Bond Prices, Yield Spreads, and Optimal Capital Structure with Default Risk

by

Hayne Leland

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WITH DEFAULT RISK

Hayne Leland
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ABSTRACT

This paper examines the value of debt subject to default risk in a continuous time framework. By considering debt with regular principal repayments (e.g. through a sinking fund), we are able to examine bonds with arbitrary maturity while retaining a time-homogeneous environment. This extends Leland's [1994] earlier closed-form results to a much richer class of possible debt structures.

We examine the term structure of yield spreads and find that a rise in interest rates will reduce yield spreads of current debt issues. It may tilt the term structure as well. Duration is also affected by default risk. The traditional Macaulay duration measure overstates effective duration, which for "junk" bonds may even be negative. While short term debt does not exploit tax benefits as completely as does long term debt, it is more likely to provide incentive compatibility between debt holders and equity holders. The agency costs of "asset substitution" are minimized when the firms use shorter term debt.

Optimal capital structure depends upon debt maturity. Optimal leverage ratios are smaller, and maximal firm values are less, when short term debt is used. The yield spread at the optimal leverage ratio increases with debt maturity.
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I. INTRODUCTION

A formula for the price of a coupon-paying bond with default risk and arbitrary maturity has proved elusive. The search for such an expression has been frustrated by the complexity of the debt instrument, whose cash flows include both coupon payments and a return of principal. When coupon payments are nonzero, Merton [1974] showed that bond values must satisfy a partial differential equation which has no known closed form solution for the general case.

A few special cases have admitted closed form solutions. Merton [1974] derived values for risky zero-coupon bonds. But coupon-paying bonds are far more common than zero-coupon bonds. Black and Cox [1979] examined coupon-paying debt, but only debt with infinite maturity. Neither Merton nor Black and Cox considered bankruptcy costs or the tax deductibility of interest payments. These are necessary ingredients for studying optimal capital structure as well as bond prices. Brennan and Schwartz [1978] used numerical methods to study debt value and optimal capital structure in a quite general setting. While instructive in providing examples, numerical techniques typically preclude the derivation of general results.
Leland [1994] derived closed form solutions for risky debt value and optimal capital structure in two environments consistent with time-homogeneous debt cash flows.\textsuperscript{1} The first environment extends the infinite-maturity debt case examined by Black and Cox [1979] to include bankruptcy costs and tax deductibility of interest payments. The asset level which triggers bankruptcy is endogenously determined, and optimal capital structure can readily be calculated.\textsuperscript{2}

Leland's second case also presumes a constant and perpetual coupon (unless default occurs). Unlike the first case, bankruptcy is triggered when the firm's net worth becomes negative. While this environment could be interpreted as infinite-life debt with a positive net worth covenant, Leland [1994] also likened it to rolling over very short term debt, or more exactly to a revolving line of credit, which is continuously renewed (at the same coupon rate) as long as assets are sufficient to repay debt principal.

While offering potential insights at both extremes of the maturity spectrum, Leland's analysis does not directly lend itself to analyzing debt with arbitrary maturities. Furthermore, the association of a positive net worth requirement with very short term debt begs the question of whether bankruptcy would be triggered \textit{endogenously} if and only if net worth becomes

\textsuperscript{1} This leads to ordinary differential equations for coupon-paying bond prices, which have closed form solutions.

\textsuperscript{2} A recent manuscript by Ross [1994] also examines capital structure in the context of a time-homogeneous model. The principal differences with Leland [1994] are that Ross assumes that cash flow ("EBIT") follows a random walk whose drift may be arbitrary; and that bankruptcy is triggered whenever cash flow is less than the required bond coupon. Ross also focuses on the "cost of capital" and on aspects of optimal recapitalization.
negative.

This paper derives a formula for the prices of a broad class of coupon-paying bonds with default risk and arbitrary maturity. We again focus on debt with time-homogeneous cash flows. But how is this possible, when finite-maturity debt cash flows seem time inhomogeneous by their very nature? The answer lies in examining debt with a time-homogeneous repayment of principal as well as a constant coupon. The real-world equivalent is a sinking fund provision. Sinking funds are quite common in corporate debt issues. They require that a fraction of the principal value of debt be retired (or "amortized") on a regular basis.

In our model, a constant fraction of currently outstanding debt is retired annually, but replaced (except when bankruptcy occurs) by newly-issued debt. The cash flow requirements for debt service in each period are therefore constant: a fixed coupon amount, and a fixed sinking fund requirement to retire current debt principal. The rate at which debt principal is retired serves as an inverse proxy for the average maturity of debt: The higher the debt retirement rate (or "rollover rate") $m$, the shorter is the average maturity $M = 1/m$ of debt. If $m = 0$, principal is never retired, and debt has infinite maturity as in

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3 See Smith and Warner [1979], for example.

4 While some sinking funds allow the retirement of debt at its current market value, we assume (as is often the case) that debt is retired at its principal value.
Leland [1994]. As \( m \to \infty \), the average maturity of debt approaches zero.\(^5\)

Our results relate bond values and yield spreads to debt maturity, firm risk, leverage, tax rates, bankruptcy costs, and riskless interest rates.\(^6\) The term structure of yield spreads may be increasing, humped, or (effectively) decreasing, depending on the degree of leverage of firms. These patterns confirm those derived for zero-coupon debt by Merton [1974] and Pitts and Selby [1983], and observed empirically by Sarig and Warga [1989].

The duration of debt with default risk exhibits some surprising properties. As risk increases, the sensitivity of risky debt to uniform shifts in interest rates (which we call *effective duration*) becomes significantly less than the traditional Macaulay [1938] measure of duration. For very risky debt, effective duration may even be negative. These findings are significant for fixed income portfolio managers attempting to immunize obligations by hedging against shifts in interest rate levels.

Duration also changes with the level of interest rates. For riskfree debt, a rise in interest rates shortens duration. This is reflected by "convexity": debt value is a convex function of

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5 For analytical convenience, we assume that a constant fraction of the remaining principal balance of each debt "vintage" is retired per unit time. This means (formally) that debt of each vintage has infinite life. However, by choosing \( m \) sufficiently large, the fraction of the principal outstanding at any future date (and the debt's duration) can be made arbitrarily small.

6 Longstaff and Schwartz [1992] examine finite-life debt whose cash flows may be time inhomogeneous. But they assume the bankruptcy-triggering asset level (our \( V_a \)) is exogenously given and has constant present value. The endogeneity of bankruptcy is key to our results, and leads to many of the surprising comparative static results derived in Section III.
the interest rate. But risky debt behaves differently: it always exhibits less convexity than riskless debt, and its value may even be a concave function of the riskfree rate. Again, this has significant implications for hedging risky bond portfolios.

Optimal capital structure is shown to depend critically upon the maturity of debt. Optimal leverage will be considerably lower when shorter term debt is used. The term structure of yield spreads at optimal leverage levels is an upward sloping function of debt maturity. Firms should issue only high quality short term debt; longer term debt typically should not be as highly rated. Maximal firm value also increases with debt maturity. This last result poses an important question: why do firms ever choose to issue short-term debt?

An answer may be found in the greater potential agency costs of long-term debt. Agency theory suggests that stockholders will wish to increase risk--by "asset substitution"--in order to transfer value from debt to equity (see, for example, Jensen and Meckling [1976] and Harris and Raviv [1990]). This conclusion relies upon the analogy between equity of a levered firm and a call option suggested by Black and Scholes [1973]. However, the analogy is inexact except in Merton's [1974] case of zero-coupon bonds. With coupon-paying bonds, bankruptcy may be triggered by asset value falling to a critical level at any time, rather than only at maturity. Furthermore, this value is endogenous, and will change with the riskiness of the asset value process.
We show that the agency problems vanish with short and intermediate term debt, when bankruptcy is not imminent. Incentives are misaligned for risky long term debt. And they become misaligned for all debt maturities as bankruptcy is neared, or when bankruptcy costs and taxes are negligible.

Our generalizations confirm many but not all of Leland's [1994] conclusions. As \( m \) increases (and debt duration becomes shorter), the endogenously determined bankruptcy-triggering asset value \( V_B \) increases, but approaches a higher value than Leland [1994] predicted. And the fact that \( V_B \) is endogenously determined for short term debt also alters its behavior.

The analytical techniques of this paper also differ from earlier work. Merton [1974], Black and Cox [1979], and Leland [1994] solve (partial) differential equations to determine bond values. A martingale approach is used here to derive risk-neutral expected values.\(^7\)

II. RISKY DEBT WITH ARBITRARY AVERAGE MATURITY

As in Merton [1974], Black and Cox [1976], and Brennan and Schwartz [1978], the firm has productive assets whose market value \( V \) follows a continuous diffusion process with constant proportional volatility:

\(^7\) Such techniques, of course, have been widely used to study related problems since the work of Cox and Ross [1976] and Harrison and Kreps [1979]. Longstaff and Schwartz [1992] also use a martingale approach.
\[ \frac{dV}{V} = \mu(V,t)dt + \sigma dz \]

where \(dz\) is a standard Brownian motion. The process continues without time limit unless \(V\) falls to a bankruptcy-triggering value \(V_B\), which is endogenously determined and depends upon the amount of debt issued.

We examine stationary debt policies. At each moment in time, the firm has debt with constant total principal \(P\), paying a constant total coupon rate \(C\). The firm continuously rolls over a fraction \(m\) of debt. That is, it continuously retires outstanding debt principal at the rate \(mP\), and replaces it with new debt of equal coupon, principal, and seniority.\(^8\)

At each instant \(\tau\), the firm issues new debt principal equal to a constant \(p\), paying a coupon rate \(c\).\(^9\) Since \(mP\) is the amount of debt principal retired at each instant, it follows that

\[ p = mP \]

Let \(p(\tau,t)\) denote the principal outstanding at time \(t\) of debt issued at time \(\tau \leq t\). Note \(p(\tau,\tau) = p\) for all \(\tau\). As \(t\) passes, the principal of the debt issued at any time \(\tau \leq t\) is retired at a fractional rate \(m\):

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\(^8\) Although the new debt is assumed to have the same coupon and principal (to preserve stationarity), the price at which the new debt can be sold depends upon current asset value \(V\). Additional equity will have to be raised if the new debt selling price is lower than its principal value.

\(^9\) Formally, \(p\) is a rate of principal issue; the actual amount of debt issued over the instant \(dt\) is \(p dt\). However, in equations (2), (4), (5), (9), and (10) we shall drop the \(dt\) terms and speak of the "rate" as an "amount." The strict reader may wish to supply multiplicands of \(dt\) to both sides of these equations.
\[
\frac{\partial p(\tau,t)/\partial t}{p(\tau,t)} = -m
\]

implying both the outstanding principal and coupon of debt of each vintage \( \tau \) declines exponentially with time:

\[
p(\tau,t) = e^{-m(t-\tau)}p
\]

\[
c(\tau,t) = e^{-m(t-\tau)}c
\]

Confirming equation (2), when \( m > 0 \), at any time \( t \) the total principal outstanding \( P \) and coupon rate \( C \) are:

\[
P = \int_{-\infty}^{t} p(\tau,t)d\tau = \int_{-\infty}^{t} e^{-m(t-\tau)}pd\tau = \frac{P}{m}
\]

\[
C = \int_{-\infty}^{t} c(\tau,t)d\tau = \int_{-\infty}^{t} e^{-m(t-\tau)}cd\tau = \frac{C}{m}
\]

Note that remaining units of debt from all prior issues have the same value per unit, since units of all vintages pay the same coupon, and the remaining units of all vintages will be retired at the same fractional rate \( m \). Thus a unit of bonds issued five years ago will look exactly like (and will carry the same price) as a unit of bonds issued today, except that there
will be fewer units of the older vintage bonds. All units have the same seniority.  

The inverse of the rollover rate \( m \) also serves as a parameter of the average maturity \( M \) of (riskfree) debt, and its duration \( Z \). Let the current time be \( t = 0 \). The fraction of currently outstanding debt principal which is redeemed at time \( t \) in the future is \( me^{-mt} \). If bankruptcy never occurs, the average maturity \( M \) of debt is

\[
M = \frac{\int_{0}^{\infty} (me^{-mt}) dt}{\int_{0}^{\infty} e^{-mt} dt} = \frac{1}{m}
\]

Let \( Z \) represent the Macaulay measure of duration. That is,

\[
Z = \frac{\int_{0}^{\infty} te^{-Rt}[e^{-mt}(c + mp)] dt}{\int_{0}^{\infty} e^{-Rt}[e^{-mt}(c + mp)] dt} = \frac{1}{m + R}
\]

where \( R \) is the yield to maturity of the debt (equal to \( r \) when debt is riskfree), and \( e^{-mt}(c + mp) \) is the cash flow (coupon plus amortized principal) accruing to currently-issued debt, at time \( t \) in the future. If \( m = 0 \), average maturity \( M \) is infinite and \( Z = 1/R \), the duration of a consol with yield \( R \). As \( m \to \infty \), both average maturity \( M \) and duration \( Z \) approach zero.

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10 Because all outstanding debt units are homogeneous, we can treat the initial (at \( t = 0 \)) total principal of debt \( P \) (and coupon \( C \)) as control variables, rather than simply the current flow \( p \) (and \( c \)). By assumption, however, once \( P \) and \( C \) are fixed they are expected to remain constant thereafter. The reader may note a similarity with the example of light bulbs whose longevity is exponentially distributed. At any moment looking forward, all light bulbs currently operating will have the same distribution of remaining life, regardless of how long they have already operated.
We now derive the value of debt as a function of $m$ and $V_B$, where $V_B$, the asset value which will trigger bankruptcy if reached, is less than the current asset value $V$. Let $d(0)$ represent the value of the debt which is currently issued. Cash flows to these debtholders will include future coupon payments and fractional repayments of principal, in amounts $e^{-mt}(c + mp)$, unless bankruptcy occurs. Let $\alpha$ be the fraction of asset value lost in the event of bankruptcy. Then $(1-\alpha)V_B$ is the amount in total that bondholders receive if bankruptcy occurs. We presume absolute priority, in that bondholders receive all remaining asset value, and stockholders receive nothing, when the firm becomes bankrupt.

Using risk neutral valuation, and denoting the density of the first passage time $t$ to $V_B$ from $V$ as $f(t;V,V_B)$, gives a value to currently issued debt

$$d(0)=\int_0^\infty e^{-rt}e^{-mt}(c+mp)[1-F(t;V,V_B)]dt + \int_0^\infty e^{-rt}(e^{-mt}p/P)(1-\alpha)V_B f(t;V,V_B)dt \tag{9}$$

The first term in equation (9) represents the discounted expected value of the continuously declining coupon plus principal repayment (which will be paid with probability $(1-F)$, where $F$ is the cumulative distribution function of the first passage time); the second term represents the expected present value of the fraction of the bankruptcy value of the firm which will go to owners of bonds issued at time zero, if bankruptcy occurs at time $t$. Recalling that $p/P = m$, integrating by parts, and simplifying gives
\[ d(0) = \frac{c + mp}{r + m} \left[ 1 - \int_{t=0}^{\infty} e^{-(r+m)t} f(t; V, V_B) dt \right] + m(1 - \alpha) V_B \left[ \int_{t=0}^{\infty} e^{-(r+m)t} f(t; V, V_B) dt \right] \]

In Appendix A, it is shown that

\[ \int_{t=0}^{\infty} e^{-(r+m)t} f(t; V, V_B) dt = (\frac{V}{V_B})^{-y} \]

where

\[ y = \frac{(r - \delta - 0.5a^2) + [(r - \delta - 0.5a^2)^2 + 2(m + r)a^2]^{1/2}}{a^2} \]

and \( \delta \) is the (constant) proportional payout rate of asset value \( V \) by the firm.\(^{11}\) Coupons are paid at rate \( C \) to bondholders; in addition, \( p - d \) is the net cash outflow associated with redeeming a fraction \( m \) of the principal \( P \), less the market value \( d \) of floating new debt of equal coupon and principal but whose value fluctuates with \( V \). Thus \((\delta V - C - p + d)\) is the payout rate to stockholders. Note this payout rate declines as \( V \) falls and may become negative (i.e. new equity must be issued to meet bond requirements).\(^{12}\)

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\(^{11}\) Following Brennan and Schwartz [1978] and Leland [1994], we assume that the underlying asset with value \( V \) is a traded asset, or is perfectly correlated with a traded asset. This implies that the drift \( \mu(A, t) \) of the asset process in equation (1) equals the riskfree rate less the payout rate, \( r - \delta \).

However, it might be argued that an unlevered firm inefficiently exploits tax benefits, and will not be traded. (The author thanks Fischer Black for raising this point). In this case, rather than returning a "fair" risk adjusted return of \( r - \delta \), the untraded asset may offer a lower risk adjusted return, e.g. \( r - \delta - \lambda \). Our payout rate \( \delta \) can then be interpreted as the sum of an actual payout rate, plus an underperformance rate.

\(^{12}\) Bankruptcy will occur at an endogenously-determined asset value, \( V_B \), when equity value is no longer sufficient to cover the required bond coupons plus refundings. (See Section II(iii) below). One (but not the only) environment consistent with this description is that assets generate cash flows \( \delta V \) which are always paid out collectively to stock and bond holders. Note this does not mean that bankruptcy will occur when debt service payments exceed cash flow \( \delta V \). Rather, bankruptcy is triggered when asset value has fallen to where equity can no longer be issued to meet debt service requirements. Stockholders will always want to avoid bankruptcy if they can by issuing stock to meet the current debt service-something (their diluted stock value) is worth more than nothing (their stock value if bankruptcy occurs).
The value of debt outstanding of generation \( \tau, \tau \leq 0 \), is \( e^{mt}d(0) \), since all outstanding units of debt sell for the same price, but there are fewer units outstanding of older debt vintages due to accumulated debt retirement. All units sell for the same price because all carry the same coupon, and the retirement of remaining units follows the same proportionally declining schedule. Integrating over \(-\infty \leq \tau \leq 0\) gives \( D = d(0)/m \), the total value of debt outstanding. Dividing equation \( (10) \) by \( m \), and recalling that \( P = p/m \) and \( C = c/m \), gives

\[
D = \frac{C + mP}{r + m} \left[ 1 - \left( \frac{V}{V_B} \right)^{-\gamma} \right] + (1 - \alpha) V_B \left( \frac{V}{V_B} \right)^{-\gamma}
\]

or, equivalently (since average maturity \( M = 1/m \)),

\[
D = \frac{MC + P}{Mr + 1} \left[ 1 - \left( \frac{V}{V_B} \right)^{-\gamma} \right] + (1 - \alpha) V_B \left( \frac{V}{V_B} \right)^{-\gamma}
\]

Note that as \( m \to 0 \), \( y \to x \) (where \( x \) is given by equation \( (12) \) with \( m = 0 \)), and

\[
D \to C/r + \left[ (1 - \alpha)V_B - (C/r) \right] (V/V_B)^x, \text{ the same as in Leland [1994; equation (7)].}
\]

As \( m \to \infty \), we have \( y \to \infty \), and \( D \to P \).

Equation \( (13) \) is a closed-form solution for the value of debt with arbitrary rollover rate \( m \), and therefore for debt with arbitrary average maturity \( M = 1/m \). However, the bankruptcy-triggering value \( V_B \) remains to be determined. To find this value, we must invoke the smooth-pasting conditions for equity. But first we must determine the value of the firm and
the value of equity.

\[(i) \text{ The value of the firm } v\]

The total value of the firm, \(v\), equals its asset value \(V\), plus the value of tax benefits, less the value of bankruptcy costs: \(v = V + TB - BC\), where (as in Leland [1994])

\[
TB = (\tau C/\tau)[1-(\frac{V}{V_B})^{-\tau}]
\]

(14)

\[
BC = \alpha V_B(\frac{V}{V_B})^{-\tau}
\]

(15)

implying

\[
v = V+(\frac{\tau C}{\tau})[1-(\frac{V}{V_B})^{-\tau}]-\alpha V_B(\frac{V}{V_B})^{-\tau}
\]

(16)

where \(\tau\) is the corporate tax rate.\(^{13}\) This presumers that tax benefits are received whenever the firm is solvent, an assumption we modify later. It also assumes that tax benefits depend only upon the coupon, and not whether the debt is originally sold at a discount or premium.

\(^{13}\) As discussed in Miller [1977], in the presence of personal tax rates

\[
\tau = 1 - (1-\tau_e)/(1-\tau_d),
\]

where \(\tau\) is the effective tax advantage of debt, \(\tau_e\) is the corporate tax rate, \(\tau_p\) is the personal tax rate on equity income, and \(\tau_d\) is the tax rate on debt income.
to principal value.\textsuperscript{14} Observe that $x$ (given by equation (12) with $m = 0$), not $y$, is the exponent in these equations. This is because total coupon $C$ and principal $P$ remain constant: total tax benefits and potential bankruptcy costs are \textit{not} being reduced over time at rate $m$, in contrast with the outstanding amount of debt principal of each generation of debt. It can be shown that $(V/V_B)^x$ is simply the present value of $1$ received at the first passage time of asset value to $V_B$, when commencing at $V$.

(ii) \textbf{The value of equity $E$}

The value of equity equals the firm less debt: $E = v - D$, where $v$ is given by equation (16) and $D$ by equation (13):

\begin{equation}
E = V^*(-\frac{C}{r})(\frac{V}{V_B})^{-\gamma}-\alpha V_B(\frac{V}{V_B})^{-\gamma}-(\frac{C+mP}{r+m})([1-(\frac{V}{V_B})^{-\gamma}] - (1-\alpha)V_B(\frac{V}{V_B})^{-\gamma})
\end{equation}

The dynamics of $E$, which follow directly from the dynamics of $v$ and $D$, will be important in determining the value of options on the stock of leveraged firms. However, we shall not pursue these concerns in this paper.\textsuperscript{15}

\textsuperscript{14} Current U.S. tax rules require that any difference between initial selling price and principal value of bonds be considered interest. Much of our analysis examines the value of bonds which initially sell at par. However, subsequent debt issuances may occur at selling prices other than par, since by assumption $c$ and $p$ remain constant in each debt vintage, but $V$ (and therefore debt value) may change. Our analysis ignores the potential changes in the value of tax deductions resulting from debt being sold at value other than par.

\textsuperscript{15} Toft [1993] has provided a closed form solution for option prices when $m = 0$; his analysis can (in principle) be extended to price options on firms with debt of arbitrary duration using the dynamics of $E$.  

15
(iii) The bankruptcy-triggering asset value $V_B$

Now consider the asset value $V_B$ which, if reached, will trigger bankruptcy. Bankruptcy occurs when the asset value of the firm drops to a level such that the firm can no longer raise sufficient capital to retire the required amount of debt, plus pay the current total coupon.\textsuperscript{16} Since over an interval $dt$ the required amount of debt service is infinitesimal, it follows that the value of equity when asset value falls to $V_B$ is zero: $E(V_B) = 0$. But in addition, to maximize equity value $E$, the smooth-pasting condition must be satisfied by $E$ at $V = V_B$:

\begin{equation}
\frac{\partial E(V)}{\partial V} \bigg|_{V=V_B} = 1 + \frac{\tau C x}{r V_B} + \alpha x - \left(\frac{y}{V_B}\right)\left(\frac{C + m P}{r + m}\right) + (1 - \alpha)y = 0
\end{equation}

Solving for $V_B$ gives

\begin{equation}
V_B = \frac{\left[\frac{(C + m P) y}{r} - \frac{\tau C x}{r + m}\right]}{1 + \alpha x + (1 - \alpha)y}
\end{equation}

Using (19) to substitute for $V_B$ in (13), (16), and (17) gives \textit{closed form solutions for the value of debt, the value of the firm, and the value of equity.}

\textsuperscript{16} As in Brennan and Schwartz [1978] and Leland [1994], we assume that assets in place cannot be liquidated in order to raise money to pay debtholders. Equity must be raised to meet demands (dividends, interest, and net principal repayment) in excess of the cash flows paid out to investors, $\delta V$. See also footnote 12.
For all examples considered, $E$ is strictly increasing and convex in $V$ for $V > V_B$. In principle we can solve for $V$ as a function of $E$ from (17). Using this to replace $V$ in (13) would create a debt value function in terms of equity value $E$ rather than the (possibly difficult to observe) asset value $V$. We shall not pursue this approach, however.

As $m \to 0, y \to x$ and $V_B \to (1-\tau)(C/r)(x/(1+x))$, the same as in Leland's [1994] endogenous bankruptcy case. However, as $m \to \infty, y \to \infty$ and $V_B \to P/(1-\alpha)$. This is unlike Leland [1994], where it was argued that (very) short run debt could be associated with $V_B = P$, not $P/(1-\alpha)$. If $\alpha > 0$, bankruptcy will occur at a higher asset value than $P$ when debt is very short term.

But if the firm has assets $V = V_B$ which exceed the bondholders' principal $P$, why must bankruptcy be declared? Recall that bankruptcy is triggered not because $V$ falls beneath $P$, but rather because the firm cannot raise sufficient equity to pay the current coupon plus the net cost of retiring bond principal (the cost of retiring principal at par, less the revenue from selling--perhaps at less than par--the newly-issued bonds). In this event, bankruptcy occurs and productive assets must be liquidated (or reorganized). But assets are in place; liquidation or reorganization costs a fraction $\alpha$ of their value. As $m \to \infty$, debt becomes riskfree, since post-liquidation value, $(1-\alpha)V_B$, approaches the principal value $P$ of debt. Thus an important implication of the model is that, as long as $P < V(1-\alpha)$, debt can be made essentially riskfree by making its average maturity sufficiently short.17

17 We shall see in Section III that this limiting result is exactly that--a limiting result. In many cases we examine, debt with average maturity as short as 3 months still may carry a substantial yield premium.
If liquidation costs are zero, then for very short term debt the bankruptcy-triggering asset value $V_B$ will equal the principal value $P$ of the bonds, as in Leland [1994]. But if $\alpha > 0$, the bankruptcy triggering asset value (at which the firm cannot raise sufficient funds to pay debtholders their current coupon plus net return of bond principal) will exceed $P$.

Appendix B considers the optimal $V_B$ in the case where tax deductibility is lost when $V$ falls beneath $V_T \geq V_B$. It is shown there that

$$V_B = \frac{y(C+mP)rV_T}{(r+m)[rV_T[1+\alpha x+y(1-\alpha)]+\tau Cx]}$$

When $V_T > V_B$, $V_B$ in (20) will exceed $V_B$ in (19). Thus the loss of tax deductibility raises the asset value at which bankruptcy will be declared. This in turn will lower the value of debt and equity, as can be seen from equations (13) and (17).

III. APPLICATIONS

(i) Debt Value and Debt Capacity

From equation (19) or (20), $V_B$ can be substituted into equation (13) to yield a closed form solution for total debt value, given coupon $C$ and principal value $P$. When debt is first issued, however, there is typically a further constraint on relating market value, coupon, and
principal: the coupon is set so market value $D$ equals principal value $P$. If $V_0$ is the asset value when the debt is first issued, this constraint requires that $C$ be the smallest solution to the equation

$$D(V_0; C, P) = P$$

(21)

Using (13) and (19) or (20), it is simple to find solutions to (21) numerically.\(^8\)

Figure 1 plots the value of newly-issued debt $D$ as a function of leverage $(D/v)$ for different maturities $M = 1/m$, given that principal $P$ and debt value $D$ coincide at current value $V_0 = 100$. Our base-case example assumes $r = .075$, $\sigma = .20$, $\alpha = .50$, and $\tau = .35$.\(^9\) We further assume that $V_T = 50 + 2.5C$. This implies that, at the initial asset value, the coupon rate can be as high as 20% of asset value before the tax advantage of deducting coupons is lost. All ability to tax shelter coupon payments is lost if $V$ falls to 50 or less; implicitly, the firm is generating no profits to shield interest payments at that low asset value.\(^{10}\)

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\(^8\) Our analysis in Section II considers the case where at each moment, including the present, only a small amount of debt $(mPdI)$ is issued. However, since all outstanding units from previous vintages are identical in value, we can equally well assume the total debt is issued at the current moment, and thereafter rolled over at rate $m$. However, while $C$ may be chosen so that principal equals debt value when debt is originally issued, it will not generally equal debt value thereafter. This is because we require $C$ and $P$ to remain at the same level, once they are originally set. Subsequent issues of debt will sell above par (principal) value if $V > V_\phi$ and below par when $V < V_\phi$.

\(^9\) These parameters were chosen to reflect the current U.S. environment, with the possible exception of the bankruptcy parameter $\alpha$. We require substantial bankruptcy costs for short term debt to have "reasonable" yield spreads. Actual bankruptcy costs are difficult to ascertain, although many studies indicate they may be smaller than 50% (e.g. Altman [1989]). Perhaps yield spreads will ultimately prove to be the data we use to impute bankruptcy costs, rather than vice-versa.

\(^{10}\) Without a $V_T$ which increases in the coupon paid, debt capacity using shorter term debt may be unboundedly large: $D$ is monotonically increasing in $C$ when bonds are sold at par. But clearly such a result is absurd: at some level, coupons will exceed profits and tax deductibility will be limited.
Finally, we assume that the firm pays out an amount $\delta V$ which covers the initial coupon at $V = V_0 = 100$, plus a 3 percent dividend on the initial value of equity $E_0$. That is, the payout by the firm satisfies $\delta V_0 = C + .03E_0$, or $\delta = .01C + .0003E_0$.

In Figure 1, debt capacity ($D_{max}$) is the maximal value of the debt value curve. Note that the debt capacity is smaller for shorter maturities. It can be shown that debt capacity falls as volatility $\sigma$ and/or bankruptcy costs $\alpha$ rise. Maximal debt value tends to occur at higher coupon levels (denoted $C_{max}$) for shorter term debt, but at approximately the same leverage (about 75%-80%) for debt of different maturities.

For any given maturity, as $\sigma$ increases, debt value falls when $C/V < C_{max}/V_0$, but increases with $\sigma$ when $C$ is very large relative to $V$ and the bond is "junk." This surprising result, which we revisit in Section V (and Figure 8), results from the endogeneity of bankruptcy. It is more pronounced with longer term debt.

A rise in the riskfree rate can also increase debt value when the firm is near bankruptcy. Leland [1994] derived similar results for "junk" bonds, but only for the case of long term debt; short term debt (which in his model had an exogenously specified $V_B$) exhibited no

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21 Shareholders receive $\delta V - C$, less the net debt retirement costs $p - d$ (the principal retired less the market value of newly issued debt). In our example, shareholders receive a 3% dividend on their equity value $E$ when $V = V_0$. As $V$ falls, this net dividend received by shareholders falls, and eventually becomes negative (additional equity must be issued). At $V = V_B$, the contributions required are no longer met by shareholders with limited liability, and bankruptcy occurs.

22 If debt is optimally issued originally, the coupon rate will exceed $C_{max}$ only if the bond is a "fallen angel"--$V$ has fallen beneath $V_0$. Firms would never initially offer debt carrying so large a coupon.
such anomalies. In this model, anomalies can occur even with short term debt.

(ii) The Term Structure of Yield Spreads

Figure 2 examines yield spreads \((C/D - r)\) of newly-issued debt as a function of maturity, for alternative leverage ratios. (Figure 3 presents the same data in a 3-dimensional format, using a log scale.) For high leverage levels, yield spreads are high, but decrease as maturity increases beyond 0.50 years. For moderate leverage levels, we find that yield spreads are distinctly "humped": intermediate term debt offers higher yields than either very short or very long term debt. In Figure 2, at 50% leverage, the yield spread for short term debt (maturity \(M = 3\) months) is 50 basis points, rising to 122 basis points for debt with maturity 5 years, and falling to 110 basis points for 20-year debt. Finally, for firms that have low leverage, yield spreads are low but increase with debt duration. Interestingly, these patterns are also predicted by Merton's [1974] model of zero coupon debt (without taxes or bankruptcy costs), which in turn have been verified empirically by Sarig and Warga [1989].

We turn now to the behavior of yield spreads as bankruptcy costs, asset risk, and riskfree rates change. There are two sets of comparative statics to consider. First, we ask how the

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23 With high leverage, the term structure of yield spreads decreases from a peak at very short maturity, but it still is of a "humped" shape. This is because as \(M \to 0\) debt becomes riskless, since in the cases we examine \(V > V_g - P/(1-a)\), and the firm is solvent even in the limit. In Merton [1974], the yield spread approaches infinity as maturity approaches zero in the case where \(P/V > 1\). But this case implies the firm is insolvent in the limit as \(M \to 0\), since \(V_g - P > V\).
yield spreads of current debt (with coupon and principal equal to those providing 50% leverage in the base case) change. Then, we examine how yield spreads of newly-issued debt (with coupon and principal providing leverage of 50%, and bonds selling at par in the changed environment) vary.

Yield spreads of both current and newly-issued debt are quite sensitive to the volatility $\sigma$ of underlying assets. The yield spreads of current debt with 3-month, 5-year, and 20-year maturity rise to 133, 208, and 177 basis points, respectively, if volatility rises to 25%, and fall to 7, 49, and 50 basis points, respectively, if volatility falls to 15%. This compares with yield spreads of 50, 122, and 110 basis points in the base case with 20% volatility. For newly-issued debt, yield spreads rise to 98, 217, and 198 basis points, respectively, if volatility rises to 25%, and fall to 25, 52, 45 basis points, respectively, when volatility falls to 15%.

The effect of changes in the level of riskfree interest rates on yield spreads is more surprising. For current debt, a rise in riskfree rates reduces yield spreads. If the riskfree rate increases from 7.5% to 10%, yield spreads on 3-month, 5-, and 20-year maturities fall to 18, 45, and 34 basis points, respectively, from 50, 122, and 110 basis points. A fall in the

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24 This must be an unexpected change, since we have not allowed for the possibility of randomly changing parameters. Since environmental changes mean that the bonds no longer sell at par, we have computed a yield spread to maturity $M$.

25 Since in this case the change takes place before debt is issued, it is possible that the parameters will remain fixed thereafter, as our model presumes.

26 Note that for the longer maturities, the yield spread at a 20% volatility is approximately the average of yield spreads for 15% and 25% volatilities. The yield spread for shorter term maturities seems to exhibit greater convexity in volatility, implying an average yields spread across volatilities which exceeds the yield spread at the average volatility.
riskfree rate to 5% raises yield spreads to 103, 257, and 247 basis points.

When leverage is kept at 50% by issuing new debt at par, the yield curve rotates clockwise as interest rates rise. The yield spread on 3-month debt rises to 57 basis points and falls to 39 basis points as the riskfree rate rises to 10% or falls to 5%, respectively. But for 5-year (20-year) debt, yield spreads decrease to 98 (85) basis points when rates rise, but increase to 150 (143) basis points when rates fall. This rotation of the yield spread structure for newly-issued debt is observed at leverages between 40% and 60% as well. At 60% leverage, yield spreads on newly-issued debt swing from 173/262 bps (shortest/longest maturity debt) to 281/207 bps, as riskfree interest rates (with a flat term structure) rise from 5% to 10%. The net swing of 163 basis points in long/short yield spreads actually shifts the term structure for these corporate bonds from upward to downward sloping. Of course, these results assume asset value $V$ remains fixed. It is likely that shifting interest rates may also change $V$, and the net effect on yield spreads must reflect this effect as well.

Yield spreads are quite also sensitive to bankruptcy costs, particularly for short term debt. For example, with bankruptcy costs falling to 25%, the yield spreads on current debt fall to 1, 63, and 78 basis points, for debt with maturities 3 months, 5 years, and 20 years, respectively; the figures for newly-issued debt are 4, 67, and 77 basis points, respectively, when leverage is maintained at 50%. This compares with 50, 122, and 110 basis points when
bankruptcy costs are 50%.$^{27}$

(iii) The Duration and Convexity of Risky Debt

The Macaulay [1938] measure of duration is an accurate description of the percent change of a bond price in response to a uniform change in the level of interest rates--for bonds with no default risk. A critical question follows: How sensitive are correct measures of duration to the presence of default risk? By "correct" measure of duration, we simply mean an expression which correctly predicts the percentage change in the (risky) bond value in response to a change in the riskfree rate.

In our framework, the Macaulay duration of risky bonds is $1/(m + R)$, where $R = C/D$ when bonds sell at par (i.e. $D = P$ at $V = 100$). Using the base example, we compute the change in the value of current debt for a 1% change in the riskfree interest rate. The true or effective duration of risky debt, plotted as a function of the Macaulay duration, is given in Figure 4, for different degrees of leverage. Figure 5 also plots effective duration vs. Macaulay duration, but for different levels of yield spreads rather than leverage. When leverage is 30% or less, effective duration is slightly smaller than Macaulay duration. As

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$^{27}$Although 77 basis points is the average spread for investment-grade corporate debt over Treasury bonds as reported in Kim, Ramaswamy, and Sundaresan [1993], two important aspects may limit the realism of such a comparison. First, corporate bonds tend to be less liquid than governments. A more realistic comparison might be the corporate yield spread relative to off-the-run Treasury bonds. Second, the typical corporate bond is callable, which tends to increase its spread relative to the noncallable government bonds.
leverage becomes larger, effective duration becomes much shorter than Macaulay duration: with leverage of 60%, debt of 20-year maturity has Macaulay duration of about 6.7 years but effective duration is only 2.1 years.

When leverage exceeds 75%, effective duration of short term debt becomes negative: as the riskfree rate rises, so do bond prices. At 80% leverage, duration for all maturities is negative. This reflects the importance of the endogenous bankruptcy value $V_B$. As previously observed, high risk ("junk") bonds may rise in value with the riskfree rate because the bankruptcy value $V_B$ is lower, implying bankruptcy is less imminent.

Such dramatic differences in effective duration vs. Macaulay duration suggest that immunization and related techniques using corporate bonds must explicitly reflect actual bond risk, and not rely on traditional duration-matching methods.

Riskless debt value is a convex function of the interest rate $r$. Convexity is critical for managing a duration-matching strategy. A dynamic strategy must be followed, because duration increases as interest rates fall. However, if debt values were concave rather than convex in the riskfree interest rate, the opposite kind of dynamic hedging would be required.

We find that the riskiness of debt, as well as its maturity, affects convexity. Figure 6 examines debt value as a function of the riskfree interest rate $r$. Panels 6a - 6c examine long term (20-year maturity) debt carrying yield spreads of 0, 50, and 200 basis points when $r =$
7.5%. Panels 6d - 6f examine intermediate-term debt (5-year maturity), for coupon levels also consistent with yield spreads of 0, 50, and 200 basis points at \( r = 7.5\% \).

The differences in convexity as yield spreads (and therefore bond risks) increase are dramatic, particularly for long term debt. As debt becomes increasingly risky, convexity is reduced and ultimately turns to concavity. The degree of concavity is most pronounced at lower interest rates. Again, this suggests that the hedging of risky debt portfolios requires quite different actions than the hedging of riskfree debt.

(iv) **Bankruptcy Rates and Bond Ratings**

Figure 7 considers the cumulative probability of bankruptcy over a 20 year period, for debt with average maturities of 3 months, 5 years, and 20 years. Panels 7a - 7c reflect debt at each duration bearing a 100 bp yield spread over the riskfree rate. The probability of bankruptcy over a period \( T \) is given by

\[
N\left(\frac{-b - \lambda T}{\sigma \sqrt{T}}\right) + e^{-2bl_0 \sigma^2} N\left(\frac{-b + \lambda T}{\sigma \sqrt{T}}\right)
\]

where \( b = \log(V/V_B) \), \( \lambda = \mu - \delta - .5\sigma^2 \), \( \mu \) (the rate of return to the asset \( V \), including payouts) is assumed to be 15\% per year, and other parameters are as in the base case.

For long term debt, the cumulative probability of bankruptcy is negligible over the first two years, and eventually rises to 1.5\%. Thus, approximately 1.5\% of 20-year debt which pays
100 bps over the riskfree rate will default.\textsuperscript{28} Very short term debt carrying the same yield spread implies a considerably higher probability of default over any fixed time period. This follows because the bankruptcy-triggering value $V_B$ is larger for shorter maturities.\textsuperscript{29} Reflecting the high probability of default, optimal use of short term debt dictates much lower coupon levels than those which generate a 100 bp yield spread.

Panels 7d - 7f chart bankruptcy probabilities at the optimal coupon--that is, the coupon that maximizes firm value--for debt with maturity 0.25, 5, and 20 years. Optimal leverage (considered in Section IV below) is 22.1 percent for debt with maturity 3 months, 39.9 percent for debt with maturity 5.0 years, and 49.2 percent for debt with maturity 20 years. Associated yield spreads are 0 bps, 45 bps, and 103 bps. In contrast with panels 7a - 7c, which assumed a 100 bp yield spread for all durations, panels 7d - 7f indicate that (optimal) debt with the shortest duration gives the smallest probabilities of bankruptcy, about 0.35\% over a 20-year horizon). Clearly this implies that it is optimal to issue only the highest quality short term debt, as seems to characterize the commercial paper market.\textsuperscript{30} Optimal long term debt, carrying a 103 bps annual yield spread, has a probability of default of about 1.5\% over a 20-year horizon. Such debt might receive Moody's bond ratings in the range

\textsuperscript{28} The long-term limiting probability of default is highly dependent on the drift $\mu$ assumed for the asset process $V$. For example, if $\mu = .125$ (rather than .15), the probability of default of 20-year debt would be close to 4\% rather than 1.5\%. The limiting probability of default for 0.25 year debt would rise to 30\% from 20\%.

\textsuperscript{29} Short term yield spreads are low despite the higher cumulative probabilities of bankruptcy because the bulk of short term principal will be repaid before the cumulative probabilities of bankruptcy become sizable.

\textsuperscript{30} However, commercial paper spreads are larger than predicted here, despite their high credit ratings. This may reflect the tax and liquidity advantages of short-term T-bills rather than a default premium. Interestingly, defaults in the commercial paper market have been negligible--consistent with our findings for optimal short term debt.
A to AA, reflecting a moderate yield spread.

In principle, our techniques could be used to produce bond ratings themselves. An important question is "what are we trying to measure?" with a bond rating. Is it probability of default during the debt's life, or yield spreads? The two are related but the relation is complex; the probability of default also requires the actual drift of the asset process. Yield spreads seem the more important variable to predict, since they are intimately related to market valuation. Yield spreads of newly-issued debt are given by

$$\frac{C/D - r}{\left(C + mP\right)\left[1 - (V/V_B)^{-\gamma}\right] + (r + m)(1 - \alpha)V_B(V/V_B)^{-\gamma}} = r$$

where $V_B$ is given by equation (19) or (20).

Bond ratings based on predicted yield spread ranges will reflect current asset value, risk, debt maturity, bankruptcy costs, the riskfree interest rate, and the (total) bond coupon and principal. In fact, examination of equation (22) suggests that the ratio $(V/V_B)^{-\gamma}$ is the critical statistic for determining yield spreads, and therefore bond ratings. An examination of the comparative statics of this ratio are critical to the behavior of "our" bond ratings.

Current bond rating methodologies take many of the variables listed above into consideration, although exactly how they are combined is somewhat murky. Popular rating methodologies also focus on flow measures, such as interest coverage ratios, which are not directly evident in our approach. (However, coverage ratios may be indirectly reflected in
the ratio $V/V_B$, since $V_B$ depends upon $C$ through equation (19) or (20), and $V$ will reflect cash flow variables such as earnings before taxes and interest. Finally, note that the specification of $V_T$ may also reflect cash flow considerations.)

It is important to recall why cash flows (and therefore coverage ratios), are not directly key to our analysis. Bankruptcy in our model is caused by a shortfall of equity value to raise the funds needed to service debt. Current cash flows could be negative, but if equity value remains, the firm need not be forced into bankruptcy.\footnote{For example, if $m = 0$ (infinite maturity debt) carrying a yield spread of 200 bps, then for our example we find from (20) that $V_B = 40.36$. We compute $\delta = .0793$ in this case (covering the $6.55$ coupon plus a 3% stock dividend, when $V = 100$). If $\delta$ is associated with proportional cash flow (EBIT), then cash flow just before bankruptcy is $.0793(40.36) = 3.20$. Clearly, equity financing is making up the difference between the $6.55$ coupon and the cash flow generated by the firm. This confirms that our bankruptcy-triggering condition is quite different than Ross' [1994] condition that bankruptcy occurs whenever cash flow falls beneath the required coupon payment. Of course, one could always find a $\delta$ such that the two conditions coincided; however, recall that $V_B$ itself depends upon $\delta$.} Of course, asset value $V$--which is crucial in our analysis--will reflect past and projected cash flows.

IV. OPTIMAL LEVERAGE

We now examine the leverage ratio which maximizes firm value for alternative choices of debt maturity. If there are no limits on tax deductibility (i.e. arbitrarily large coupons can create further tax benefits), there may be no limit to the optimal amount of debt issued, for shorter duration debt. Ever larger coupon payments provide ever larger tax benefits; these large coupons do not provoke bankruptcy "quickly enough" for bankruptcy costs (and
the loss of tax benefits) to offset the tax gains. This implies that there must be a limit to potential tax benefits. We model this by assuming that tax benefits are lost at an asset level \( V_T \) which increases with the coupon \( C \). Following our earlier example, we assume \( V_T = 50 + 2.5C \).

With the base case parameters of our earlier examples, we can relate firm value \( v \) to the leverage ratios, for debt maturities from 0.25 years to 20 years. Figure 9 plots this relationship. It is assumed that the current asset value \( V = 100 \), and that the principal of debt equals its market value. Observe that the leverage ratio which maximizes firm value is larger for longer duration debt. The maximal firm value is also greater. Table I reports the actual values of variables at the optimal leverage, including the volatility of equity and debt.

**TABLE I**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Coupon</th>
<th>Firm Value</th>
<th>Leverage</th>
<th>Yield Spread</th>
<th>( \sigma_{\text{Equity}} )</th>
<th>( \sigma_{\text{Debt}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25 yrs.</td>
<td>1.75</td>
<td>105.7</td>
<td>22.1%</td>
<td>0 bps</td>
<td>25.7%</td>
<td>0.0%</td>
</tr>
<tr>
<td>1.0 yrs.</td>
<td>2.25</td>
<td>107.1</td>
<td>27.9%</td>
<td>2 bps</td>
<td>27.8%</td>
<td>0.0%</td>
</tr>
<tr>
<td>5.0 yrs.</td>
<td>3.50</td>
<td>110.4</td>
<td>39.9%</td>
<td>45 bps</td>
<td>34.4%</td>
<td>1.5%</td>
</tr>
<tr>
<td>10.0 yrs.</td>
<td>4.25</td>
<td>112.0</td>
<td>45.7%</td>
<td>81 bps</td>
<td>35.3%</td>
<td>2.5%</td>
</tr>
<tr>
<td>20.0 yrs.</td>
<td>4.75</td>
<td>113.3</td>
<td>49.2%</td>
<td>103 bps</td>
<td>36.4%</td>
<td>3.7%</td>
</tr>
<tr>
<td>Infinity</td>
<td>5.60</td>
<td>115.4</td>
<td>54.6%</td>
<td>139 bps</td>
<td>37.6%</td>
<td>5.7%</td>
</tr>
</tbody>
</table>
Given these results, why would firms ever issue short term debt? At least one answer to this may be the differing agency costs associated with different debt maturities. We now show that shorter debt provides fewer incentives for increasing firm risk, and thus minimizes potential agency costs.

V. AGENCY EFFECTS: Debt Maturity and Asset Substitution

Since Black and Scholes [1973] and Jensen and Meckling [1976], it has been a tenet of financial economics that, after debt is issued, stockholders will wish to increase the riskiness of the firm's activities. This is presumed to transfer value from debt to equity--the problem of "asset substitution." The presumption follows from regarding equity as a call option on the underlying firm value, as indeed is the case when debt has no coupon, and taxes and bankruptcy costs are ignored--the case studied by Merton [1974].

Equity in our model, however, is not precisely analogous to an ordinary call option. First, there is no obvious "expiration date." Bankruptcy may occur at any time, when assets fall to the value $V_B$. Second, $V_B$ itself will change with the risk of the firm's activities, as can be seen by equation (19). Finally, the existence of tax benefits (and their potential loss in bankruptcy) implies that debt and equity holders are not splitting a claim whose value depends only on the underlying asset value.\footnote{It has been brought to my attention that many of these reasons were anticipated by Long [1974].}
Figure 8 examines the behavior of the derivative of equity value, $dE/d\sigma^2$, and the derivative of debt value, $dD/d\sigma^2$, as the underlying value $V$ changes. If debt is riskfree (as will occur with all situations, when $V \to \infty$), $dE/d\sigma^2 \to 0$. But when debt is risky, the behavior of $dE/d\sigma^2$ is somewhat complex.

Panels 8a - 8d plot the two derivatives as $V$ varies, for optimal debt levels (at $V = 100$: see Section IV) at maturities of 0.25, 5, and 20 years, plus a consol ($M = \infty$). The dotted line maps $dE/d\sigma^2$; the solid line maps $dD/d\sigma^2$. As $V \to \infty$, debt becomes risk free and the derivative of debt with respect to risk $\sigma^2$ approaches zero from below. Observe that

(i) For either short-term or intermediate-term debt, increasing risk will not benefit bondholders or shareholders, except as bankruptcy is imminent ($V \to V_B$).

(ii) The incentives for increasing risk are much more pronounced for longer term debt.$^{33}$ For very long term debt, $dE/d\sigma^2 > 0$ for all asset levels.

(iii) At all maturities, the incentives for increasing risk become positive for both stockholders and bondholders, as bankruptcy $V_B$ is approached.

However, incentives to increase risk become positive for stockholders before they become positive for bondholders.

$^{33}$ More exactly, the price measure of asset values for which it debtholders and equityholders are in conflict about raising risk is larger, the longer the maturity of debt. Bankruptcy (and resultant distortions) may occur at higher asset values for short duration debt, but the "window" for incentive incompatibility is relatively small.
The incentive compatibility problem exists only for the range of \( V \) for which \( dE/d\sigma^2 > 0 \), and \( dD/d\sigma^2 < 0 \). For very short term debt \( (M = 0.25 \text{ years}) \), this range is minuscule: \( 35.5 < V < 36 \). For intermediate term debt \( (M = 5.0 \text{ years}) \), the range is \( 45 < V < 51 \). The range extends to \( 47 < V < 74 \) with 20 year debt; for very long term debt long term debt the asset substitution problem exists whenever \( V > 48 \).

The extent of conflict between stockholders and bondholders increases when tax rates \( \tau \) and bankruptcy costs \( \alpha \) decline. This is because the outside parties have less claim on firm values, and the "game" between bondholders and stockholders approaches a zero sum game. Figures 8e and 8f illustrate the effect of increased risk on stock and bond values when both \( \alpha = 0 \) and \( \tau = 0 \), and the firm has 50% leverage. Here there is direct conflict between bondholders and stockholders, although the problem is much reduced (except near bankruptcy) when short term debt is used.

These results suggest that the "general" asset substitution problem has been overstated, except when debt is very long term, or when taxes and bankruptcy costs are minimal. The results do illustrate that incentive incompatibilities will arise as bankruptcy is approached. On the very brink of bankruptcy, incentive compatibility is again restored: both stock and bondholders want to raise risks to avoid bankruptcy costs and preserve the potential of tax shelters for debt.
VI. MULTIPLE CLASSES AND SENIORITIES OF DEBT

Multiple debt securities, with differing maturity and seniority, can be brought within the framework developed here. Each different debt issue will be valued according to equation (13), with the amount received by each class in bankruptcy reflecting its seniority. The asset value $V_B$ which triggers bankruptcy must satisfy the smooth-pasting conditions for equity, which will reflect the total value of the firm from equation (16) less the sum of debt values. Detailed results will be pursued in a future paper.

In principle, alternative sinking fund schedules could also be used (e.g., a constant fraction of initial rather than remaining principle is repaid each moment). But now units of debt issued at different times $\tau \leq t$ would no longer be identical at time $t$. Each vintage would have to be valued separately.

VII. CONCLUSIONS

This paper has developed a model of risky corporate bond prices of arbitrary maturity. Key to the model is a constant rollover rate of outstanding debt. Average debt maturity is the reciprocal of the rollover rate. By considering rollover rates from zero to infinity, we can study debt of any average maturity. The time homogeneity of a constant repayment of

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34 The amount each debt issue receives in bankruptcy will reflect its appropriate claim on asset value after bankruptcy costs. If all debt is of equal seniority, its proportion of bankruptcy value will equal its outstanding principal value relative to the total principal of all extant debt.
principal as well as coupon allows closed form solutions for pricing bonds of any maturity. We relate bond values to firm risk, leverage, bankruptcy costs, tax rates, dividends, and the riskfree interest rate.

Our results show that risky debt behaves very differently from riskless debt. Effective duration may be far shorter than Macaulay duration--and even become negative. Convexity can become concavity. This suggests that the proper hedging of fixed income portfolios must consider potential default risks.

The "term structure of yield spreads" -- the relationship between yield spread and maturity, for a given leverage -- exhibits patterns similar to those which have been observed empirically in a related context by Sarig and Warga [1989]. For low risk (low leverage) debt, yield spreads increase with maturity. This is reversed when leverage (and therefore risk) is very large. At intermediate leverage levels, yield spreads are humped, reaching a maximum at intermediate durations.

Yield spreads of current debt decrease as riskless rates rise. For newly-issued debt, a rise in riskless rates will to "tilt" yield spreads negatively: a rise in riskless rates will increase yield spreads of short term debt, but decrease spreads of long term debt. Our techniques also allow computation of the probabilities of default occurring over any horizon. This requires knowledge of the actual (not risk-neutral) drift of the asset value process, however.
Optimal capital structure depends upon debt maturity. Short term debt requires lower leverage ratios than those which are optimal for long term debt. Yield spreads increase markedly with maturity, at the optimal leverage ratio. An important conclusion is that firms should issue higher-rated short term debt than long term debt.

The fact that longer term debt generates higher firm value poses the interesting challenge of why firms issue short term debt. A possible answer to this question lies in the lessening of agency problems, specifically the problem of asset substitution. In contrast with conventional wisdom, our results show that stockholders of firms issuing short-term debt generally will not have an incentive to raise firm risk.

Our model has the virtue of simplicity. In several dimensions it might be thought simplistic. We ignore the possibility that riskfree rates may vary stochastically. And the firm always replaces retired debt which with the same amount of new debt--the same coupon, and the same principal. Extensions to include randomly varying riskfree rates and optimal dynamic adjustments remain for the future.\footnote{First steps in this direction have been taken by Fischer, Heinkel and Zechner [1989].}
APPENDIX A

Let $f(t)$ be the density of the first passage time for a Brownian motion reaching a barrier from above. Let the barrier be at zero, and the starting point $k = \ln(V/V_B)$. Let $\mu = r - \delta - \sigma^2/2$. Then

\begin{equation}
(23) \quad f(t) = \frac{k}{\sigma \sqrt{2\pi t^3}} e^{-\frac{1}{2t} \frac{k+\mu t}{\sigma \sqrt{t}}} dt
\end{equation}

Let $K = k/\sigma(2\pi t^3)^{1/2}$, and $\lambda = [\mu^2 + 2(r+m)\sigma^2]^{1/2}/\sigma^2$. Then

\begin{equation}
(24) \quad \int_{t=0}^{\infty} e^{-(r+m)t} f(t) dt = \int_{t=0}^{\infty} Ke^{-\frac{1}{2} \frac{k+\mu t}{\sigma \sqrt{t}}} dt
\end{equation}

\begin{align*}
= & \int_{t=0}^{\infty} Ke^{-\frac{1}{2} \frac{k^2}{\sigma^2 t} + \frac{1}{\sigma^2} (\mu^2 + 2(r+m)\sigma^2 t + 2k\mu)} dt \\
= & \int_{t=0}^{\infty} Ke^{-\frac{1}{2} \frac{k^2}{\sigma^2 t} + \frac{1}{\sigma^2} (\mu^2 + 2(r+m)\sigma^2 t + 2k\mu)} dt \\
= & \int_{t=0}^{\infty} Ke^{-\frac{1}{2} \frac{k+\lambda \sigma^2 t}{\sigma \sqrt{t}}} e^\frac{k+2\lambda}{2\sigma^2} dt \\
= & \frac{V}{V_B} \left( \frac{\lambda - \frac{k}{\sigma^2}}{\sigma^2} \right) \int_{t=0}^{\infty} g(t) dt
\end{align*}

where $g(t)$ is the first passage time of a process starting at $k$ with volatility $\sigma$ and drift $\lambda \sigma^2$.

\textsuperscript{36} The author thanks Klaus Toft for assistance in this derivation.

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For such a process, it is well known that the limiting probability of absorption (as $T \to \infty$) at $V_B$, when starting at $V$, is

\[
\int_{t=0}^{\infty} g(t) dt = e^{-2\lambda \sigma^2 t} = \left(\frac{V}{V_B}\right)^{-2\lambda}
\]

Substituting this into the last line of equation (24) gives

\[
\int_{t=0}^{\infty} e^{-(r+m)t} f(t) dt = \left(\frac{V}{V_B}\right)^{-\lambda + \frac{\mu}{\sigma^2}} = \left(\frac{V}{V_B}\right)^{-y}
\]

where

\[
y = \lambda + \frac{\mu}{\sigma^2} = \frac{(r-\delta - .5\sigma^2 + [(r-\delta - .5\sigma^2 + 2(r+m)\sigma^2)]^{1/2})}{\sigma^2}
\]
APPENDIX B

Following the Appendix of Leland [1994], the value \( v \) of the firm when \( V_B \leq V \leq V_T \) is

\[
(28) \quad v = V + A_1 V + A_2 V^{-x} - \alpha V_B \left( \frac{V}{V_B} \right)^{-x}
\]

where from equations (49) and (50) in Leland [1994] we have

\[
(29) \quad A_1 = \left( \frac{\tau C}{r} \right) \left( \frac{x}{x + 1} \right) \left( \frac{1}{V_T} \right)
\]

\[
(30) \quad A_2 = -\left( \frac{\tau C}{r} \right) \left( \frac{x}{x + 1} \right) \left( \frac{V_B^{x+1}}{V_T} \right)
\]

Equity value \( E = v - D \), where \( D \) is given in equation (13). Taking the derivative of \( E \) with respect to asset value \( V \) gives

\[
(31) \quad \frac{\partial E}{\partial V} = 1 + A_1 - x A_2 V^{-x-1} + \alpha x \left( \frac{V}{V_B} \right)^{-x-1}
\]

\[
-\left( \frac{V}{V_B} \right) \left[ \frac{C + m P}{r + m} \right] (1 - \alpha) V_B \left( \frac{V}{V_B} \right)^{-y-1}
\]

Evaluating the derivative at \( V = V_B \) and invoking the smooth-pasting condition gives

\[
(32) \quad 1 + A_1 - x A_2 V_B^{-x-1} + \alpha x - \left( \frac{V}{V_B} \right) \left( \frac{C + m P}{r + m} \right) (1 - \alpha) V_B = 0
\]

Substituting for \( A_1 \) and \( A_2 \) from (29) and (30) and solving for \( V_B \) gives the desired result

\[
(33) \quad V_B = \frac{y(C + m P) r V_T}{(r + m)[r V_T (1 + \alpha x + (1 - \alpha) y)] + \tau C x}
\]
Following Leland's [1994] Appendix A, it can then be shown that, when $V > V_T$,

\begin{equation}
\nu = V + \frac{tC}{r} + B_2 V^{-x} - \alpha V_B \left( \frac{V}{V_B} \right)^{-x}
\end{equation}

where

\begin{equation}
B_2 = \left( -\frac{\tau C}{r} \right) \left( \frac{x}{x+1} \right) \left( \frac{1}{V_T} \right) \left( V_B^{x+1} + V_T^{x+1} \right)
\end{equation}
REFERENCES


FIGURE 1: Debt Value
FIGURE 2: Term Structure of Yield Spreads
Figure 3: YIELD SPREADS (log scale)
Figure 4: DURATION (given Leverage)

<table>
<thead>
<tr>
<th>Leverage (%)</th>
<th>80</th>
<th>70</th>
<th>60</th>
<th>50</th>
<th>40</th>
</tr>
</thead>
</table>

- Effective Duration (Yrs)
- Macaulay Duration (Yrs)
Figure 5: DURATION (given Yield Spread)

Effective Duration (Yrs)

Macaulay Duration (Yrs)

Yield Spread (bps)
- ▲ 600
- ▲ 400
- ▲ 200
- ▲ 100
- ▲ 50
Figure 6
Convexity of Debt Values

6a
6b
6c
6d
6e
6f
Figure 7

Probability of Bankruptcy

FIGURE 7a: Maturity 0.25 Years
Yield Spread 100 bps

Pr(Bankruptcy)

Years

FIGURE 7b: Maturity 5 Years
Yield Spread 100 bps

Pr(Bankruptcy)

Years

FIGURE 7c: Maturity 20 Years
Yield Spread 100 bps

Pr(Bankruptcy)

Years

FIGURE 7d: Maturity 0.25 Years
Leverage 22%

Pr(Bankruptcy)

Years

FIGURE 7e: Maturity 5 Years
Leverage 40%

Pr(Bankruptcy)

Years

FIGURE 7f: Maturity 20 Years
Leverage 49%

Pr(Bankruptcy)

Years
Figure 8: Effect of Increase in Risk σ on Bond and Equity Values

Figure 8a: Maturity 0.25 Years

Figure 8b: Maturity 5.0 Years

Figure 8c: Maturity 20 Years

Figure 8d: Infinite Maturity Debt
FIGURE 8e: Maturity 0.25 Years

dE/d(Sigma) — — —

FIGURE 8f: Maturity 5 Years

dE/d(Sigma) — — —

\[ \frac{dE}{d \Sigma} \quad \frac{dD}{d \Sigma} \]
FIGURE 9: Firm Value

Maturity (Yrs)
- Consol
- 20
- 10
- 5
- 2
- 1
- 0.5
- 0.25