illustrates the convexity and the smooth pasting condition at \( V = V_b. \) Using the option interpretation, the equityholders own the underlying assets minus the after tax value of the promised payments on the bond in addition to a long perpetual cash-at-hit option with a rebate of \( A - V_b. \) Figure 1 illustrates that for the perpetual capital structure considered in this paper, equity value is not necessarily a convex function of the value of the firm’s assets. If the debt’s covenants contain moderately strict net-worth requirements, the equity value may be a concave function of asset values. This stands in sharp contrast to the usual interpretation of equity as a long call option on the firm’s assets.

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\(^{8}\text{This rate of asset liquidation implies zero dividends to equityholders when the value of the underlying assets is 100. If } V > 100, \text{ equityholders receive positive dividends. If } V < 100, \text{ new equity must be issued to cover the after-tax payments on the debt.}\)
3 Options on Equity with Default Risk

Based on the capital structure theory reviewed in the previous section, I now turn to evaluating options written on equity with default risk. In this section, I assume that the equity price process is given by equation (2). If the value of the firm’s assets, \( V \), falls below the bankruptcy trigger, \( V_b \), bankruptcy is declared. The value of the option does not depend on the mechanism that leads to bankruptcy; only the size of \( V_b \) determines the option’s value. However, the option’s sensitivities to changes in the interest rate, proportional payout rate, volatility, coupon payments, and corporate tax rate depend on the way bankruptcy is triggered; when bankruptcy is optimally chosen by equityholders, changes in these variables change the firm value at which bankruptcy is optimal. This, in turn, affects the option’s sensitivities to changes in these parameters. On the other hand, if bankruptcy is exogenously determined by a net-worth covenant, changes in the interest rate, proportional payout rate, volatility, coupon payments, and corporate tax rate do not change the bankruptcy trigger. These changes still have an effect on option prices, but they do not occur through a change in the bankruptcy trigger.

3.1 Derivation

The equity value process is derived by Leland (1973) under the assumption that it is possible to trade a security with a price process which is perfectly correlated with the value of the firm’s underlying assets. This assumption, coupled with time independence of debt, leads to closed form solutions for debt and equity values. In this economy there is only one fully hedgeable source of risk. Consequently, the economy is dynamically complete and call and put values can be determined by calculating the discounted expected payoff under the equivalent martingale measure.\(^9\)

3.1.1 Call Options

Using the martingale approach, the value of a call option written on equity with default risk can be expressed as

\[
\text{CALL} = e^{-rT} E^{Q}[\max(E-K,0) \mathbf{1}_{(s \geq T)}],
\]

where $s$ is the stopping time, defined as the first time the value of the firm’s assets, $V$, reaches the bankruptcy trigger, $V_b$, or $T$, whichever comes first: $s = \inf\{t \geq 0 : V(t) \leq V_b\} \wedge T$. $E^{Q}$ is the expectation under the equivalent martingale measure and $1(t)$ is the indicator function. If $s < T$, equity becomes worthless prior to the option’s expiration. This implies an option value of zero. The expectation in (10) can be evaluated using techniques from the pricing of barrier options.\textsuperscript{10}

$$\text{CALL} = Ve^{-\delta T} \left[ N\left(y + \sigma \sqrt{T}\right) - \left(\frac{V_b}{V}\right)^{\frac{2b}{\sigma^2}} \frac{2}{\sigma^2} N\left(y + \sigma \sqrt{T} + \left(\frac{2b}{\sigma^2}\right)\right) \right] + (A - V_b) \left(\frac{V_b}{V}\right)^{x} \left[ N\left(y - x \sigma \sqrt{T}\right) - \left(\frac{V_b}{V}\right)^{\frac{2b}{\sigma^2}} \left(\frac{2b}{\sigma^2}\right)\right] N\left(y - x \sigma \sqrt{T} + \left(\frac{2b}{\sigma^2}\right)\right) \right]$$

\begin{align}
+ (A - V_b) \left(\frac{V_b}{V}\right)^{x} \left[ N\left(y - x \sigma \sqrt{T}\right) - \left(\frac{V_b}{V}\right)^{\frac{2b}{\sigma^2}} \left(\frac{2b}{\sigma^2}\right)\right] N\left(y - x \sigma \sqrt{T} + \left(\frac{2b}{\sigma^2}\right)\right) \right]
\end{align}

\begin{align}
- (A + K)e^{-\tau T} \left[ N\left(y - \left(\frac{V_b}{V}\right)^{\frac{2b}{\sigma^2}} \left(\frac{2b}{\sigma^2}\right)\right) \right],
\end{align}

where $y$ is the smallest solution to the following equation in $e$:\textsuperscript{11}

$$K + A = Ve^{(r - \delta - \frac{1}{2}\sigma^2)T + \sigma \sqrt{T}(-\epsilon)} + BV - e^{-x(r - \delta - \frac{1}{2}\sigma^2)T + x \sigma \sqrt{T}(-\epsilon)} , \tag{12}$$

where

\begin{align}
b &= \ln\left(\frac{V_b}{V}\right), \tag{13} \\
B &= (A - V_b)\frac{V_b}{V}. \tag{14}
\end{align}

The implicit equation, (12), equates equity value and striking price at the option’s expiration date, or equivalently it is used to find the critical value of $V(T)$ such that the call option finishes exactly at-the-money. If $A > V_b \geq (1 - \tau)Cx/(r(x + 1))$, there are two solutions to (12). However, the larger solution does not have any economic meaning; to reach this solution the bankruptcy boundary must be crossed. To see this, note that when the bankruptcy trigger, $V_b$, is between

\textsuperscript{10}See Rubinstein and Reiner (1991) and Rich (1994). For standard barrier options, we have to distinguish between cases where the striking price is above or below the barrier. For this down-and-out option, hitting the barrier implies an equity value of zero. Consequently, for options with $K \geq 0$, the barrier will always be at or below the asset value that equates equity and striking price.

\textsuperscript{11}When $V(T) \equiv V(0)e^{(r - \delta - \frac{1}{2}\sigma^2)T + \sigma \sqrt{T}(-\epsilon)}$, $-\epsilon$ represents a standard normal random variable, measurable with respect to $\mathcal{F}_T$, which completely describes the value of the firm’s asset at time $T$. I use the minus sign in this definition to avoid an excessive number of minus signs in the formula for the call option price.
(1 − τ)C \frac{x}{(r(1 + x))} and A, then B is positive. Outside the bankruptcy region, E is increasing in the value of the firm's assets, and therefore decreasing in \( \epsilon \). Hence, one searches for the smallest value of \( \epsilon \) that satisfies the implicit equation (12). If \( V_b > A \), then B is negative. In this case the implicit equation has only one solution for \( \epsilon \).

Although slightly more complicated, we see that the structure of the option pricing formula (11) is related to that of the Black-Scholes equation. The equity option's value is written as a function of the value of the firm's assets. The equity value (2) contains two stochastic terms which contribute to the option's payoff if it expires in-the-money. These two stochastic terms are equivalent to the terms on the right-hand side of the implicit equation (12). The structure of the first term on the first line of formula (11) is similar to that of the first term in the Black-Scholes equation; it represents the value of the payoff from the first term on the right-hand side of the implicit equation (12), the value of the underlying assets, if the option expires in-the-money. The second term on the first line adjusts for the possibility of hitting the bankruptcy trigger. The second line of (11) represents the value of the payoff in the risk neutral economy from the second term on the right-hand side of the implicit equation (12) in the case of exercise. This is the value of a security which pays the owner a perpetual cash-at-hit option with a payoff \((A - V_b)\) if the value of equity is higher than the option's striking price at the expiration date. Again, the second term on the second line of (11) adjusts for the possibility of bankruptcy. Finally, the last line of (11) is the value of the striking price and the after-tax risk-free promised debt payments which are the option owner's liability if the option finishes in the money. In a more stringent notation, the first line of the equation (11) is the value of a contingent claim that pays \( V \) if \( V > V_b \forall t \leq T \) and \( E \geq K \) at expiration. The second line is the value of a security that pays \((A - V_b)(V_b/V)^\epsilon\) if \( V > V_b \forall t \leq T \) and \( E \geq K \) at expiration. Recall that \((A - V_b)p_b\) represents the value of the right to default on the debt, minus the value of the firm's assets in the case of bankruptcy. Finally, the last line is the value of a security that pays \(-(A + K)\) at time \( T \) if \( V > V_b \forall t \leq T \) and \( E \geq K \) at expiration.

One can derive an approximate pricing boundary for the case where the value of the firm's underlying assets is large relative to the striking price and the bankruptcy trigger. In this case, the probability of default is small. When the probability of default is close to zero the equityholders will, over the lifespan of the option, pay \((1 - \tau)C\) per unit of time to the bondholders, while receiving \( \delta V \) from the firm. Now consider two portfolios: portfolio 1 contains one call option and
portfolio 2 contains one share of stock and riskless bonds of value

\[
\frac{C(1 - \tau)(1 - e^{-rT})}{r} - V(1 - e^{-\delta T}) - Ke^{-rT}.
\]

The first term is the present value of the after-tax coupon payments due before the option’s expiration. The second term represents minus the value of the assets which will be liquidated from time 0 to time \( T \). Given that we do not cross the bankruptcy trigger, the value of portfolio 1 (the call option) will always be larger than that of portfolio 2. This implies that the following approximate pricing boundary,

\[
\text{CALL} \geq E + \frac{C(1 - \tau)(1 - e^{-rT})}{r} - V(1 - e^{-\delta T}) - Ke^{-rT},
\]

holds for \( V \gg V_b \). The term \( V(1 - e^{-\delta T}) - (C(1 - \tau)(1 - e^{-rT}))/r \) represents the value of the dividends payable to equityholders between time 0 and time \( T \). If this quantity equals zero, the usual lower bound for call option prices results. If \( (C(1 - \tau)(1 - e^{-rT}))/r > V(1 - e^{-\delta T}) \), we encounter a case where the option may be more valuable than the underlying asset. Consider the extreme case where \( \delta = 0 \) and \( K = 0 \). When the possibility of bankruptcy is negligible, the only difference between a call option and equity is the equityholders’ obligation to service the debt. Consequently, the call option may be more valuable than equity.

When the bankruptcy trigger is exogenously given, the derived option value is a function of the underlying asset value, asset value at which bankruptcy is triggered, coupon payments, striking price, riskfree interest rate, payout rate, volatility, time to maturity, and the corporate tax rates, that is,

\[
\text{CALL} = \text{CALL}(V, V_b, C, K, r, \delta, \sigma, T, \tau).
\]

It is easily verified that option prices are homogeneous of degree one in \( V, V_b, C, \) and \( K \).

Once the capital structure is determined, equity values do not depend on the size of the bankruptcy costs. Consequently, option prices are also independent of bankruptcy costs. However, this is not necessarily the case if deviations from absolute priority rules result in payments to equityholders in the event of default.

If bankruptcy is endogenously determined by equityholders at the optimal asset value \( V_b = \)
\( C(1 - \tau)x/(r(x + 1)) \), option prices are only functions of the underlying asset value, coupon rate, striking price, riskfree interest rate, payout rate, volatility, time to maturity, and finally the corporate tax rate, that is,

\[
\text{CALL} = \text{CALL}(V, C, K, r, \delta, \sigma, T, \tau).
\]

In this formulation, the pricing formula is homogeneous of degree one in \( V, C \), and \( K \).

It is important to note the difference between the option pricing formula with and without optimal bankruptcy. First, optimal bankruptcy is declared when \( V_b = ((1 - \tau)C\times)/r(x + 1)) \), which is less than the present value of the promised after-tax payments. Second, even if the net-worth covenant requires bankruptcy to be declared at \( V_b = (1 - \tau)C\times/(r(x + 1)) \), the option pricing formula with exogenously determined bankruptcy exhibits distinct differences from the pricing formula with optimal bankruptcy. While option prices are the same regardless of the bankruptcy triggers, their sensitivities to changes in coupon payments, interest rate, proportional payout, volatility, and the corporate tax rate are different. For example, an increase in the corporate tax rate has several effects when bankruptcy is optimally determined by equityholders. First, the equity value changes simply because the value of the tax shield changes. Second, optimal bankruptcy will be declared at a lower asset value. The lower bankruptcy trigger increases equity values and thus influences option values. Finally, option values also increase because the knock-out barrier, \( V_b \), is at a lower level.\(^{12}\) Leverage changes, that is, changes in \( C \) affect the option value in similar ways. To keep the value of the underlying assets constant, the market value of the additional debt must be paid out to equityholders as dividends.\(^{13}\) In this case, the change in the option value is affected directly by the change in equity value given by equation (2) when \( C \) changes.\(^{14}\) In addition, the optimal bankruptcy trigger changes, producing a secondary effect due to the changing knock-out boundary. The same interpretation holds for changes in interest rate, proportional payout rate and volatility; there is a direct effect from the change in equity value and a secondary effect from the change in the knock-out boundary.

The call option pricing formula simplifies to well known formulas in two special cases. First,

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\(^{12}\)Note that this analysis assumes that \( V \) is unaffected by changes in the corporate tax-rate.

\(^{13}\)While this is clearly unrealistic for large changes in the capital structure, this may not be an unrealistic assumption for marginal changes in leverage.

\(^{14}\)Alternatively, if shares of stock are retired using the proceeds from the newly issued debt, the change in equity value cannot be derived from equation (2) because the number of outstanding shares changes.
if the bankruptcy trigger is determined by a net-worth covenant that requires the firm's assets to be worth at least \( V_b = C(1 - \tau)/r \), then the call option pricing formula (11) simplifies to the Black-Scholes type formula for a down-and-out call option, where the option is written on the underlying asset value, has a striking price of \( A + K \), and a knock-out boundary at \( V_b = A \),

\[
\text{CALL} = Ve^{-sT} \left[ N(z_1) - \left( \frac{\delta}{\sigma} \right)^{2b+2} N \left( z_1 + \left( \frac{2b}{\sigma \sqrt{T}} \right) \right) \right] - (A + K)e^{-rT} \left[ N(z_2) - \left( \frac{\delta}{\sigma} \right)^{2b} N \left( z_2 + \left( \frac{2b}{\sigma \sqrt{T}} \right) \right) \right],
\]

(16)

where

\[
z_1 = \frac{\ln \left( \frac{V}{K + A} \right) + (r - \delta + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}},
\]

and

\[
z_2 = z_1 - \sigma \sqrt{T}.
\]

The special case above simplifies because the equity value is a linear function of the asset value above the knock-out barrier.

Second, if the firm is unleveraged and bankruptcy is impossible, the call value (11) simplifies to the Black-Scholes formula,

\[
\text{CALL} = Ve^{-sT}N(z_1) - Ke^{-rT}N(z_2),
\]

(17)

where \( A = 0 \) in the expression for \( z_1 \) and \( z_2 \).

Figures 2 and 3 show call option values and deltas as functions of equity values for three different bankruptcy triggers: \( V_b = C(1 - \tau)x/(r(x + 1)) = 43.8 \) (optimal bankruptcy), \( V_b = (1 - \tau)C/r = 75 \), and \( V_b = C/r = 100 \). The remaining parameters for figures 2 and 3 are: \( C = 5 \), \( r = 0.05 \), \( \tau = 0.25 \), \( \delta = \frac{(1-\tau)c}{100} = 0.0375 \), \( \sigma = 0.2 \), \( T = 0.5 \), and \( K = 20 \). Option values are calculated as functions of equity values by inverting (2) which yields \( V \) as a function of \( E \) for \( E > 0 \). The resulting asset values, along with the remaining parameters, are then used in the call option pricing formula (11). The functions plotted in figure 2 and 3 correspond to the equity value functions from figure 1. Note, the value of the call option for the firm which declares bankruptcy optimally at \( V_b = 43.8 \) is lower than that of firms with higher bankruptcy triggers. This
phenomenon can be explain by the higher equity volatility of the firms with higher bankruptcy triggers. The higher volatility is caused by two effects. First, given higher asset values, a constant $\sigma$ implies larger absolute changes in asset values. Second, the larger derivative of the equity value function with respect to asset value for the firm with the higher bankruptcy trigger translates the larger asset volatility into an even larger equity volatility.

Figure 3 illustrates the importance of modeling the firm’s capital structure correctly. For most equity values the option’s delta varies substantially for options on equity in firms with different bankruptcy triggers. For example, if the firm’s debt is subject to a strict net-worth covenant, the equity call option’s delta is far from zero even when the value of equity is close to zero. This is in sharp contrast to the Black-Scholes model where the call option’s delta converges to zero as the value of equity goes to zero. For the model developed in this paper, the option’s delta does not go to zero as the equity value goes to zero if the bankruptcy trigger is larger than the optimal $V_0$.

### 3.1.2 Put Options

The value of a put option can also be derived using the martingale approach. Put values are given by

$$\text{PUT} = e^{-rT}E^Q[\max(K - E, 0)1_{(s \geq T)}] + K e^{-rT} \int_0^T f(t) dt,$$  \hspace{1cm} (18)

where $f(t)$ is the first passage time density for $V$ reaching $V_0$.\(^{15}\) The first term represents the payoff of the put if bankruptcy never occurs. The second term is the value of the strike in the case of

---

Figure 2: Call Option Values as Functions of Equity Value for Three Different Bankruptcy Triggers: $V_b = C/r = 100$, $V_b = (1 - \tau)C/r = 75$, and $V_b = (1 - \tau)Cx/(r(x + 1)) = 43.8$ (Optimal Bankruptcy). The following input parameters are used: $C = 5$, $r = 0.05$, $\sigma = 0.2$, $\tau = 0.25$, $\delta = (1 - \tau)C/100 = 0.0375$, $K = 20$, and $T = 0.5$. 
Figure 3: Call Option Deltas as Functions of Equity Value for Three Different Bankruptcy Triggers: $V_b = C/r = 100$, $V_b = (1 - \tau)C/r = 75$, and $V_b = (1 - \tau)C\delta/(\sigma(\sigma + 1)) = 43.8$ (Optimal Bankruptcy). The following input parameters are used: $C = 5$, $r = 0.05$, $\sigma = 0.2$, $\tau = 0.25$, $\delta = (1 - \tau)C/100 = 0.0375$, $K = 20$, and $T = 0.5$. 
bankruptcy. Evaluating the expectation in equation (18) yields

\[
PUT = (A + K)e^{-rT} \left[ (N(-y) - N(-y^*)) - \left( \frac{V_0}{V} \right)^{\frac{2b}{\sigma^2}} N \left( -y - \left( \frac{2b}{\sigma_0 \sqrt{T}} \right) \right) - N \left( -y^* - \left( \frac{2b}{\sigma_0 \sqrt{T}} \right) \right) \right] \\
- Ve^{-rT} \left[ N(-y - \sigma_0 \sqrt{T}) - N(-y^* - \sigma_0 \sqrt{T}) - \left( \frac{V_0}{V} \right)^{\frac{2b}{\sigma^2}} \times N \left( -y - \sigma_0 \sqrt{T} - \left( \frac{2b}{\sigma_0 \sqrt{T}} \right) \right) - N \left( -y^* - \sigma_0 \sqrt{T} - \left( \frac{2b}{\sigma_0 \sqrt{T}} \right) \right) \right] \\
-(A - V_b) \left( \frac{V_0}{V} \right)^{x} \left[ N(-y + x\sigma_0 \sqrt{T}) - N(-y^* + x\sigma_0 \sqrt{T}) - \left( \frac{V_0}{V} \right)^{\frac{2b}{\sigma^2}} \times N \left( -y + x\sigma_0 \sqrt{T} - \left( \frac{2b}{\sigma_0 \sqrt{T}} \right) \right) - N \left( -y^* + x\sigma_0 \sqrt{T} - \left( \frac{2b}{\sigma_0 \sqrt{T}} \right) \right) \right] \\
+ Ke^{-rT} \left[ N(-y^*) + \left( \frac{V_0}{V} \right)^{\frac{2b}{\sigma^2}} N \left( y^* + \frac{2b}{\sigma_0 \sqrt{T}} \right) \right] ,
\]

where \( y^* \) solves the implicit equation (12) for \( K = 0 \). Equivalently, \( y^* \) is the value of \( \epsilon \) for which \( E(T) = 0 \), that is,

\[
y^* = \frac{\ln \left( \frac{V_0}{V_b} \right) + (r - \delta - \frac{1}{2} \sigma^2)T}{\sigma_0 \sqrt{T}} . \tag{19}
\]

The put option formula is slightly more complicated than the formula for calls; for each term in the call option formula, one must subtract an extra term to adjust for the probability that \( \epsilon \) is so large that bankruptcy occurs. In addition, one must add the present value of a security that pays \( K \) at expiration if bankruptcy ever occurs.

The put option pricing formula appears in the form

\[
PUT = PUT(V, V_b, C, K, r, \delta, \sigma, T, \tau) ,
\]

but under the assumption of optimal bankruptcy by equityholders, \( V_b \) equals \( \frac{(1-\tau)C^e}{\tau(z+1)} \), and the formula can be rewritten as

\[
PUT = PUT(V, C, K, r, \delta, \sigma, T, \tau) .
\]

The put option formula with exogenous bankruptcy is homogeneous of degree one in \( V, V_b, C, \) and

\[\text{16} \text{ I assume the striking price is paid at the option's expiration date if the firm defaults.} \]
\( K \), whereas the formula with endogenous bankruptcy is homogeneous of degree one in \( V, C, \) and \( K \).

Like the formula for calls, the pricing formula for puts can be simplified in two special cases. First, if bankruptcy is triggered when \( V_0 = \frac{(1-r)t}{r} C \), the pricing formula simplifies to

\[
\text{PUT} = (A + K)e^{-rT} \left[ (N(-y) - N(-y^*)) - \left( \frac{V_0}{V} \right)^{\frac{y^*}{\sigma \sqrt{T}}} N \left( y^* - \left( \frac{2k}{\sigma \sqrt{T}} \right) \right) \right]
\]

\[
- V e^{-\delta T} \left[ N(-y - \sigma \sqrt{T}) - N(-y^* - \sigma \sqrt{T}) - \left( \frac{V_0}{V} \right)^{\frac{y^*}{\sigma \sqrt{T}}} N \left( y^* - \sigma \sqrt{T} - \left( \frac{2k}{\sigma \sqrt{T}} \right) \right) \right] \times \]

\[
+ K e^{-rT} \left( N(-y^*) + \left( \frac{V_0}{V} \right)^{\frac{y^*}{\sigma \sqrt{T}}} N \left( y^* + \frac{2k}{\sigma \sqrt{T}} \right) \right)
\]

with

\[
y = \frac{\ln \left( \frac{V}{A+K} \right) + (r - \delta - 0.5\sigma^2)t}{\sigma \sqrt{T}}
\]

This is the value of a down-and-out put option on the firm’s assets with a striking price equal to \( A + K \), a knock-out barrier at \( V_0 = A \), and finally, a rebate of \( K \) payable at time \( T \).

Second, if the company is debt free and bankruptcy is not possible, the formula simplifies to the Black-Scholes put pricing formula,

\[
\text{PUT} = K e^{-rT} N(-z_2) - V e^{-\delta T} N(-z_1), \tag{20}
\]

with \( A = 0 \) in the expression for \( z_1 \) and \( z_2 \).

### 3.2 Comparison to Black-Scholes Values

Recently, practitioners and academics such as Dupire (1993a,1993b), Derman and Kani (1994a,-1994b), and Rubinstein (1994) have expressed much interest in the term structure of implied volatilities and the volatility skew. The philosophy behind the cited studies is to use the observed implied volatility surface, which represents prices of traded options, to derive a stochastic process for the underlying asset price. Derman (1993) indicates that the identified stochastic process can
eventually be used to price and hedge more exotic option structures.

The model developed in this paper, on the other hand, relates observed Black-Scholes implied volatility structures to structural characteristics such as leverage and the debt's protective covenants. Leverage effects have previously been introduced in the option pricing literature by Geske (1979), Cox and Ross (1976), and Rubinstein (1983). However, this paper also conjecture a relationship between the volatility skew and protective covenants in the capital structure.\(^{17}\) This is done by analyzing the pricing biases Black-Scholes formula exhibits, while assuming that the true model for option prices is the one presented in this paper. Since the Black-Scholes model and the model presented in this paper are functions of different, but closely related sets of parameters, we must construct sets of consistent input variables for both.

First, we choose six different sets of fundamental input parameters which represent a variety of capital structures. The parameters illustrate 20%, 40%, and 60% leveraged firms with either unprotected (bankruptcy declared optimally by equityholders) or protected debt (bankruptcy triggered by a strict net-worth covenant) in the capital structure. Firms with unprotected debt declare bankruptcy at \(V_b = (1 - \tau)C_x/(r(x + 1))\),\(^{18}\) whereas if the capital structure contains protected debt, bankruptcy is triggered (by the definition used in this paper) when the firm value reaches \(V_b = C/r\), the present value of the promised payments. Parameter values for the six capital structures are shown in table 1.

Second, prices for options on leveraged equity are computed for different times to maturity and striking prices using the parameters in table 1. Assume for the following analysis that these prices are the true observed option prices.

Third, the parameters for the securities in the capital structure and the option pricing model presented in this paper are converted into input variables used in Black-Scholes equation: Times to maturity, the riskfree interest rate, and striking prices are identical for the two option pricing models, and are therefore used directly in the Black-Scholes equation. The equity value is determined by equation (2). The payout rate \(\delta\) is chosen such that the value of the proportional payout over the next six month equals the after-tax value of the coupon payments. This implies that the net payout to equityholders over the next six month has a value of zero. The payout rate is found by searching

\(^{17}\)The author would like to thank Hayne Leland for suggesting investigation of this issue.
\(^{18}\)For a derivation of this optimal bankruptcy condition, see Leland (1994).
for the $\delta$ which equates equity value and the value of a call option with a striking price of zero and six months to maturity. Note that this payout rate is close to, but not necessarily identical to $C(1 - \tau)/V$.\textsuperscript{19} Since the value of the net-payout on equity is zero, the appropriate dividend yield to use in Black-Scholes equation for comparative purposes is zero.

Finally, we avoid calculating the last input parameter, the equity volatility, by representing the pricing biases by a matrix of implied Black-Scholes volatilities, where the implied volatilities are calculated for different striking prices and times to maturity. To facilitate cross-sectional comparisons, the matrix of implied volatilities are normalized by the implied volatility for the six month at-the-money option.

Table 2 shows the resulting six matrices of relative implied volatilities for call options calculated from Black-Scholes equation, when the option value is determined by the model for options on leveraged equity. The implied volatilities for each of the six capital structures listed in table 1 are calculated for options with striking prices ranging from 80% to 120% of the equity value, and for maturities of three months, six months, nine months, and one year.

Table 2 illustrates several interesting features of the option pricing model developed in this paper. First, the volatility skew is more pronounced for highly leveraged firms than for moderately leveraged firms. This is explained directly by leverage effects; as the value of the firm’s assets decreases, equity volatility increases. This makes in-the-money call and out-of-the-money put options relatively more valuable. On the other hand, as asset values increase, equity volatility decreases. This makes out-of-the-money call and in-the-money put options relatively less valuable.

\textsuperscript{19}When bankruptcy is eminent, the computed payout rate may be quite different from $C(1 - \tau)/V$. This is caused by the high likelihood of bankruptcy.
<table>
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<tr>
<th></th>
<th>Unprotected Debt</th>
<th>Protected Debt</th>
</tr>
</thead>
<tbody>
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<td>Leverage</td>
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<td>20%  40%  60%</td>
</tr>
<tr>
<td>C</td>
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<td>1.056 2.233 3.387</td>
</tr>
<tr>
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<td>0.0078 0.0170 0.0292</td>
<td>0.0078 0.0166 0.0246</td>
</tr>
<tr>
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<td>10.94 22.65 36.13</td>
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</tr>
<tr>
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<tr>
<td>D</td>
<td>21.05 44.22 68.59</td>
<td>21.00 42.80 60.81</td>
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Table 1: Parameter Values for 20%, 40%, and 60% Leveraged Capital Structures with Protected and Unprotected Debt. All other parameters are constant: \( V = 100, r = 0.05, \sigma = 0.2, \alpha = 0.2, \) and \( \tau = 0.25 \). For unprotected debt, bankruptcy is declared optimally by equityholders when \( V = V_b = (1 - \tau)Cx/(r(x + 1)) \). For protected debt, bankruptcy is triggered when the firm value reaches \( V_b = C/r \), the present value of the promised payments. The liquidation rate, \( \delta \), is determined such that the value of equity six month forward equals the current equity value.
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<th>protected Debt</th>
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<td>1.001 1.000 0.999 0.998</td>
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<td>0.993 0.992 0.991 0.990</td>
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<tr>
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Table 2: Relative Implied Black-Scholes Volatilities as Functions of Striking Price over Equity Price and Time to Maturity for 20%, 40%, and 60% Leveraged Firms with Protected and Unprotected Debt (0% Dividend Yield). The relative implied Black-Scholes volatilities are defined as the implied Black-Scholes volatility divided by the implied volatility for the six month at-the-money option. Input parameter values are taken from Table 1.  

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Second, equity options on firms with protected debt tend to have more pronounced volatility skews than options on firms with unprotected debt. This can be explained by returning to figure 1. The smooth pasting condition of the equity value of the firm with unprotected debt results in a decreasing absolute equity volatility as $E \rightarrow 0$. This decrease in the absolute volatility partially offsets the increase in the relative volatility caused by leverage effects. Consequently, the volatility skew becomes less pronounced. However, with protected debt in the capital structure, a move in the value of the underlying assets translates into an even larger move in the equity price when $V$ is close to $V_b$. The relative volatility is therefore amplified both by leverage effects, and by an increased absolute volatility, thus magnifying the volatility skew.

Third, implied volatilities are decreasing in the option’s time to maturity for all reported capital structures except for the 60% leveraged firm with protected debt. In this case, a leverage ratio of 60% with protected debt is far beyond optimal leverage. The value of equity and debt for this capital structure is 101.35, which is less than the value of the firm with a leverage ratio of 40% (107.00).

Fourth, as leverage decreases, protective covenants have less of an effect on the anatomy of the volatility skew. In the case of an unleveraged firm, the implied volatility structure is a flat surface; the model for options on leveraged equity and the Black-Scholes model are equivalent.
Table 3: Parameter Values for 20%, 40%, and 60% Leveraged Capital Structures with 3% Dividend Yield. All other parameters are constant: $V = 100$, $r = 0.05$, $\sigma = 0.2$, $\alpha = 0.2$, and $\tau = 0.25$. For unprotected debt, bankruptcy is declared optimally by equityholders at $V = V_b = (1 - \tau)C x / (r(x + 1))$. For protected debt, bankruptcy is triggered when the firm value reaches $V_b = C/r$, the present value of the promised payments. The asset liquidation rate, $\delta$, is determined such that the value of equity six months forward equals $E \ast e^{-0.03T}$, $T = 0.5$. 

<table>
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<th>Unprotected Debt</th>
<th>Protected Debt</th>
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<td>20%  40%  60%</td>
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Table 4: Relative Implied Black-Scholes Volatilities as Functions of Striking Price over Equity Price and Time to Maturity for 20%, 40%, and 60% Leveraged Firms with Protected and Unprotected Debt (3% Dividend Yield on Equity). The relative implied Black-Scholes volatilities are defined as the implied Black-Scholes volatility divided by the implied volatility for the six month at-the-money option. Input parameter values are taken from table 3.
In Tables 3 and 4 we consider the effects of dividends on the volatility structures. Table 3 shows parameter values for the same six capital structures with an asset liquidation rate $\delta$, defined such that the value of the dividend payments over the next six months equals $E \times (1 - e^{-0.03T})$, $T = 0.5$. Comparing tables 2 and 4, we note that the volatility structures are slightly less downward sloping in the striking price for options on equity with a 3% dividend yield than for those with a zero dividend yield. However, the difference is small, and we conclude that reasonable dividend payments do not seem to affect the volatility structures significantly.

Figure 4 shows the implied volatility structure for the 40% leveraged firm with protected debt and a 3% dividend yield. The details of the capital structure are given in table 3. This figure is intended to represent an "average" capital structure in the U.S. economy. We notice that equity options on this moderately leveraged firm have a significantly skewed Black-Scholes implied volatility structure. The implied volatilities range from 31% for out-of-the-money options to 35% for at-the-money options with an average volatility of approximately 33%.

Although the model is developed for options on individual equities, we may also conjecture that leverage effects exist in the aggregate market. Rubinstein (1994) shows that implied volatility structures for SP500 index options have been significantly skewed since the crash of 87. One explanation is that the option market suffers from crash-o-phobia. An aggregate leverage effect provides an alternative explanation, which may have become more important as the US equity market plunged and more debt was issued in the late nineteen eighties.
Figure 4: Implied Volatility Structure for a 40% Leveraged Firm with Protected Debt. The following input parameters are used: $V = 100, C = 5, r = 0.05, \delta = 0.0356, \sigma = 0.2, \tau = 0.25$ (effective corporate tax rate). $V = 100 \Rightarrow E = 62.94$. The proportional payout rate is defined such that the value of the net payout to equityholders over the next six months is equivalent to a dividend yield of 3%.
3.3 Interest-Only Strips

The previous sections of this paper discussed the valuation of debt, equity, and options written on the equity. However, it is also appropriate to evaluate corporate interest-only strips within the framework presented.

First assume that the proportional payout, $\delta$, is zero. If we consider the evaluation of a call option with a striking price of zero, owning the call option is equivalent to holding equity without the obligation to service debt between now and the maturity of the option. This enables us to allocate the coupon stream into security classes that receive coupon payments in different time intervals, and find the value of each security class using the option pricing formula derived in this paper. Consider the allocation of the coupon stream into two security classes. The first interest-only security, denoted $IO_{0,T}$, gives the owner the right to receive coupon payments from now until time $T$. The owner of the second security, $IO_{T,\infty}$, receives the coupon payments from time $T$ to infinity. The two security classes receive payments only if the firm is solvent. The only difference between a call with zero strike and equity is the after-tax value of the coupon payments on debt. Therefore, the value of the first security class, $IO_{0,T}$, equals the difference between the value of a call with maturity $T$ and equity, divided by $(1 - \tau)$

$$IO_{0,T} = \frac{\text{CALL}(V, V_b, C, K = 0, r, \delta = 0, \sigma, T, \tau) - E(V, V_b, C, r, \delta = 0, \sigma, \tau)}{(1 - \tau)}.$$  

(21)

The value of the second security class, $IO_{T,\infty}$, is simply the value of a risky perpetual coupon stream, less the value of the first security class

$$IO_{T,\infty} = \frac{C}{r} (1 - p_b) - IO_{0,T}.$$  

(22)

Finally, to account for the total value of the risky debt, recall that the debtholders get $(1 - \alpha)V_b$ in the case of bankruptcy. This is simply a cash-at-hit option, and the value of this option, which for debt with infinite maturity is the value of the principal only security, PO, is

$$PO = (1 - \alpha)V_b \left( \frac{V_b}{V} \right)^x.$$  

(23)

The total value of the three security classes equals the total value of the risky debt.
Given a proportional payout rate greater than zero, the value of the interest-only strip can be derived by noting that the value of $IO_{0,T}$ equals the value of a long riskfree annuity with a continuous payout of $C$ per unit of time, minus the value of a claim that pays

$$\frac{C}{r} \left(1 - e^{-r(T-t)}\right)$$

if bankruptcy occurs; this equation represents the value of the annuity's remaining cash-flow at time $t$ when bankruptcy is triggered. The value of this portfolio is

$$IO_{0,T} = \frac{C}{r} (1 - e^{-rT}) - \int_0^T \frac{C}{r} (1 - e^{-r(T-t)}) e^{-rt} f(t) dt,$$  \hspace{1cm} (24)

where $f(t)$ is the first passage time density for a Brownian motion with starting point $\ln(V/V_b)$, absorbing barrier at zero, drift $\mu = r - \delta - \frac{1}{2} \sigma^2$, and volatility $\sigma$. Performing the integration over $t$ yields the value of $IO_{0,T}$,

$$IO_{0,T} = \frac{C}{r} (1 - e^{-rT}) - \frac{C}{r} \left(\frac{V_b}{V}\right)^{\frac{\mu-m}{\sigma^2}} \left[N\left(\frac{b-mT}{\sigma \sqrt{T}}\right) + \left(\frac{V_b}{V}\right)^{\frac{\mu}{\sigma^2}} N\left(\frac{b+mT}{\sigma \sqrt{T}}\right)\right] +$$

$$\frac{C}{r} e^{-rT} \left[N\left(\frac{b-\mu T}{\sigma \sqrt{T}}\right) + \left(\frac{V_b}{V}\right)^{\frac{\mu}{\sigma^2}} N\left(\frac{b+\mu T}{\sigma \sqrt{T}}\right)\right],$$  \hspace{1cm} (25)

where we recall the definitions of $b$ and $m$ in (13) and (3). With the value of the $IO_{0,T}$ security for the case with a proportional payout in hand, we can value $IO_{T,\infty}$ using (22).

For illustrative purposes, I compare the value of a risky coupon stream $IO_{0,T}$ relative to a riskfree coupon stream with the same maturity in figure 5. Figure 5 plots the discount at which corporate IOs trade relative to riskfree IOs as a function of time to maturity and the underlying asset value. The discount is defined as

$$\text{DISCOUNT} = \frac{IO_{0,T}}{\frac{C}{r} (1 - e^{-rT})},$$

where the numerator is the value of a risky coupon stream, and the denominator represents the value of riskfree coupon payments from time 0 to $T$. In figure 5, we use the following input parameters: $C = 5$, $r = 0.05$, $\sigma = 0.2$, $\tau = 0.25$, $\delta = (1 - \tau)C/100 = 0.375$ ($\delta$ is defined so that the value
Figure 5: Value of Risky Relative to Riskfree Coupon Stream as a Function of Time to Maturity and the Firm’s Asset Value. The following input parameters are used: $V_b = (1 - \tau)C x / (r(x + 1)) = 43.8$ (optimal bankruptcy), $C = 5$, $r = 0.05$, $\sigma = 0.2$, $\tau = 0.25$ (effective corporate tax rate), and $\delta = (1 - \tau)C / r = 0.375$. 
of the liquidated assets, $\delta V$, equals the after-tax value of the promised debt payments, $(1 - \tau)C$, when $V = 100$, and $V_0 = 43.8$ (optimal bankruptcy). When the firm is approaching bankruptcy, the value of the corporate IO relative to the riskfree IO is very close to zero. This is obviously caused by the high likelihood of bankruptcy. For an asset value of 100, which corresponds to a leverage ratio of approximately 70%, short-term IOs trade at almost 100% of riskfree IOs, but for long-term IOs, discounts can be substantial. For corporate IO's with 30 years to maturity, the price is 77.6% of the price of a riskfree annuity.

## 4 Conclusion

We have derived equity option pricing formulas which are consistent with the firm’s overall capital structure in an economy where corporate taxes and bankruptcy costs exist. Combined with Leland’s (1994) work this yields a comprehensive set of mutually consistent security price processes expressed in closed form. Equity option prices and sensitivities to changes in input variables in this model depend directly on structural characteristics such as corporate tax rates, coupon payments, and the mechanism behind bankruptcy. For example, equity options are relatively more valuable if bankruptcy is triggered by a protective covenant than if bankruptcy is declared optimally by equityholders. Furthermore, if the firm is highly leveraged, option prices’ sensitivities to changes in equity prices depend on the debt’s protective covenants. This implies that if a conventional model such as the Black-Scholes model is used to derive the option’s delta, then the hedge portfolio is significantly misspecifed. Even when the underlying firm is moderately leveraged, the proposed model results in an implied Black-Scholes volatility structure which is decreasing in the striking price. Consequently, we can test the leverage effect hypothesis:

*The volatility skew is more pronounced for options on equity in a leveraged firm than for options on equity in an unleveraged firm, ceteris paribus.*

The developed model also predicts that this volatility skew is more pronounced for options on equity in a firm where the capital structure contains debt with strict net-worth covenants than for options on equity in a firm with debt without these protective covenants. We can test this prediction by investigating the validity of the covenant effect hypothesis:
The volatility skew is more pronounced for options on equity in a firm where debt is protected by strict net-worth covenants than for options on equity in a firm with debt without these covenants, ceteris paribus.

Also, with suitable approximations the model's parameters may be estimated, and its validity tested, by determining all the unobservable parameters from a joint system of pricing equations and a time series of asset prices. If these estimates are obtained for a significant number of firms, then the result may be used to shed light on cross-sectional differences in bankruptcy triggers, bankruptcy costs, and effective corporate tax rates.

Finally, although the model in this paper is developed for options on individual equities, it is reasonable to conjecture that the leverage effect also exists for options on market indices. This is supported by Rubinstein (1994) who finds that implied volatility structures for SP500 index options have been significantly skewed in the period since the crash of 1987.
References


