Anatomy of an ARM: Index Dynamics and Adjustable Rate Mortgage Valuation

by

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Abstract

This paper analyzes the dynamics of the commonly used indices for Adjustable Rate Mortgages, and systematically compares the effects of their time series properties on adjustable rate mortgage prepayment and value. Our ARM valuation methodology allows us simultaneously to capture the effects of the dynamics of the index, discrete coupon adjustment, and caps and floors. It allows us either to calculate an optimal prepayment strategy for mortgage holders, or to use an empirical prepayment function. We find that the dynamics of the ARM indices, including both their average levels and their speeds of adjustment to interest rate shocks, introduce significant variation in the value of the prepayment option across ARMs. Valuation methodologies that ignore the time series properties of the index with respect to current rates will therefore systematically misprice adjustable rate mortgages.
1 Introduction

Recent surveys of major thrifts and mortgage bankers (See Inside Mortgage Finance) indicate that there are twelve commonly used indices for adjustable rate mortgages in the U.S. This finding is a significant change from 1985 surveys by the Federal Home Loan Mortgage Corporation and the United States League of Savings Institutions. These found that the one year constant maturity Treasury index accounted for between two thirds and eighty percent of all ARM lending ([10], [1]). There is no available information on the index market share of the outstanding stock of ARMs or the periodic flow of ARM originations.

Despite the variety of available ARM indices, it is remarkable that most contingent claims ARM valuation strategies do not explicitly account for the time series characteristics of the underlying index. Instead, published ARM valuation models implicitly assume that the ARM coupon resets with the contemporaneous term structure (See for example Kau et al. [12], McConnell and Singh [15]). There are no previous studies that systematically compare the effects of the times series properties of different ARM indices on the valuation of ARMs in a contingent claims framework with endogenous prepayment.

The results of the Ott [20] ARM duration study, and numerous recent studies of the time series properties of EDCOFI, suggest that the commonly used indices do not adjust instantaneously to changes in contemporaneous spot rates. The strength of this empirical evidence suggests that we should reconsider the instantaneous reset assumptions found in most ARM valuation models. These models should instead be based on empirically tested specifications of the time series properties of the indices relative to observed term structure data.

This paper compares the time series dynamics of the most commonly used ARM indices, includes these dynamics in an ARM valuation model, and determines their impact on prepayment option and mortgage value. We build upon techniques developed by Kau et al. [12], Kishimoto [13], and Stanton and Wallace [25]. We use finite difference techniques to solve the pricing equation, taking into account all the main contractual features of the ARM index. A major advantage of this strategy is that it allows us to either determine endogenously the optimal prepayment policy for mortgage holders, or to use an empirically derived prepayment function. A second advantage is that it enables us to price ARMs in which lags in the index interact with other contract elements, such as caps, to induce path dependence in the mortgages’ cash flows.

The paper is in three sections. The empirical specification for ARM indices is discussed in section 2. Section 3 discusses the valuation methodology, and analyzes the effects of index dynamics, caps and margins on the value of ARM contracts. Section 4 concludes the paper.

2 Dynamics of ARM Indices

The ARM indices that dominate the market are:

1. The one year constant maturity Treasury yield,

2. The Federal Housing Finance Board (FHFB) national average contract interest rate,
3. The Eleventh District Cost-of-Funds Index (EDCOFI),

4. The five year Treasury note rate,

5. One year LIBOR.

The one year Treasury rate reflects the average yields of all existing Treasury securities with one year of maturity remaining. The yield is determined from the closing market bids on actively traded Treasury Securities in the over-the-counter market, as disclosed by the five leading U.S. government securities dealers. The index is computed as a weekly average, and the Federal Reserve Board publishes this yield in its weekly H-15 statistical release. The five year constant maturity index is computed in the same way for existing Treasury securities with five years of remaining maturity.

The FHFB contract interest rate is the weighted average of initial mortgage interest rates paid by home buyers for loans originated during the first five business days of every month. The weights are determined by the type, size and location of the lender. The index is constructed by the Federal Housing Finance Board and reported on a monthly basis. The Eleventh District Cost-of-funds Index is computed from the book values of liabilities for all insured savings and loan (S&L) institutions in the Eleventh District (institutions in California, Nevada, and Arizona). The index is the ratio of the month-end total interest expenses on savings accounts, advances, and purchased funds to the average book value of these liabilities from the beginning to the end of the month. The ratio is adjusted with an annualizing factor so that the interest expenses are comparable across months.

The historical values of the ARM indices from July 1981 through May 1993 are plotted against the 3-month Treasury rate in Figure 1. The plot shows that EDCOFI and the FHFB average contract rate display considerably less volatility than the Treasury and LIBOR series. EDCOFI appears to lag the Treasury series by several months. This should be expected, given that it is based on book yields, which can only change when a liability matures. The FHFB average contract rate looks rather like EDCOFI, with a spread of approximately 200 basis points. The FHFB average contract also lags the Treasury rates.

Considering the construction of EDCOFI and the FHFB average contract rate, and our plots of these indices relative to market rates, a partial adjustment model is a reasonable representation for the movements of EDCOFI, the FHFB average contract rate, and one year LIBOR. For a given index, \( I_t \), the model can be written as

\[
I_t = \alpha + \beta r_t + \gamma I_{t-1} + \epsilon_t, \tag{1}
\]

where \( r_t \) is an instantaneous spot rate, and \( \epsilon_t \) is an error term. The coefficient \( \beta \) indicates the effect of the spot rate on the index each period, and \( \gamma \) indicates the speed at which the index adjusts. The extremes in the adjustment dynamics would be \( \beta = 0 \), where the index does not move at all with market rates, and \( \gamma = 0 \), where the index moves perfectly with the spot rate (the usual implicit assumption).

---

3 EDCOFI is the ratio of the month end total interest expenses on savings accounts, advances, and purchased funds to the average book value of these liabilities from the beginning to the end of the month, calculated for all insured savings and loan (S&L) institutions in the Eleventh District (California, Nevada, and Arizona).

4 All of the data series, except EDCOFI, were obtained from CITIBASE. The EDCOFI series data were obtained from the Office of Thrift Supervision.

5 See Ott [20], Cornell [3], Passmore [21], Roll [24] and Stanton and Wallace [25] for further discussion and justification of this specification.
Ignoring the error term, if the index starts at a value \( I_0 \), and the interest rate remains at a constant level \( r \), the value of the index at any later time is given by\(^6\)

\[
I_t = (1 - \gamma^t) \left( \frac{\alpha + \beta r}{1 - \gamma} \right) + \gamma^t I_0. \tag{2}
\]

This is a weighted average of the long run value of \( I_t \) and its initial value. The speed of convergence is governed by the value of \( \gamma \). The half-life, the number of periods required to reach halfway between the two values, is the solution to

\[
\gamma^{t_{1/2}} = \frac{1}{2}, \tag{3}
\]

which is

\[
t_{1/2} = -\frac{\log(2)}{\log(\gamma)}. \tag{4}
\]

Note that substituting \( \gamma = 0 \) into equation 2 yields the correct result for instantaneous adjustment,

\[
I_t = \alpha + \beta r. \tag{5}
\]

Because we are interested in the adjustment of observed ARM indices to the instantaneous spot rate, we estimate the partial adjustment models using the three month Treasury rate as a proxy for the instantaneous spot rate.\(^7\) The estimation results are reported in Table 1. All the indices are estimated in levels.\(^8\) We estimate the partial adjustment model for EDCOFI using dummies for January and February to account for seasonality. Because of problems with both serial correlation and heteroscedasticity, we estimate the partial adjustment model for EDCOFI using the Newey and West [17] instrumental variable procedure to obtain a heteroscedasticity and autocorrelation consistent covariance matrix. The partial adjustment model for the FHKB average contract rate was estimated using OLS, because neither heteroscedasticity nor serial correlation violations was observed. The one year LIBOR partial adjustment model was estimated using instruments for the first lag of one year LIBOR, and then using Yule-Walker estimation methods and an AR(2) specification for second stage estimation of the model. The \( R^2 \), Breusch-Pagan [2] tests for heteroscedasticity, and the Durbin tests for AR(1) errors are also reported.

\(^6\)This solution can be verified by inserting it into equation 1, with \( \epsilon_t \) set to zero.

\(^7\)This choice was made because the three month Treasury rates offered the shortest term rate with reasonable large trading volume. The one month Treasury rates reflect very erratic trading volume over our period of analysis.

\(^8\)Augmented Dickey-Fuller [6] tests of the form

\[
\Delta x_t = \mu + \gamma^* x_{t-1} + \sum_{j=1}^{p-1} \phi_j \Delta x_{t-j} + \epsilon_t
\]

were performed on all series using twelve lagged differences to control for possible seasonality. We were unable to reject the null hypothesis that there are unit roots in market rates and in the indices. Phillips and Perron [23] nonparametric unit root testing procedures were also applied, with the same result. Tests for the cointegration of the indices and Treasury rates, using Johansen [11], showed that they are not cointegrated. However, because the series are relatively short, and it is well known that the low power of standard unit roots tests often leads to acceptance of the null hypothesis of a unit root in many economic time series (Kwiatkowski et al. [14], Faust [8]), we rely on our strong priors that our interest rate series are mean reverting rather than explosive, and undertake all our estimation in levels of interest rates.
EDCOFI responds a little more quickly to the three month Treasury rate than the FHFB average contract rate. One year LIBOR responds faster than either, keeping very close track of the three month Treasury rate. This is shown in Figure 2, which shows the effect of an instantaneous 1% shift in the riskless interest rate on each of the three indices. Each index starts at its long run level (the level it would reach if \( r \) stayed at 7.5% for ever), and the graph shows what happens when \( r \) jumps from 7.5% to 8.5%. Besides the obvious lags in EDCOFI and the FHFB rate, one other interesting feature of the graph is the difference between the levels of the three series. EDCOFI is approximately 0.6% higher than \( r \) in this region; LIBOR is approximately 1.2% higher, and the FHFB rate is almost 3% higher. This means that for a given margin, a loan based on the FHFB rate will have significantly higher payments (and hence value) than one based on either of the other two series. The pricing impact of the lags in the indices’ adjustment processes can only be evaluated in an option pricing framework that accounts for the adjustment frequency, caps and prepayment characteristics of ARM contracts.

3 Valuation

This section develops an algorithm for valuing adjustable rate mortgages. The algorithm can handle all of the important features of the contract, including the model for movements in EDCOFI developed in section 2. We can either use an empirical prepayment function (as commonly used in Wall Street valuation models), or derive endogenously the optimal prepayment strategy for mortgage holders. This latter strategy has the advantage that it is robust to possible changes in the economic environment, such as changes in the interest rate process, which would have an unquantifiable effect on an empirical prepayment function. In addition, using the optimal prepayment strategy allows us to determine an upper bound for the value of the prepayment option possessed by mortgage holders. The algorithm is based on techniques developed by Kau et al. [12], Kishimoto [13], and Stanton and Wallace [25].

3.1 Main Features of an ARM Contract

**Coupon rate**, \( C_t \). The coupon rate on an ARM changes at each “reset date”. The coupon determines the monthly cash flows on the mortgage until the next reset date. The monthly cash flow equals that on a fixed rate mortgage with the same time to maturity, same remaining principal balance, and same coupon rate as the ARM.

**Underlying Index**, \( I_t \). The adjustment rule for the coupon rate specifies a particular index to which the rate is tied.

**Margin**, \( m \). At each coupon reset date, the new rate is set by adding a margin, \( m \) (e.g. 2%), to the prevailing level of the underlying index (subject to certain caps, discussed below).

**“Teaser” rate**, \( C_0 \). It is common for the initial coupon rate to be lower than the “fully indexed” rate given by adding the margin to the initial level of the index. The initial rate, \( C_0 \), is often referred to as a “teaser” rate.

**Annual cap**, \( \Delta \). ARM contracts usually specify a maximum adjustment in the coupon rate at each reset period (e.g. 2% per year).

**Lifetime caps**, \( \bar{C} \) and \( \underline{C} \). ARM contracts usually specify an overall maximum coupon rate over the life of the loan, \( \bar{C} \) (e.g. the initial rate plus 6%), and a minimum coupon rate over the life of the
loan, $C$.

**Reset Frequency.** The coupon rate on an ARM contract adjusts at prespecified intervals. This interval is usually every 6 months or one year. In this paper, we assume yearly adjustment. If month $t$ is a coupon reset date, the new coupon rate is given by

$$C_t = \max \left[ C, C_{t-1} - \Delta, \min \left[ I_t + m, C_{t-1} + \Delta, C \right] \right]$$

(6)

### 3.2 Interest Rates

To value the mortgage, we need to make assumptions about the process governing interest rate movements. We use the Cox, Ingersoll and Ross [4] one-factor model. In this model, the instantaneous risk-free interest rate, $r_t$, satisfies the stochastic differential equation

$$dr_t = \kappa (\mu - r_t) \, dt + \sigma \sqrt{r_t} \, dz_t.$$  

(7)

This equation says that, on average, the interest rate $r$ converges toward the value $\mu$. The parameter $\kappa$ governs the rate of this convergence. The volatility of interest rates is $\sigma \sqrt{r_t}$. One further parameter, $\lambda$, which summarizes risk preferences of the representative individual, is needed to price interest rate dependent assets.

The parameter values used here are those estimated by Pearson and Sun [22], using data from 1979–1986. These values are

$$\kappa = 0.29368,$$
$$\mu = 0.07935,$$
$$\sigma = 0.11425,$$
$$\lambda = -0.12165.$$  

The long run mean interest rate is 7.9%. Ignoring volatility, the time required for the interest rate to drift half way from its current level to the long run mean is $\ln(1/2)/(-\kappa) \approx 2.4$ years.

### 3.3 Other factors affecting ARM value

The value of an ARM depends not only on the current interest rate, $r_t$, but on the whole path of interest rates since its issue. This determines the current coupon rate, $C_t$, the current level of the underlying index, $I_t$ (which in turn determines future movements in the coupon rate), and the current remaining principal balance, $F_t$. These three variables summarize all relevant information about the history of interest rates. By adding these as extra state variables, we return to a Markov setting where all prices can be written as a function only of the current values of a set of underlying state variables.

Write $B_t$ for the value of a non-callable bond which makes payments equal to the promised payments on the ARM. The mortgage holder’s position can be decomposed into a short position in $B_t$ (the scheduled payments on the mortgage) plus a long position in a call option on $B_t$, with (time varying) exercise price $F_t$. Writing $M_t$ for the market value of the mortgage, and $O_t$ for the value of the prepayment option, we have

$$M_t = B_t - O_t$$

(8)
Since $B_t$ does not depend on the mortgage holder’s prepayment decision, minimizing his or her liability value is equivalent to maximizing the value of the prepayment option, $O_t$. Write

$$B_t \equiv B(r_t, I_t, C_t, F_t, t),$$
$$O_t \equiv O(r_t, I_t, C_t, F_t, t).$$  \hspace{1cm} (9) \hspace{1cm} (10)

All values are homogeneous of degree one in the current remaining principal amount, $F_t$. Thus, if each month we value a mortgage with $\$1$ remaining principal, we can scale up or down as necessary for different principal amounts. Define normalized asset values (values per $\$1$ of remaining principal) by

$$\bar{B}_t = B_t / F_t,$$
$$\bar{O}_t = O_t / F_t,$$
$$\bar{B}(r_t, I_t, C_t, t),$$
$$\bar{O}(r_t, I_t, C_t, t).$$ \hspace{1cm} (11) \hspace{1cm} (12)

### 3.4 Valuation with one State Variable

Given the interest rate model defined by equation 7, write $V(r; t)$ for the value of an asset whose value depends only on the current level of $r_t$ and time, and which pays coupons or dividends at some rate $\delta(r_t, t)$. This value satisfies the partial differential equation\(^9\)

$$\frac{1}{2} \sigma^2 r V_{rr} + [\kappa \mu - (\kappa + \lambda)r] V_r + V_t - r V + \delta = 0,$$ \hspace{1cm} (13)

which can be solved for $V$, subject to appropriate boundary conditions.

Natural boundaries for the interest rate, $r$, are 0 and $\infty$. Rather than working directly with $r$, define the variable $y$ by

$$y = \frac{1}{1 + \gamma r},$$ \hspace{1cm} (14)

for some constant $\gamma > 0,^{10}$ The infinite range $[0, \infty)$ for $r$ maps onto the finite range $[0, 1]$ for $y$. The inverse transformation is

$$r = \frac{1 - y}{\gamma y}.$$ \hspace{1cm} (15)

Equation 14 says that $y = 0$ corresponds to "$r = \infty$" and $y = 1$ to $r = 0$. Next, rewrite equation 13 using the substitutions

$$U(y; t) \equiv V(r(y); t),$$
$$V_r = U_y \frac{dy}{dr},$$
$$V_{rr} = U_y \frac{d^2y}{dr^2} + U_{yy} \left( \frac{dy}{dr} \right)^2.$$ \hspace{1cm} (16) \hspace{1cm} (17) \hspace{1cm} (18)

\(^9\)We need to assume some technical smoothness and integrability conditions (see, for example, Duffie [7]).

\(^{10}\)The larger the value of $\gamma$, the more points on a given $y$ grid correspond to values of $r$ less than, say, 20%. Conversely, the smaller the value of $\gamma$, the more points on a given $y$ grid correspond to values of $r$ greater than, say, 4%. We are most interested in values of $r$ in an intermediate range. Therefore, as a compromise between these two competing objectives, we choose $\gamma = 12.5$. The middle of the range, $y = 0.5$, then corresponds to $r = 8\%$. 

7
to obtain
\[
\frac{1}{2}\gamma^2 y^4 \sigma^2 r(y) U_{yy} + \left(-\gamma y^2 \left[ \kappa \mu - (\kappa + \lambda) r(y) \right] + \gamma^2 y^3 \sigma^2 r(y) \right) U_y + U_t - r(y) U + \delta = 0. \quad (19)
\]

We can solve equation 19 using a finite difference algorithm. Finite difference algorithms replace derivatives with differences, and approximate the solution to the original partial differential equation by solving the set of difference equations that arise. We use the Crank-Nicholson algorithm.\(^{11}\)

Represent the function \(U(y, t)\) by its values on the finite set of points,
\[
y_j = j \Delta y, \quad (20)
\]
\[
t_k = k \Delta t, \quad (21)
\]
for \(j = 0, 1, \ldots, J\), and for \(k = 0, 1, \ldots, K\). \(\Delta y\) and \(\Delta t\) are the grid spacings in the \(y\) and \(t\) dimensions respectively. \(\Delta y = 1/J\), and \(\Delta t\) is chosen for convenience to be one month, making a total of 360 intervals in the time dimension. Write
\[
U_{j,k} \equiv U(y_j, t_k), \quad (22)
\]
for each \((j, k)\) pair. The Crank-Nicholson algorithm rewrites equation 19 in the form
\[
MU_k = D_k, \quad (23)
\]
where \(M\) is a tridiagonal matrix, \(U_k\) is the vector \(\{U_{0,k}, U_{1,k}, \ldots, U_{I,k}\}\), and \(D_k\) is a vector whose elements are functions of \(U_{j,k+1}\). This system of equations relates the values of the asset for different values of \(y\) at time \(t_k\) to its possible values at time \(t_{k+1}\). To perform the valuation, we start at the final time period, when all values are known, and solve equation 23 repeatedly, working backwards one period at a time.

### 3.5 Extension to multiple state variables

In general, when asset prices depend on more than one state variable plus time, solution of the resultant partial differential equation becomes numerically burdensome. In this case, the additional variables, \(I_t\) and \(C_t\), are functions of the path of interest rates, and so they introduce no additional risk premia. This allows us to extend the Crank-Nicholson finite difference algorithm to handle the multiple state variable case. The extensions required are:

1. Allow values to depend on \(C_t\) and \(I_t\) as well as \(r_t\) and \(t\), allowing for dependence between the processes governing movements in these variables.

2. Scale values to correspond to $1 remaining principal.

3. Handle caps, floors and teaser rates.

\(^{11}\)See, for example, McCracken and Dorn [16].
In addition to the finite sets of values for $y$ and $t$ defined above, define a finite set of values for $I$ and $C$ by

$$I_l = l \Delta I,$$  \hspace{1cm} (24)$$

$$C_m = m \Delta C,$$  \hspace{1cm} (25)$$

for $l = 0, 1, \ldots, L$, and for $m = 0, 1, \ldots, M$. $\Delta I$ and $\Delta C$ are the grid spacings in the $I$ and $C$ dimensions respectively. We are now solving for values on the points of a 4-dimensional grid, whose elements are indexed by the values of $(j, k, l, m)$. Write the value of an asset whose cash flows depend on these state variables as

$$U_{j,k,l,m} \equiv U(y_j, t_k, I_l, C_m),$$  \hspace{1cm} (26)$$

for each $(j, k, l, m)$. $I$ and $C$ are functions of the path of interest rates. Over the next instant, the movement in $r$ completely determines the movements in both $I$ and $C$. Assume that movements in EDCOFI are described by the equation

$$I_{t+1} = g(I_t, r_{t+1}),$$  \hspace{1cm} (27)$$

so that EDCOFI this month is a deterministic function of EDCOFI last month, plus the short term riskless rate this month (the models estimated above are of this type). Define $l^*$ by

$$I^*_{t+1, j, l, m} \approx g(I_t, r_{j+1}),$$  \hspace{1cm} (28)$$

$$I^*_{t+1, j, l, m} \approx g(I_t, r_{j}),$$  \hspace{1cm} (29)$$

$$I^*_{t+1, j, l, m} \approx g(I_t, r_{j-1}).$$  \hspace{1cm} (30)$$

In words, $l^*$ gives the closest index to the value of $I$ next period given the current values of $r$, $I$ and $C$, and three possible values of $r$ next period (up, the same, and down). Assuming that next month is a coupon reset date (since otherwise, the coupon rate next month will just be the same as the coupon rate this month), define $m^*$ similarly, to give the index of $C$ next period given the current values of $r$, $I$ and $C$, and the value of $r$ next period. $m^*$ is determined by the interplay between the current coupon $C_t$, the index $I_t$, the margin $m_t$, and the caps $\overline{C}$, $\underline{C}$ and $\Delta$. Note that the effects of caps, floors and teaser rates are all automatically captured in this definition of $m^*$.

We can now generate a set of finite difference equations for each pair $(l, m)$. For example, the approximation for the time derivative now becomes

$$U_t(y_j, t_k, I_l, C_m) \approx \left( U_{j,k+1, I^*_{t+1, j, l, m}, m} - U_{j,k,l,m} \right) / \Delta t,$$  \hspace{1cm} (31)$$

if $t_{k+1}$ is not a coupon reset period, and

$$U_t(y_j, t_k, I_l, C_m) \approx \left( U_{j,k+1, I^*_{t+1, j, l, m}, m^*_{j, l, m}} - U_{j,k,l,m} \right) / \Delta t,$$  \hspace{1cm} (32)$$

if $t_{k+1}$ is a coupon reset period. This allows us to write down one set of systems of equations like equation 23 for each $(l, m)$ pair. These equations are independent of each other, so we can solve them for each $(l, m)$ pair in turn, looping over $l$ and $m$ to calculate values at every grid point at time $t_k$. A simplified version of this is shown graphically in Figure 3. Each horizontal plane corresponds
to a grid of values in \((r, t)\) space. There is a separate such grid for each value of \(I_t\) (as shown in the figure), and for each value of \(C_t\).\(^{12}\) As in the standard Crank-Nicholson algorithm, we value the asset by solving a set of difference equations, just like equation 23, for each \((r, t)\) plane. The difference equation for the value of the asset at any point involves its values at six points, corresponding to the current time, \(t\), and the following time period, \(t+1\), and interest rates \(r_t, r_{t-1}\) and \(r_{t+1}\). Note that the values at time \(t\) all sit on the current \((r, t)\) plane, while the values for next period may be on other planes, from equations 31 and 32 (for example, in Figure 3, if the interest rate moves from \(r_t\) to \(r_{t+1}\) next period, the index moves from \(I_j\) to \(I_{j+1}\)). Similarly, if the interest rate moves from \(r_t\) to \(r_{t-1}\) next period, the index moves from \(I_j\) to \(I_{j-1}\)). We can solve the equations for each \((r, t)\) plane separately, rather than having to consider them all simultaneously. This is because the interaction between different \((r, t)\) planes only occurs in the values at date \(t+1\). By the backward nature of the solution methodology, when we are calculating values at time \(t\), we can regard all values at date \(t+1\) as known, so this only affects the calculation of the right hand side of equation 23.

The final step in the process is to deal with the normalization of asset prices to correspond to a remaining principal balance of $1. This is possible because, at any time, we know exactly how much principal will be repaid over the next one month. Given a coupon rate \(C_t\) and a current remaining principal \(F_t\), the usual amortization formula tells us the value of \(F_{t+1}\), regardless of any possible movements in \(r_t\), \(I_t\), or \(C_t\). The values stored in the grid for next period correspond to $1 in remaining principal next period. These need only to be multiplied by \(F_{t+1}/F_t\) (a function only of \(C_t\)) to make them correspond to $1 of remaining principal today.

### 3.6 Valuation Results

The extended Crank-Nicholson algorithm described above was used to value 30 year adjustable rate mortgages. Starting in month 360, the algorithm works backward to solve equation 23 one month at a time, calculating the normalized bond value, \(\bar{B}_t\). For the option, the same process gives the value conditional on its remaining unexercised for the next month. This value must then be compared with the option's intrinsic value (\(\max[0, \bar{B}_t - 1]\)), to determine whether prepayment is optimal. \(\bar{O}_t\) is set to the higher of these two values, and the mortgage value is calculated from the relationship

\[
\bar{M}_t = \bar{B}_t - \bar{O}_t.
\]

Figures 4–15 show the results for different underlying indices and different contract terms. For ease of comparison, every mortgage shown has an annual coupon reset frequency, and a lifetime cap of 13.5%. Figure 4 shows the values of the bond (the promised coupon payments, with no prepayment option), the mortgage, and the prepayment option, for different values of the interest rate \(r\), with the current value of the FHFB rate set to 8.5%. The coupon rate adjusts annually to equal the prevailing value of the FHFB rate (with no additional margin). The prepayment option has a value of at least 4% of the remaining principal balance on the loan. Its value decreases as interest rates increase. Figure 5 shows the values of the bond, the mortgage, and the prepayment option for different values of the FHFB rate, keeping the riskless interest rate equal to 7.5%. The higher the current value of the FHFB rate (and the current coupon rate on the mortgage), the higher the value of the underlying instrument and the value of the prepayment option. There is a discontinuity in the

\(^{12}\)Imagine Figure 3 repeated in the direction perpendicular to the page.
slope of the graph at the point where the FHFB rate equals 13.5%. Below this value, as the FHFB rate increases, the graph shows the value of mortgages with progressively higher values of both the underlying index and a higher index. At 13.5% the cap becomes binding, and from then on, while the index increases, the coupon rate remains fixed at 13.5%. The graphs of both bond value and option value are almost flat beyond this point. This is because further increasing the FHFB rate does not increase the coupon rate, merely the expected time before the cap ceases to bind.

Figures 6–15 focus on the impact on the value of the prepayment option of the index used, and the size of the reset margin. Figures 6 and 7 look again at mortgages based on the FHFB rate, with reset margins ranging from −0.5% to 1% over the FHFB rate. For each reset margin, the value of the prepayment option decreases in r for a given value of the FHFB rate, and for a given r the value increases in the FHFB rate, almost flattening off after 13.5%. The value increases in the reset margin, but note that in both Figures 6 and 7, the values for different reset margins for high values of r, or high values of the FHFB rate, appear to converge or even cross slightly. The reason for this is that once the cap is binding, due to the high mean level of the FHFB rate and its slow movement relative to shifts in the term structure, it is likely to stay binding for a long time. With a binding coupon cap, the size of the margin is irrelevant; all mortgages have a coupon rate of 13.5%, and the value of the underlying bond is almost independent of the reset margin. The value of the prepayment option is high. For a margin of 1%, and a value of 7.5% for r, the prepayment option is worth at least 10% of the remaining principal balance on the mortgage. This is a function of the slow movement in the index, and also the generally high level of this index relative to the other indices studied (see Figures 1 and 2).

Figures 8 and 9 show the value of the prepayment option contained in ARMs based on EDCOFI. Comparing these graphs with those for FHFB loans, the spread between option values for loans with different reset margins is larger for loans based on EDCOFI. This is for two reasons. First, EDCOFI reacts faster to movements in r than does the FHFB rate. Second, its mean level is lower. Together, these imply that even if the coupon cap is binding today, it is likely not to do so in the near future, with the result that different margins imply significantly different bond, and hence option, values. The overall value of the options is lower than for FHFB, for the same reset margin. This is because FHFB is in general about 2% higher than EDCOFI (see Figure 1). Prepayment option values for EDCOFI loans with a 2% margin, and FHFB loans with a 0% margin, are similar across much of the range of possible index values. For high levels, the FHFB option is more valuable, as the effect of slow movements in the underlying index becomes more important.

Figures 10 and 11 show the value of the prepayment option for LIBOR based loans. Figures 12 and 13 show its value for ARMs based on the one year T-Bill rate, and Figures 14 and 15 look at loans based on the five year T-Note rate. The graphs for LIBOR and the one year T-Bill rate look very similar. This is because LIBOR tracks the short term interest rate closely (see Table 1 and Figure 1). The values for LIBOR loans are higher, since LIBOR is on average higher than the one year T-Bill rate (see Figure 2). For a given reset margin, the prepayment options contained in loans based on the five year T-Note rate are rather more valuable. This is because the five year rate is in general substantially higher than the one year rate, so the underlying bond is more valuable, and prepayment is more likely to be optimal (see Figure 1).

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13 The lines crossing is an artifact of the discrete approximation to the true asset value.
4 Summary

This paper analyzes the valuation of adjustable rate mortgages based on the most commonly used indices,

1. The one year constant maturity Treasury yield,
2. The Federal Housing Finance Board (FHFB) national average contract interest rate,
3. The Eleventh District Cost-of-Funds Index (EDCOFI),
4. The five year Treasury note rate,
5. One year LIBOR.

We find that a simple partial adjustment model closely describes the behavior of EDCOFI, the FHFB average contract rate, and one year LIBOR, and we develop an ARM valuation methodology which allows us simultaneously to capture the effects of index dynamics, discrete coupon adjustment, and caps and floors. Our methodology allows us either to calculate an optimal prepayment strategy for mortgage holders, or to use an empirical prepayment function. We use the conduct a systematic comparison of the properties of ARMs based on the different indices, and show in particular that the prepayment options embedded in most of these ARMs usually have significant value, a fact which is often overlooked. More generally, we find that the value of the prepayment option, and hence of the mortgage, is significantly affected by both its contract terms and by the dynamics of the index underlying the mortgage. This includes both the average level of the index\textsuperscript{14} and its speed of adjustment to interest rate shocks.\textsuperscript{15} Our valuation methodology allows us for the first time to quantify these interacting effects. Previous approaches, which ignore the time series properties of the index with respect to current rates, will systematically misprice adjustable rate mortgages.

\textsuperscript{14}For example, the mean level of the FHFB rate is higher than that of EDCOFI, leading to higher expected payments on a loan backed by the FHFB rate, all else being equal.

\textsuperscript{15}For example, while the mean level of EDCOFI is not very different from that of the one year Treasury rate, its slow adjustment to changes in interest rates mean that the prepayment option embedded in EDCOFI based ARMS are more valuable than those embedded in ARMs indexed to the one year Treasury rate, all else being equal.
References


<table>
<thead>
<tr>
<th>Independent Variables</th>
<th>EDCOFI</th>
<th>FHFB Average Contract Rate</th>
<th>One Year LIBOR (Weekly Avg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>.3306** (2.065)</td>
<td>.366*** (6.062)</td>
<td>.6688** (2.068)</td>
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<tr>
<td>January dummy</td>
<td>-.0632** (-2.094)</td>
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<td></td>
</tr>
<tr>
<td>February dummy</td>
<td>.1517*** (3.441)</td>
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<tr>
<td>First lag of EDCOFI</td>
<td>-.8430*** (17.415)</td>
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<tr>
<td>First lag of FHFB Avg. Contract Rate</td>
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<tr>
<td>First lag of One Year LIBOR</td>
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<td>.1361** (2.622)</td>
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<tr>
<td>Three month T-Bill rate</td>
<td>.1263*** (3.539)</td>
<td>.0928*** (11.340)</td>
<td>.9148*** (16.002)</td>
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<td>Autoregressive Parameters</td>
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<td>( (1 - .969 B + .214 B^2 ) ( (11.56) ) ( (2.55) )</td>
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<tr>
<td>( R^2 )</td>
<td>.994</td>
<td>.997</td>
<td>.857</td>
</tr>
<tr>
<td>Breusch-Pagan test for heteroscedasticity, ( \chi_m^2 )</td>
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<td>4.2</td>
<td>8.7</td>
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<tr>
<td>Durbin test for AR(1) (t-statistic)</td>
<td>.684</td>
<td>-.195</td>
<td>.803</td>
</tr>
</tbody>
</table>

* Significant at the 10% level.  
** Significant at the 5% level.  
*** Significant at the 1% level.

Table 1: Estimates for Adjustment Models of ARM Indices, July 1981 – May 1993 (t-statistics in parentheses).
Figure 1: EDCOFI and other indices, July 1981 – May 1993.
Figure 2: Example of the lags in the different indices' responses to movements in the term structure. The graph shows the movement in the different indices resulting from a jump in the short term riskless interest rate from 7.5% to 8.5%.
Figure 3: Extended Crank Nicholson algorithm.
Figure 4: Bond and option values for different values of $r$. Current FHFB rate is 8.5%. Coupon rate currently equals 8.5%, and resets annually to the prevailing value of FHFB rate. Coupon rate has a lifetime cap of 13.5%. Remaining principal is $100.

Figure 5: Bond and option values for different values of FHFB rate. Current short term riskless interest rate is 7.5%. Coupon rate equals current value of FHFB rate, and resets annually to the prevailing value of FHFB rate. Coupon rate has a lifetime cap of 13.5%. Remaining principal is $100.
Figure 6: Prepayment option values for different values of $r$. Current FHFB rate is 8.5%. Coupon rate currently equals 8.5% plus appropriate margin. Coupon resets annually to the prevailing value of FHFB rate, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 7: Prepayment option values for different values of FHFB rate. Current short term riskless interest rate is 7.5%. Coupon rate equals current value of FHFB rate, plus appropriate margin. Coupon resets annually to the prevailing value of FHFB rate, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 8: Prepayment option values for different values of $r$. Current EDCOFI is 8.5%. Coupon rate currently equals 8.5% plus appropriate margin. Coupon resets annually to the prevailing value of EDCOFI, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 9: Prepayment option values for different values of EDCOFI. Current short term riskless interest rate is 7.5%. Coupon rate equals current value of EDCOFI, plus appropriate margin. Coupon resets annually to the prevailing value of EDCOFI, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 10: Prepayment option values for different values of $r$. Current LIBOR is 8.5%. Coupon rate currently equals 8.5% plus appropriate margin. Coupon resets annually to the prevailing value of LIBOR, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 11: Prepayment option values for different values of LIBOR. Current short term riskless interest rate is 7.5%. Coupon rate equals current value of LIBOR, plus appropriate margin. Coupon resets annually to the prevailing value of LIBOR, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 12: Prepayment option values for different values of $r$. Current coupon rate is 8.5% plus appropriate margin. Coupon resets annually to the prevailing value of one year T-Bill rate, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 13: Prepayment option values for different values of one year T-Bill rate: Coupon rate equals current value of one year T-Bill rate, plus appropriate margin. Coupon resets annually to the prevailing value of one year T-Bill rate, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 14: Prepayment option values for different values of $r$. Current coupon rate is 8.5% plus appropriate margin. Coupon resets annually to the prevailing value of five year T-Note rate, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.
Figure 15: Prepayment option values for different values of five year T-Note rate. Coupon rate equals current value of five year T-Note rate, plus appropriate margin. Coupon resets annually to the prevailing value of five year T-Note rate, plus appropriate margin, subject to a lifetime cap of 13.5%. Remaining principal is $100.