Double Lookbacks
by
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Abstract

A new class of options, *double lookbacks*, where the payoffs depend on the maximum and/or minimum prices of one or two traded assets is introduced and analyzed. This class of double lookbacks includes calls and puts with the underlying being the difference between the maximum and minimum prices of one asset over a certain period, and calls or puts with the underlying being the difference between the maximum prices of two correlated assets over a certain period. Analytical expressions of the joint probability distribution of the maximum and minimum values of two correlated geometric Brownian motions are derived and used in the valuation of double lookbacks. Numerical results are shown, and prices of double lookbacks are compared to those of standard lookbacks on a single asset.

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1 Introduction

Exotic options designed as contingent claims on equity indices, currencies, and the term structure of interest rates, have achieved enormous success in global financial markets during the past decade. Such exotic products, while seemingly complicated to small investors, have provided institutional investors new vehicles to meet their various financial needs which include hedging, risk management and speculation (when investors have specific views on future market movements). Theoretical advances by academics as well as practitioners have helped market participants create more exotic products and understand the economic benefits of such products, and consequently, contributed in an important way to the surge in popularity of exotic options.

In this paper we introduce a new class of exotic options: lookback options based on two traded assets. A standard one-asset lookback call (or put) gives its holder the right to buy (or sell) the underlying stock at its historical minimum (or maximum) price over a certain period. Analytical solutions for standard lookback options have been found by Goldman, Sosin, and Gatto (1979) and Goldman, Sosin, and Shepp (1979). Lookback options are appealing because they offer investors the opportunity (at a price, of course) of buying a stock at its lowest price and selling a stock at its highest price.

In our two-asset generalization, we consider options whose payoffs depend on the extremal (i.e., maximum and/or minimum) prices of one and/or two stocks over a given period. For example, we consider call or put options on the spread between the maximum and minimum price of Xerox stock over a given interval of time; an option to receive the maximum of General Motors' stock price (or return) at the maximum of Ford's stock price (or return) over a given period; an option to receive the minimum of IBM's stock price (or return) at the minimum of Digital's stock price (or return) over a given period. We refer to these options as double lookbacks.

The economic motivation for double lookbacks is not difficult to perceive. An option on the spread between the maximum and minimum price of a single stock over a given interval of time captures in part the idea of an option on price volatility, and is conceptually simpler. Such an option might be of interest to traders who want to bet on price volatility or hedge an existing position which is sensitive to price volatility. Double lookbacks involving two assets allow investors to bet on the difference between the extreme values of two assets or two indices. Since the double lookback is an exchange between two extreme values, it is cheaper and therefore more attractive than a lookback option that exchanges one asset for the extreme value of that asset. If an investor wants to take a long position in the maximum of one asset and a short position in the value of another asset, then a semi double lookback, which is an option on the difference between the
maximum price of one asset over a given period and the terminal price of another asset, would be an appropriate investment vehicle.

The main contribution of the paper is to derive analytical expressions for the distributions necessary to price double lookback options. In the case of a single asset, we derive the analytical expression for the probability distribution of the maximum and the minimum prices of one asset following a geometric Brownian motion. In the case of two assets, we derive the analytical expression for the probability distribution of the maximum and/or minimum prices of two assets following two correlated geometric Brownian motions.\(^1\) Numerical procedures are readily available for evaluating all of the double lookback options discussed above.

It is important to point out that the analysis of this paper can also be used to value knockout options based on two traded assets. The analytical solution for the one-asset knockout is well known, see Merton (1973) and Rubinstein (1992). For a two-asset example, we may consider an option on the difference of two asset prices subject to a knockout condition based on either one or both assets' prices not reaching the boundary. We refer to these options as *double knockouts*.

The paper is organized as follows. Section 2 sets up the economy and lists the options in which we are interested. In Section 3, we derive the relevant distribution functions for lookbacks on a single asset, and in Section 4, we derive the distributions needed for lookbacks on two assets. Section 5 contains results on semi-lookbacks. Numerical examples are provided and compared in Sections 3-5. We discuss how our analysis can be used to value double knockouts in Section 6. Section 7 concludes.

## 2 Arbitrage-free Pricing of Lookbacks

Consider a Black-Scholes economy in which stock prices are log-normal, the interest rate is constant, and continuous trading without transaction costs, taxes, or other market frictions is permitted. There are three assets: one riskfree bond and two risky stocks. The prices of the bond and the stocks are:

\[
B(t) = e^{rt}
\]

\[
S_1(t) = S_1(0)e^{(\mu_1-q_1-\frac{\sigma_1^2}{2})t+\sigma_1w_1(t)}
\]

\[
S_2(t) = S_2(0)e^{(\mu_2-q_2-\frac{\sigma_2^2}{2})t+\sigma_2w_2(t)}
\]

where \(r\) is the riskless rate, \(\mu_i\) the expected instantaneous return of stock \(i\), \(q_i\) the dividend yield of stock \(i\), \(\sigma_i\) the volatility of stock \(i\), and \(w_i\) a standard Brownian motion with \(\text{cov}(dw_1, dw_2) = \rho dt\).

\(^1\)As far as we know, this distribution function has not appeared in the literature.
Throughout this paper, we assume that we are pricing the lookback options at date 0, the options expire at date $T$, and that the lookback period runs from $t^*$ to $T$. Note that $t^*$ may be either positive or negative. For $i = 1, 2$ and $t \geq t^*$, define the running minimum and maximum of stock price $S_i$ by:

$$S_i(t) = \min_{t^* \leq s \leq t} S_i(s)$$

$$\bar{S}_i(t) = \max_{t^* \leq s \leq t} S_i(s).$$

We are interested in studying several types of European double lookback options. The payoffs of these options at expiration date $T$ are:

- **(Lookback Spread)** call or put on the spread between the maximum and minimum of a single stock price:

$$\max \left[ 0, \left( \overline{S}_1(T) - \underline{S}_1(T) \right) - K \right]$$

$$\max \left[ 0, K - \left( \overline{S}_1(T) - \underline{S}_1(T) \right) \right]$$

- **(Double Maxima)** call or put on the difference between the maximum of $S_1$ and the maximum of $S_2$:

$$\max \left[ 0, \left( a \overline{S}_1(T) - b \overline{S}_2(T) \right) - K \right]$$

$$\max \left[ 0, K - \left( a \overline{S}_1(T) - b \overline{S}_2(T) \right) \right]$$

where $a > 0$ and $b > 0$ are parameters to be chosen by investors. In practice, if $t^* = 0$, it may make sense to pick $a$ and $b$ such that $aS_1(0) = bS_2(0)$. For example, $a = \frac{1}{\overline{S}_1(0)}$ and $b = \frac{1}{\overline{S}_2(0)}$. When $K = 0$, the double maxima call is equivalent to an option to buy the maximum of $S_1$ at the maximum of $S_2$.

- **(Double Minima)** call or put on the difference between the minimum of $S_1$ and the minimum of $S_2$:

$$\max \left[ 0, \left( a \underline{S}_1(T) - b \underline{S}_2(T) \right) - K \right]$$

$$\max \left[ 0, K - \left( a \underline{S}_1(T) - b \underline{S}_2(T) \right) \right]$$

When $K = 0$, the double minima call is equivalent to an option to sell the minimum of $S_1$ for the minimum of $S_2$.

- **(Double Lookback Spread)** call or put on the spread between the maximum $S_1$ and the minimum of $S_2$:

$$\max \left[ 0, \left( \overline{S}_1(T) - \underline{S}_2(T) \right) - K \right]$$

$$\max \left[ 0, K - \left( \overline{S}_1(T) - \underline{S}_2(T) \right) \right].$$
For comparison purposes, we will also look at the following options:

- option to buy the maximum of \( S_1 \) at \( S_1(T) \) or \( S_2(T) \):
  \[
  \max \left[ 0, \bar{S}_1(T) - S_1(T) \right] 
  \]
  \[
  \max \left[ 0, a\bar{S}_1(T) - b\bar{S}_2(T) \right] 
  \]
  with \( a = 1 \) and \( b = S_1(0)/S_2(0) \) if \( t^* = 0 \).

- option to sell \( S_1 \) or \( S_2 \) for the minimum of \( S_2 \):
  \[
  \max \left[ 0, S_2(T) - \bar{S}_2(T) \right] 
  \]
  \[
  \max \left[ 0, aS_1(T) - b\bar{S}_2(T) \right] 
  \]
  where \( a = S_2(0)/S_1(0) \) and \( b = 1 \) if \( t^* = 0 \).

The premiums of these options should be compared with those of double maxima and double minima calls with \( K = 0 \).

Following Harrison and Kreps (1979) and Harrison and Pliska (1981), the Black-Scholes economy is known to be viable and dynamically complete, Thus, each of the above derivative securities can be replicated through dynamic trading in the stock(s) and bond. Furthermore, there exists a probability measure \( Q \) (the equivalent martingale or risk-neutral measure) under which the discounted price \( V^*(t) = V(t)/B(t) \) of any derivative security is a martingale. Under this risk neutral probability, the stock price processes are:

\[
S_1(t) = S_1(0)e^{(r-q_1-\frac{\sigma_1^2}{2})t + \sigma_1 w_1^*(t)} 
\]
\[
S_2(t) = S_2(0)e^{(r-q_2-\frac{\sigma_2^2}{2})t + \sigma_2 w_2^*(t)} 
\]

where \( w_1^* \) and \( w_2^* \) are standard Brownian motions under \( Q \) with the same constant correlation \( \rho \) as under the original probability measure. Because the discounted derivative price \( V^*(t) \) is a martingale, its value at date 0 can be determined by taking the conditional expectation of its terminal value:

\[
V^*(0) = E^Q[V^*(T)]. 
\]

For a Black-Scholes economy with constant interest rates, this equation can be rewritten as

\[
V(0) = e^{-rT}E^Q[V(T)]. 
\]
Thus we see that each of the valuation problems we face consists of evaluating a conditional expectation, which in turn is simply a matter of integrating the payout over an appropriate density function.

If the lookback period begins at $t^* > 0$, then by iterating the expectation we find

$$V(0) = e^{-rT} E^Q \left[ E^Q_t [V(T)] \right].$$

The inner expectation is of the same form as when $t^* \leq 0$, and hence can be evaluated using the same densities derived in the next section. The outer expectation is simply an integral over the distribution of $S(t^*)$, and can be evaluated using standard techniques. For the rest of this paper, we concentrate on the case where $t^* \leq 0$.

For ease of notation, we define $X_i(t)$, the continuously- compounded return of stock $i$, by

$$X_i(t) = \log S_i(t)/S_i(0) = \alpha_i t + \sigma_i w_i^*(t), \quad (t^* \leq 0 \leq t)$$

where $\alpha_i = r - q_i - \sigma_i^2/2$. Also, define the running minima and maxima of $X_i$ by

$$m_i = \min_{t^* \leq s \leq 0} X_i(s),$$

$$M_i = \max_{t^* \leq s \leq 0} X_i(s),$$

$$X_i(t) = \min_{0 \leq s \leq t} X_i(s),$$

$$\overline{X}_i(t) = \max_{0 \leq s \leq t} X_i(s).$$

Then the payoffs and valuation integrals for the double lookback options become:

- **(Lookback Spread)**

  $$V(T) = \max \left[ 0, \left( S_1(0) e^{\max(M_1, \overline{X}_1(T))} - S_1(0) e^{\min(m_1, X_1(T))} \right) - K \right]$$

  $$V(0) = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(X_1(T) \in dx_1, \overline{X}_1(T) \in dx_1) V(T)$$

  with corresponding formulas for the put option

- **(Double Maxima)**

  $$V(T) = \max \left[ 0, a S_1(0) e^{\max(M_1, \overline{X}_1(T))} - b S_2(0) e^{\max(M_2, \overline{X}_2(T))} - K \right]$$

  $$V(0) = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(X_1(T) \in dx_1, \overline{X}_2(T) \in dx_2) V(T)$$

  with corresponding formulas for the put option
• (Double Minima)

\[
V(T) = \max\left[0, aS_1(0)e^{\min(m_1, X_1(T))} - bS_2(0)e^{\min(m_2, X_2(T))} - K\right]
\]

\[
V(0) = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{P}(X_1(T) \in d\alpha_1, X_2(T) \in d\alpha_2) V(T)
\]

with corresponding formulas for the put option.

• (Double Lookback Spread)

\[
V(T) = \max\left[0, \left(aS_1(0)e^{\max(M_1, X_1(T))} - bS_2(0)e^{\min(m_1, X_2(T))}\right) - K\right]
\]

\[
V(0) = e^{-r(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{P}(\bar{X}_1(T) \in d\alpha_1, X_2(T) \in d\alpha_2) V(T)
\]

with corresponding formulas for the put option.

In the next section, we derive the necessary density functions. Performing the integrations indicated above, using numerical techniques if necessary, leads directly to the option prices.

3 Lookbacks on One Asset

In this section we consider lookback spread options where the payoffs depend upon the maximum and minimum of an asset whose price follows a geometric Brownian motion. Specifically, we derive the joint distribution function of the maximum and minimum of a Brownian motion with constant drift.

Before proceeding we state below as a lemma the distribution functions of the maximum and minimum of a Brownian motion with constant drift. We also record below as a theorem the analytical formulae for standard lookbacks based on either the maximum or the minimum of an asset over a given period. The proofs of these lemma and theorem can be found in Conze and Viswanathan (1991). Let us define

\[
G(x_1, t; \alpha_1) = \Phi \left( \frac{x_1 - \alpha_1 t}{\sigma_1 \sqrt{t}} \right) - e^{\frac{2\alpha_1^2}{\sigma_1^2 t}} \Phi \left( \frac{-x_1 - \alpha_1 t}{\sigma_1 \sqrt{t}} \right).
\]

**Lemma 1** The distribution functions for the maximum and the minimum of a Brownian motion with constant drift \( \alpha_1 \) is given by:

\[
\mathbb{P}(\bar{X}_1(t) \leq x_1) = G(x_1, t; \alpha_1), \quad x_1 \geq 0
\]

\[
\mathbb{P}(X_1(t) \geq x_1) = G(-x_1, t; -\alpha_1), \quad x_1 \leq 0
\]
Theorem 1 Let \( C_{LB} \) (or \( P_{LB} \)) be the price at time 0 of a standard lookback call (or put) which pays \([S_1(T) - S_2(T)] \) (or \([S_1(T) - S_2(T)] \)) at the expiration date \( T \). Then,

\[
C_{LB} = Se^{-qT}N\left(d_1\right) - e^{-rT}mN\left(d_1 - \sigma\sqrt{T}\right) + e^{-rT} \frac{\sigma^2}{2(r-q)} S^{1-\frac{2(r-q)}{\sigma^2}} m^{\frac{2(r-q)}{\sigma^2}} N(-d_1 + \frac{2(r-q)}{\sigma}\sqrt{T}) - \frac{\sigma^2}{2(r-q)} Se^{-qT} N(-d_1)
\]

\[
P_{LB} = -Se^{-qT}N(-d_2) + e^{-rT}MN(-d_2 + \sigma\sqrt{T}) - e^{-rT} \frac{\sigma^2}{2(r-q)} S^{1-\frac{2(r-q)}{\sigma^2}} M^{\frac{2(r-q)}{\sigma^2}} N(d_2 - \frac{2(r-q)}{\sigma}\sqrt{T}) + \frac{\sigma^2}{2(r-q)} Se^{-qT} N(d_2)
\]

where \( S = S_1(0), m = S_2(0), M = S_3(0) \) and

\[
d_1 = \frac{\ln(S/m) + (r-q)T + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}
\]

\[
d_2 = \frac{\ln(S/M) + (r-q)T + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}
\]

Similarly, let \( C_M \) (or \( P_m \)) be the price at time 0 of a call written on the maximum (or a put on the minimum) with a strike \( K \). Then,

\[
C_M = \begin{cases} 
Se^{-qT}N(d) - e^{-rT}KN(d - \sigma\sqrt{T}) \\
-e^{-rT} \frac{\sigma^2}{2(r-q)} S^{1-\frac{2(r-q)}{\sigma^2}} K^{\frac{2(r-q)}{\sigma^2}} N\left(d - \frac{2(r-q)}{\sigma}\sqrt{T}\right) + \frac{\sigma^2}{2(r-q)} Se^{-qT} N(d), & \text{if } K > M \\
e^{-rT}(M - K) + Se^{-qT}N(d_2) - e^{-rT}MN(d_2 - \sigma\sqrt{T}) \\
-e^{-rT} \frac{\sigma^2}{2(r-q)} S^{1-\frac{2(r-q)}{\sigma^2}} M^{\frac{2(r-q)}{\sigma^2}} N\left(d_2 - \frac{2(r-q)}{\sigma}\sqrt{T}\right) + \frac{\sigma^2}{2(r-q)} Se^{-qT} N(d_2), & \text{if } K < M 
\end{cases}
\]

\[
P_m = \begin{cases} 
-Se^{-qT}N(-d) + e^{-rT}KN(-d + \sigma\sqrt{T}) \\
e^{-rT} \frac{\sigma^2}{2(r-q)} S^{1-\frac{2(r-q)}{\sigma^2}} K^{\frac{2(r-q)}{\sigma^2}} N\left(-d + \frac{2(r-q)}{\sigma}\sqrt{T}\right) - \frac{\sigma^2}{2(r-q)} Se^{-qT} N(-d), & \text{if } K < m \\
e^{-rT}(K - m) - Se^{-qT}N(-d_1) + e^{-rT}mN(-d_1 + \sigma\sqrt{T}) \\
e^{-rT} \frac{\sigma^2}{2(r-q)} S^{1-\frac{2(r-q)}{\sigma^2}} m^{\frac{2(r-q)}{\sigma^2}} N\left(-d_1 + \frac{2(r-q)}{\sigma}\sqrt{T}\right) - \frac{\sigma^2}{2(r-q)} Se^{-qT} N(-d_1), & \text{if } K > m 
\end{cases}
\]

where

\[
d = \frac{\ln(S/K) + (r-q)T + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}
\]

We note that standard lookback call and put prices are equal to the corresponding Black-Scholes values, with strikes set at the current maximum or minimum, plus a premium. This premium reflects the opportunity that the minimum (or maximum) can go down (or up) further. The prices for calls on the minimum and puts on the maximum have a similar interpretation.

We now derive a similar set of joint density/distribution functions for the maximum and minimum of a single Brownian motion.
Lemma 2 (i) The joint density/distribution of the maximum, minimum, and end point of a Brownian motion with a constant drift, for \( x \in [x_1, x_2] \), \( x_1 \leq 0, x_2 \geq 0 \), is given by:

\[
\mathcal{P}(X_1(t) \in dx, X_1(t) \geq x_1, \overline{X}_1(t) \leq x_2) = \exp \left( \frac{\alpha_1 x - \alpha_1^2 t}{\sigma_1^2} \right) \sum_{n=-\infty}^{\infty} \frac{1}{\sigma_1 \sqrt{t}} \left[ \phi \left( \frac{x - 2n(x_2 - x_1)}{\sigma_1 \sqrt{t}} \right) - \phi \left( \frac{x - 2n(x_2 - x_1) - 2x_1}{\sigma_1 \sqrt{t}} \right) \right] dx,
\]

where \( \phi(z) = \exp(-z^2/2)/\sqrt{2\pi} \) is the standard normal density. This density can be expressed in the equivalent form

\[
\mathcal{P}(X_1(t) \in dx, X_1(t) \geq x_1, \overline{X}_1(t) \leq x_2) = \frac{2}{x_2 - x_1} \exp \left( \frac{\alpha_1 x - \alpha_1^2 t}{\sigma_1^2} \right) \sum_{n=-\infty}^{\infty} \exp \left( -\frac{n^2 \pi^2 \sigma_1^2 t}{2(x_2 - x_1)^2} \right) \sin n\pi \left( \frac{-x_1}{x_2 - x_1} \right) \sin n\pi \left( \frac{x - x_1}{x_2 - x_1} \right) dx.
\]

(ii) The joint distribution of the maximum and minimum of a Brownian motion with constant drift, for \( x_1 \leq 0, x_2 \geq 0 \), is given by:

\[
\mathcal{P}(X_1(t) \geq x_1, \overline{X}_1(t) \leq x_2) = \sum_{n=-\infty}^{\infty} \frac{2}{\sigma_1^2} \left\{ \left[ N \left( \frac{x_2 - \alpha_1 t - 2n(x_2 - x_1)}{\sigma_1 \sqrt{t}} \right) - N \left( \frac{x_1 - \alpha_1 t - 2n(x_2 - x_1) - 2x_1}{\sigma_1 \sqrt{t}} \right) \right] \right. \\
- \left. e^{\frac{2x_1 \sigma_1^2}{\sigma_1^2}} \left[ N \left( \frac{x_2 - \alpha_1 t - 2n(x_2 - x_1) - 2x_1}{\sigma_1 \sqrt{t}} \right) - N \left( \frac{x_1 - \alpha_1 t - 2n(x_2 - x_1) - 2x_1}{\sigma_1 \sqrt{t}} \right) \right] \right\}.
\]

This distribution can be written in the equivalent form

\[
\mathcal{P}(X_1(t) \geq x_1, \overline{X}_1(t) \leq x_2) = \frac{\alpha_1^2}{\sigma_1^2} \sum_{n=1}^{\infty} \frac{2n\pi}{n^2 \pi^2 + \left( \frac{\alpha_1 (x_2 - x_1)}{\sigma_1^2} \right)^2} e^{-\frac{n^2 \pi^2 \sigma_1^2 t}{2(x_2 - x_1)^2}} \sin \frac{n\pi(-x_1)}{x_2 - x_1} dx_2 \leq x_2, X_1(t) \in dx).
\]

PROOF: To prove (i), the density function can be obtained by using a reflection principle argument. Karatzas and Shreve (1991) give the zero drift result, and our result just shifts by a Girsanov factor. Alternatively, one can obtain this result by solving the Fokker-Planck equation using a method of images procedure (see, for example, Wilmott, Dewynne, and Howison (1993)).

To get the second density function, define \( g(x) dx = \mathcal{P} \left( X_1(t) > x_1, \overline{X}_1(t) < x_2, X_1(t) \in dx \right) \). Then \( g(x) \) satisfies the following Fokker-Planck equation with absorbing boundaries:

\[
\frac{\partial g}{\partial t} = \frac{1}{2} \sigma_1^2 \frac{\partial^2 g}{\partial x^2} - \alpha_1 \frac{\partial g}{\partial x}
\]

\[
g(x, 0) = \delta(x), \quad g(x_1, t) = g(x_2, t) = 0
\]

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where $\delta(x)$ denotes the Dirac delta function with a spike at $x = 0$. A routine separation of variables technique leads directly to the answer given. See Gardiner (1990) for the zero-drift solution.

Integrating the expressions in part (i) over $x \in [x_1, x_2]$ leads immediately to the joint distributions in (ii) for the minimum and maximum of a Brownian motion. \hfill \blacksquare

**Remark 1** One should note that the two seemingly different formulas for the joint density/distribution are in fact equal: the second is the Fourier sine transform of the first. In practice, one or the other may lead to more useful numerical approximations depending upon the problem.

In Figure 1, we present a surface and contour plot of the joint density function
\[
\frac{\partial^2 P(X_1(t) \geq x_1, X_1(t) \leq x_2)}{\partial x_1 \partial x_2}.
\]
The parameter values chosen are $\sigma = 0.2 \text{ yr}^{-1/2}$, $r = 0.05 \text{ yr}^{-1}$, $q = 0$, and $t = 1 \text{ yr}$, and we plot the minimum over the range $[-0.7, 0]$ and the maximum over the range $[0, 0.7]$. Given the joint density function, the price of lookback spread can be obtained by integrating the final payoffs with respect to the density. Specifically, define
\[
V_{SP}(x_1, x_2) = \max \left[ 0, \left( S_1(0)e^{\max(M_1, x_2)} - S_1(0)e^{\min(m_1, x_1)} \right) - K \right].
\]

**Theorem 2** The price at time 0 of a lookback spread call, $C_{SP}$, is given by
\[
C_{SP} = e^{-rT} \int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 V_{SP}(x_1, x_2) \frac{\partial^2 P(X_1(t) \geq x_1, X_1(t) \leq x_2)}{\partial x_1 \partial x_2}
\]
with corresponding formulas for the put option.

Given Theorem 2, we can evaluate the price of a lookback spread option by direct numerical quadrature. In Table 1, we list the prices for lookback spread call and put options for various parameter values. These prices have the right sensitivities, i.e., higher volatility leads to higher option premiums and larger strike price leads to smaller call premium but larger put premium. When the volatility is 20%, the premium for the lookback spread $(S - \bar{S})$ is 31.5% of the initial stock price.

One should note that by having an explicit form for the density function, the computational time to evaluate these options is significantly reduced compared to Monte-Carlo or lattice methods. Typically only a few terms in the infinite series of Lemma 2 are needed to obtain convergence.
<table>
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<th>\sigma</th>
<th>K</th>
<th>Call</th>
<th>Put</th>
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</thead>
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<td>0.00</td>
</tr>
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</tr>
<tr>
<td>0.9</td>
<td>0.60</td>
<td>48.63</td>
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</tr>
</tbody>
</table>

Table 1: Values for call and put lookback spread options with payoffs of, respectively, \((S - S - K)^+\) and \((K - S + S)^+\). We assume that \(S(0) = S(0) = S(0) = 100\), \(r = 0.05\) yr\(^{-1}\), \(q = 0\), \(t = 1\) yr.

4 Lookbacks on Two Assets

In this section we analyze double lookback options whose payoffs depend on the extreme values of two assets following correlated geometric Brownian motions. We first derive the joint distribution functions necessary for the evaluation of these options, and then we discuss the use of these distributions in pricing.

We begin with the following lemma for the joint distribution of the extreme values and terminal values of two correlated Brownian motions.

Lemma 3 (i) For \(x_1 \geq m_1, x_2 \geq m_2\), where \(m_1 \leq 0, m_2 \leq 0\),

\[
\mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2; X_1(t) \geq m_1, X_2(t) \geq m_2) = p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, r) \, dx_1 \, dx_2,
\]

where

\[
p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) = e^{\alpha_1 x_1 + \alpha_2 x_2 + \beta t}q(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho),
\]

\[
q(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) = \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \sum_{n=1}^{\infty} e^{-\frac{r^2 + \beta^2}{2}} \sin \frac{n \pi \theta_0}{\beta} \sin \frac{n \pi \theta_1}{\beta} \, \left( \frac{rr_0}{t} \right),
\]

and

\[
a_1 = \frac{\alpha_1 \sigma_2 - \rho \alpha_2 \sigma_1}{(1 - \rho^2) \sigma_1^2 \sigma_2}
\]
\[
\begin{align*}
a_2 &= \frac{\alpha_2 \sigma_1 - \rho \alpha_1 \sigma_2}{(1 - \rho^2) \sigma_1 \sigma_2^2} \\
b &= -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2} \sigma_1^2 a_1^2 + \rho \sigma_1 \sigma_2 a_1 a_2 + \frac{1}{2} \sigma_2^2 a_2^2 \\
\tan \beta &= -\frac{\sqrt{1 - \rho^2}}{\rho}, \quad \beta \in [0, \pi], \\
z_1 &= \frac{1}{\sqrt{1 - \rho^2}} \left[ \left( \frac{x_1 - m_1}{\sigma_1} \right) - \rho \left( \frac{x_2 - m_2}{\sigma_2} \right) \right] \\
z_2 &= \frac{x_2 - m_2}{\sigma_2} \\
z_{10} &= \frac{1}{\sqrt{1 - \rho^2}} \left[ \frac{m_1}{\sigma_1} + \rho m_2 \right] \\
z_{20} &= \frac{m_2}{\sigma_2} \\
r &= \sqrt{z_1^2 + z_2^2} \\
\tan \theta &= \frac{z_2}{z_1}, \quad \theta \in [0, \beta], \\
r_0 &= \sqrt{z_{10}^2 + z_{20}^2} \\
\tan \theta_0 &= \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0, \beta].
\end{align*}
\]

(ii) For \( x_1 \geq m_1, \ x_2 \leq M_2, \) where \( m_1 \leq 0, \ M_2 \geq 0, \) we have

\[
P(X_1(t) \in dx_1, X_2(t) \in dx_2; X_1(t) \geq m_1, X_2(t) \leq M_2) = p(x_1, -x_2, t; m_1, -M_2, \alpha_1, -\alpha_2, \sigma_1, \sigma_2, -\rho) \ dx_1 \ dx_2.
\]

(iii) For \( x_1 \leq M_1, \ x_2 \leq M_2, \) where \( M_1 \geq 0, \ M_2 \geq 0, \) we have

\[
P(X_1(t) \in dx_1, X_2(t) \in dx_2; X_1(t) \leq m_1, X_2(t) \leq M_2) = p(-x_1, -x_2, t; -M_1, -M_2, -\alpha_1, -\alpha_2, \sigma_1, \sigma_2, \rho) \ dx_1 \ dx_2
\]

PROOF: For notational convenience, we denote the density as \( p(x_1, x_2, t). \) We know that \( p \) satisfies the Fokker-Planck equation

\[
\frac{\partial p}{\partial t} = -\alpha_1 \frac{\partial p}{\partial x_1} - \alpha_2 \frac{\partial p}{\partial x_2} + \frac{1}{2} \sigma_1^2 \frac{\partial^2 p}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{1}{2} \sigma_2^2 \frac{\partial^2 p}{\partial x_2^2}
\]

with the following initial condition

\[
p(x_1, x_2, t = 0) = \delta(x_1) \delta(x_2)
\]

and absorbing boundary conditions

\[
p(x_1 = m_1, x_2, t) = 0 \\
p(x_1, x_2 = m_2, t) = 0.
\]
We proceed to explicitly solve this PDE.

First, we note that we can eliminate the drift terms by the following transformation. Define

\[ p(x_1, x_2, t) = e^{a_1 x_1 + a_2 x_2 + bt} q(x_1, x_2, t) \]

where \( a_1, a_2, \) and \( b \) are defined as above. Then \( q(x_1, x_2, t) \) satisfies

\[ \frac{\partial q}{\partial t} = \frac{1}{2} \sigma_1^2 \frac{\partial^2 q}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 q}{\partial x_1 \partial x_2} + \frac{1}{2} \sigma_2^2 \frac{\partial^2 q}{\partial x_2^2} \]

with boundary conditions

\[ q(x_1, x_2, t = 0) = \delta(x_1) \delta(x_2) \]
\[ q(x_1 = m_1, x_2, t) = 0 \]
\[ q(x_1, x_2 = m_2, t) = 0. \]

Next, we note that this PDE can be simplified by a suitable transformation of coordinates, to eliminate the cross-partial derivative and normalize the Brownian motions. Explicitly, if we define new coordinates \( z_1 \) and \( z_2 \), as given above, then

\[ q(x_1, x_2, t) = \frac{h(z_1(x), z_2(x), t)}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \]

and \( h(z_1, z_2, t) \) satisfies

\[ \frac{\partial h}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 h}{\partial z_1^2} + \frac{\partial^2 h}{\partial z_2^2} \right) \]

with boundary conditions

\[ h(z_1, z_2, t) = \delta(z_1 - z_{10}) \delta(z_2 - z_{20}) \]
\[ h(L_1, t) = h(L_2, t) = 0, \]

where

\[ L_1 = \{(z_1, z_2) : z_2 = 0\} \]
\[ L_2 = \{(z_1, z_2) : z_2 = -\frac{1 - \rho^2}{\rho} z_1\} \]

These boundary conditions along \( L_1 \) and \( L_2 \) are more conveniently expressed in polar coordinates. Introducing polar coordinates \((r, \theta)\) corresponding to \((z_1, z_2)\) as defined above, we obtain

\[ \frac{\partial h}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} + \frac{1}{r^2} \frac{\partial^2 h}{\partial \theta^2} \right) \]
with boundary conditions

\[ h(r, \theta, t = 0) = \frac{1}{r_0} \delta(r - r_0) \delta(\theta - \theta_0) \]
\[ h(r, \theta = 0, t) = 0 \]
\[ h(r, \theta = \beta, t) = 0. \]

To solve this PDE for \( h(r, \theta, t) \), we look for separable solutions of the form \( R(r)\Theta(\theta)T(t) \). Plugging this in to the PDE, we find

\[
\frac{T'}{T} = \frac{1}{2} \left( \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} \right) = -\lambda^2/2
\]

where the separation constant is negative because the solutions must decay as \( t \to \infty \). Hence, we have

\[ T(t) \sim e^{-\lambda^2 t/2} \]

and

\[
\left( r^2 \frac{R''}{R} + r \frac{R'}{R} + \lambda^2 r^2 \right) + \left( \frac{\Theta''}{\Theta} \right) = 0.
\]

Defining \( \Theta''/\Theta = -k^2 \), we find

\[ \Theta(\theta) \sim A \sin k\theta + B \cos k\theta. \]

The boundary conditions require that \( \Theta(0) = \Theta(\beta) = 0 \), and hence \( k \) must be real, \( B = 0 \), and

\[ \sin k\beta = 0. \]

This last requirement restricts \( k \) to discrete values of the form

\[ k_n = \frac{n\pi}{\beta}, \quad n = 1, 2, \ldots \]

Thus the most general angular solution consistent with the boundary conditions is

\[ \Theta(\theta) \sim \sin \frac{n\pi \theta}{\beta}, \quad n = 1, 2, \ldots \]

Finally, the radial part of the solution is

\[ r^2 R'' + r R' + (\lambda^2 r^2 - k_n^2) R = 0. \]

Defining \( y = \lambda r \), we can rewrite this in the standard form

\[
y^2 \frac{d^2 R}{dy^2} + y \frac{dR}{dy} + (y^2 - k_n^2) R = 0.
\]
This is Bessel's equation, with the well-known fundamental solutions $J_{k_n}(y)$ and $I_{k_n}(y)$. Since $I_{k_n}(0)$ diverges, and we require $R(0)$ to be well-behaved, the $I_{k_n}(x)$ solution is not permitted. Hence the general radial solution is

$$R(r) \sim J_{k_n}(\lambda r).$$

In summary, then, the most general solution to the PDE for $h(r, \theta, t)$ consistent with the absorbing boundary conditions $h(r, 0, t) = h(r, \beta, t) = 0$, is given by

$$h(r, \theta, t) = \int_0^\infty \sum_{n=1}^\infty c_n(\lambda) e^{-\lambda^2 t \beta^2} \sin \left( \frac{n\pi \theta}{\beta} \right) J_{\frac{n\pi}{\beta}}(\lambda r) d\lambda.$$ 

Our goal now is to find the coefficients $c_n(\lambda)$ which fit the initial condition $h(r, \theta, 0) = r_0^{-1}\delta(r - r_0)\delta(\theta - \theta_0)$.

To find $c_n(\lambda)$, multiply the previous equation at $t = 0$ by $\sin \left( \frac{n\pi \theta}{\beta} \right)$ and integrate over $\theta$. We find

$$r_0^{-1}\delta(r - r_0) \sin \left( \frac{n\pi \theta_0}{\beta} \right) = \frac{\beta}{2} \int_0^\infty d\lambda c_n(\lambda) J_{\frac{n\pi}{\beta}}(\lambda r).$$

Next, multiply this equation by $r J_{\frac{n\pi}{\beta}}(\lambda r)$ and integrate over $r$. Using the well-known completeness relation

$$\int_0^\infty x J_{\nu}(ax) J_{\nu}(bx) dx = a^{-1}\delta(a - b),$$

we find

$$c_n(\lambda) = \frac{2\lambda}{\beta} \sin \left( \frac{n\pi \theta_0}{\beta} \right) J_{\frac{n\pi}{\beta}}(\lambda r_0).$$

Plugging this expression back into the general formula for $h$, we find

$$h(r, \theta, t) = \int_0^\infty \left( \frac{2\lambda}{\beta} \right) \sum_{n=1}^\infty e^{-\lambda^2 t \beta^2} \sin \left( \frac{n\pi \theta_0}{\beta} \right) \sin \left( \frac{n\pi \theta}{\beta} \right) J_{\frac{n\pi}{\beta}}(\lambda r_0) J_{\frac{n\pi}{\beta}}(\lambda r) d\lambda.$$ 

The $\lambda$ integral can be performed explicitly using the fact that [Gradshteyn and Ryzhik, p. 718]

$$\int_0^\infty ye^{-c^2 x} J_\nu(ax) J_\nu(bx) dx = \frac{1}{2c^2} e^{-a^2 x^2} I_\nu \left( \frac{ab}{2c^2} \right).$$

Doing so leads to the final expression that

$$h(r, \theta, t) = \left( \frac{2}{\beta t} \right) \sum_{n=1}^\infty e^{-r^2 + r_0^2} \sin \left( \frac{n\pi \theta_0}{\beta} \right) \sin \left( \frac{n\pi \theta}{\beta} \right) I_{\frac{n\pi}{\beta}} \left( \frac{rr_0}{t} \right).$$

This completes the proof of part (i). The proofs of parts (ii) and (iii) follow immediately by symmetry.

**Remark 2** For $\alpha_1 = \alpha_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, our result coincides with Caslow (1959, p. 279 or p. 379).
For certain special values of the correlation coefficient \( \rho \), the distribution functions in Lemma 3 can be simplified. We state our result in the following lemma.

**Lemma 4** Suppose the same assumptions hold as in Lemma 3(i), except that the correlation \( \rho \) can take on only the special values

\[
\rho_n = -\cos \left( \frac{\pi}{n} \right), \quad n = 2, 3, \ldots.
\]

Then the solution to the Fokker-Planck equation is

\[
p(x_1, x_2, t) = e^{a_1 x_1 + a_2 x_2 + b_4} \frac{h(x_1, x_2, t)}{\sigma_1 \sigma_2 \sqrt{1 - \rho_n^2}},
\]

where \( h \) is a finite sum of bivariate normal densities

\[
h(x_1, x_2, t) = \sum_{k=0}^{n-1} \left[ g_k^+(x_1, x_2, t) + g_k^-(x_1, x_2, t) \right].
\]

and

\[
g_k^\pm(x_1, x_2, t) = \pm(2\pi)^{-1} \exp \left( -\frac{1}{2} \left[ \left( x_1 - r_0 \cos \left( \frac{2k\pi}{n} \pm \theta_0 \right) \right)^2 + \left( x_2 - r_0 \sin \left( \frac{2k\pi}{n} \pm \theta_0 \right) \right)^2 \right] \right).
\]

**Proof of Lemma 4:** Follow the proof of Lemma 3(i) until the PDE for \( h(x_1, x_2, t) \) is derived. When \( \rho_n = -\cos(\pi/n) \), note that the angle between the lines \( L_1 \) and \( L_2 \) in Lemma 3 takes one of the special values

\[
\beta_n = \pi/n, \quad n = 2, 3, \ldots.
\]

For these angles, a method of images solution to the PDE is possible, as follows. Note that

\[
g^\pm(x_1, x_2, t; a_1, a_2) = \pm(2\pi)^{-1} \exp \left( -\frac{1}{2} \left[ (x_1 - a_1)^2 + (x_2 - a_2)^2 \right] \right)
\]

satisfies the PDE with initial condition

\[
g^\pm(x_1, x_2, 0; a_1, a_2) = \pm \delta(x_1 - a_1) \delta(x_2 - a_2).
\]

Furthermore, since the PDE is linear in \( h \), any linear combination of these \( g^\pm \)'s, with different values of \( (a_1, a_2) \) also satisfies the PDE. We want to find that particular linear combination which also satisfies the boundary and initial conditions.

Consider the case \( n = 3 \), as illustrated in Figure 2. For this correlation value, we have \( \beta_3 = \pi/3 \). Let a circle enclosing a '+' or '-' represent the solution \( g^\pm \), with the location of center of the symbol indicating the value of \( (a_1, a_2) \). The first hextant (shaded in Fig. 2 and enclosed by solid radii representing the rays \( L_1 \) and \( L_2 \)) is the region \( \theta \in [0, \pi/3] \), the region where we want to find
a solution to the PDE. The '+' in the first hextant is located at \((z_{10}, z_{20})\), which makes an angle \(\theta_0\) with respect to the \(z_1\)-axis, and is at a distance \(r_0\) from the origin. Since this is the only symbol in the first hextant, the delta function initial condition is satisfied. The other symbols are located as follows (each at distance \(r_0\) from the origin): the '−' symbols occur at angles \(\theta_0 + 2\pi/3\) and \(\theta_0 + 4\pi/3\), and the '−' symbols occur at angles \(−\theta_0\), \(−\theta_0 + 2\pi/3\), and \(−\theta_0 + 4\pi/3\). We claim that \(h(z_1, z_2, t)\) is given by the sum of these six densities, each with unit weighting. As already seen, this linear combination satisfies the PDE and initial condition, and hence we only need to show that the absorbing boundary conditions are satisfied. But as is easily seen from the symmetry of the diagram, the six densities cancel in pairs along the rays \(L_1\) and \(L_2\). Hence the absorbing boundary conditions are satisfied, and the sum of these six Gaussians is the unique solution for \(h(z_1, z_2, t)\).

The solution for other values of \(n\) follows in a similar fashion, leading to the result given.

**Remark 3** Note that the special correlation values in Lemma 4 are all negative. Thus for the double minima or double maxima densities, the result may be of limited usefulness, since we expect most assets to be positively correlated. For the density involving the minima of one asset and the maxima of another, the Lemma is more interesting, since it is applicable to assets with positive correlations of the form \(\cos(\pi/n)\).

For purposes of evaluating double lookback options, we need only the distribution of the terminal extreme values of the Brownian motions. These distributions are related to the above results by the following corollary:

**Corollary 1**

(i) The joint distribution of the minima of two correlated Brownian motions, for \(m_1 \leq 0, m_2 \leq 0\), is given by

\[
\mathcal{P}(X_1(t) \geq m_1, X_2(t) \geq m_2) = \int_{m_1}^{\infty} \int_{m_2}^{\infty} \mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, X_1(t) \geq m_1, X_2(t) \geq m_2).
\]

(ii) The joint distribution of the minimum of one Brownian motion and the maximum of another, for \(m_1 \leq 0, M_2 \geq 0\), is given by

\[
\mathcal{P}(X_1(t) \geq m_1, X_2(t) \leq M_2) = \int_{m_1}^{\infty} \int_{-\infty}^{M_2} \mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, X_1(t) \geq m_1, X_2(t) \leq M_2).
\]

(iii) The joint distribution of the maxima of two correlated Brownian motions, for \(M_1 \geq 0, M_2 \geq 0\), is given by

\[
\mathcal{P}(X_1(t) \leq M_1, X_2(t) \leq M_2) = \int_{-\infty}^{M_1} \int_{-\infty}^{M_2} \mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, X_1(t) \leq M_1, X_2(t) \leq M_2).
\]
Remark 4 For the general density given in Lemma 3, we do not believe that the integrals in the Corollary 1 can be further simplified. However, by a simple factorization we can obtain an alternative way of expressing these integrals:

\[ P(X_1(t) \geq m_1, X_2(t) \geq m_2) = e^{(m_1+m_2)h} f(r_0, \theta_0, t) \]

where

\[ f(r_0, \theta_0, t) = \frac{2}{\beta t} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi \theta_0}{\beta}\right) e^{-\frac{r_0^2}{2t}} \int_0^\beta \sin\left(\frac{n\pi \theta}{\beta}\right) g_n(\theta) d\theta \]

with

\[ g_n(\theta) = \int_0^\infty e^{-\frac{r^2}{2t}} e^{br \cos \theta + b_2 r \sin \theta} I_{\frac{n\pi}{\beta}}(\frac{r\theta_0}{t}) dr. \]

(Note: All parameters are as defined in Lemma 3.)

For the special correlation values discussed in Lemma 4, the distribution functions of Corollary 1 can be computed explicitly. We state our result in the following Corollary. Note that we consider here only the min-max case where the special correlation values \( \rho_n = \cos(\pi/n) \) are positive.

Corollary 2 The joint distribution of the minimum of one Brownian motion and the maximum of another, with constant correlation \( \rho_n = \cos(\pi/n) \), \( n = 2, 3, \ldots \), for \( x_1 \geq 0, x_2 \leq 0 \), is given by

\[ P(X_1(t) \leq x_1, X_2(t) \geq x_2) = \sum_{k=0}^{n-1} \left[ H(r_0, \frac{2k\pi}{n} + \theta) - H(r_0, \frac{2k\pi}{n} - \theta) \right] \]

where

\[ H(r_0, \frac{2k\pi}{n} \pm \theta) = \exp \left[ A_0 + A_1 \zeta_1 + A_2 \zeta_2 + \left( \frac{1}{2} A_1^2 + \frac{1}{2} A_2^2 + b \right) t \right] \times \]

\[ N_2 \left( \frac{\zeta_2 + A_2 t}{\sqrt{t}}, -\sqrt{1 - \rho_n^2} (\zeta_1 + A_1 t) - \rho_n (\zeta_2 + A_2 t), -\rho_n \right) \]

\[ \zeta_1 = -r_0 \cos \left( \frac{2k\pi}{n} \pm \theta \right) \]

\[ \zeta_2 = r_0 \sin \left( \frac{2k\pi}{n} \pm \theta \right) \]

\[ \tan \theta = \frac{\sqrt{1 - \rho_n^2}}{\rho_n - (x_1 \sigma_2/x_2 \sigma_1)} \]

and

\[ A_0 = a_1 x_1 + a_2 x_2 \]

\[ A_1 = a_1 \sigma_1 \sqrt{1 - \rho_n^2} \]

\[ A_2 = a_1 \sigma_1 \rho_n + a_2 \sigma_2 \]
and $N_z(x,y,\rho)$ is the standard bivariate normal distribution for correlation \(\rho\). The definitions of \(r_0, a_1, a_2, \) and \(b\) are the same as in Lemma 3.

**Proof of Corollary 2**: Note that

\[
\mathcal{P}(\bar{X}_1(t) \leq x_1, \bar{X}_2(t) \geq x_2) = \int_{-\infty}^{x_1} \int_{x_2}^{\infty} \mathcal{P}(X_1(t) \in dy_1, X_2(t) \in dy_2, \bar{X}_1(t) \leq x_1, \bar{X}_2(t) \geq x_2) \\
= \int_{-\infty}^{x_1} dy_1 \int_{x_2}^{\infty} dy_2 p(-y_1, y_2, t; -x_1, x_2, -\alpha_1, \alpha_2, \sigma_1, \sigma_2, -\rho) \\
= \int_{-\infty}^{x_1} dy_1 \int_{x_2}^{\infty} dy_2 p(y_1, y_2, t; -x_1, x_2, -\alpha_1, \alpha_2, \sigma_1, \sigma_2, -\rho)
\]

where we have used the results of Lemma 3(ii). The density on the last line is given by Lemma 4. Direct evaluation of the integrals leads to the result given.

We now state a theorem that summarizes the evaluation of various double lookback options discussed in Section 2. We define

\[
\begin{align*}
V_{D_{\text{max}}}(x_1, x_2) &= \max \left[0, aS_1(0)e^{\max(M_1,x_1)} - bS_2(0)e^{\max(M_2,x_2)} - K \right] \\
V_{D_{\text{min}}}(x_1, x_2) &= \max \left[0, aS_1(0)e^{\min(m_1,x_1)} - bS_2(0)e^{\min(m_2,x_2)} - K \right] \\
V_{D_{\text{LS}}}(x_1, x_2) &= \max \left[0, aS_1(0)e^{\max(M_1,x_1)} - bS_2(0)e^{\min(m_1,x_2)} - K \right]
\end{align*}
\]

for some constants \(a\) and \(b\).

**Theorem 3** The call prices \(C_{D_{\text{max}}}, C_{D_{\text{min}}}\) and \(C_{D_{\text{LS}}}\), respectively, for double maxima, double minima and double lookback spread options are determined as follows,

\[
\begin{align*}
C_{D_{\text{max}}} &= e^{-rT} \int_{0}^{\infty} dx_1 \int_{0}^{\infty} dx_2 V_{D_{\text{max}}}(x_1, x_2) \frac{\partial^2 \mathcal{P}(\bar{X}_1(t) \leq x_1, \bar{X}_2(t) \leq x_2)}{\partial x_1 \partial x_2} \\
C_{D_{\text{min}}} &= e^{-rT} \int_{-\infty}^{0} dx_1 \int_{-\infty}^{0} dx_2 V_{D_{\text{min}}}(x_1, x_2) \frac{\partial^2 \mathcal{P}(\bar{X}_1(t) \geq x_1, \bar{X}_2(t) \geq x_2)}{\partial x_1 \partial x_2} \\
C_{D_{\text{LS}}} &= e^{-rT} \int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 V_{D_{\text{LS}}}(x_1, x_2) \frac{-\partial^2 \mathcal{P}(\bar{X}_1(t) \geq x_1, \bar{X}_2(t) \leq x_2)}{\partial x_1 \partial x_2}
\end{align*}
\]

Formulas for corresponding puts can be obtained similarly.

This theorem states that the prices of double lookbacks can be obtained by integrating their final payoffs with respect to the corresponding distribution function. In general, evaluation will require a four-dimensional numerical quadrature. There exist standard numerical techniques evaluate such integrals and we expect that this methodology will be more computationally efficient than Monte-Carlo or lattice techniques.
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Table 2: Values for call and put double lookback spread options with payoffs of, respectively, $(\bar{S}_1 - S_2 - K)^+$ and $(K - \bar{S}_1 + S_2)^+$. We assume that $S_1(0) = S_2(0) = \bar{S}_1(0) = S_2(0) = 100$, $r = 0.05 \text{ yr}^{-1}$, $q_1 = q_2 = 0$, $t = 1 \text{ yr}$. Note that the correlation coefficients correspond to $\rho = \cos \frac{\pi}{8}$, $\cos \frac{\pi}{4}$, and $\cos \frac{\pi}{7}$.

In Figure 3, we present a surface and contour plot of the joint density function $-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1 \partial x_2} P(X_1(t) \leq x_1, X_2(t) \geq x_2)$, as given in Corollary 2. The parameter values chosen for the picture are $\sigma_1 = \sigma_2 = 0.2 \text{ yr}^{-1/2}$, $t = 1 \text{ yr}$, $r = 0.05 \text{ yr}^{-1}$, $q_1 = q_2 = 0$, and $\rho = 0.5$. Using the results of Theorem 3, we employ this density in a numerical quadrature evaluation of double lookback spread options.

In Table 2, we give numerical prices for various parameter values. Once again, these prices have the desired sensitivities with respect to $\sigma_1$, $\sigma_2$ and $K$. We note that as the correlation between the two assets increases, the option premiums decrease for both calls and puts. This should be intuitive, as higher correlation leads to lower volatility in $\bar{S}_1 - S_2$. Also, if we hold $\sigma_1 = \sigma_2 = 0.2$, then the option premiums for the double lookback spread in Table 2 is more expensive than those of the lookback spread in Table 1 (for fixed strike). But, as $\rho$ increases to 1, the option premiums converge to those in Table 1.

5 Semi-Lookbacks on Two Assets

To make an interesting comparison of various types of lookback options, we now introduce another class of lookback options whose payoffs depend on the extreme value of one asset and the final value
of another asset. For example, consider an option to buy the maximum of $S_1$ at the final value of $S_2$ or an option to sell the minimum of $S_2$ at the final value of $S_1$. We call these options semi-lookbacks. In this section, we present the probability density functions necessary for the valuation of semi-lookbacks. We define

$$P(X_1(t) \in dx_1, X_2(t) \in dx_2) \equiv f_{+0}(x_1, x_2, t) \, dx_1 \, dx_2$$

$$P(X_1(t) \in dx_1, X_2(t) \in dx_2) \equiv f_{0-}(x_1, x_2, t) \, dx_1 \, dx_2$$

The following lemma is useful for the valuation of semi-lookback options.

**Lemma 5** The density function of the maximum of one Brownian motion and the end point of another Brownian motion is given by:

$$f_{+0}(x_1, x_2, t) = \frac{1}{\sigma_1 \sigma_2 \pi t^2} \exp(A) \left[ t \sqrt{1 - \rho^2} \exp\left(- \frac{B^2}{2}\right) - d \sqrt{2\pi t} N(-B) \right]$$ \hspace{1cm} (7)

with constants

$$A = \frac{2\alpha_1 x_1}{\sigma_1^2} - \frac{\alpha_2^2 t}{2\sigma_1^2} - \frac{c^2 \rho^2}{2t(1 - \rho^2)} + \frac{d^2}{2t(1 - \rho^2)}$$

$$B = \frac{1}{\sqrt{t(1 - \rho^2)}} \left( \frac{x_1}{\sigma_1} + \frac{\alpha_1 t}{\sigma_1} \left( 1 - \rho^2 \right) + c \rho^2 \right)$$

where

$$c = \frac{x_2 - \alpha_2 t}{\sigma_2 \rho} - \frac{2x_1 - \alpha_1 t}{\sigma_1}$$

$$d = \frac{\alpha_1 t(1 - \rho^2)}{\sigma_1} + c \rho^2$$

Similarly,

$$f_{0-}(x_1, x_2) = f_{+0}(-x_2, x_1, t; -\alpha_2, \alpha_1, \sigma_2, \sigma_1, -\rho).$$

**Proof:** Set

$$B_1(t) = w_1^*(t)$$

$$B_2(t) = -\frac{\rho}{\sqrt{1 - \rho^2}} w_1^*(t) + \frac{1}{\sqrt{1 - \rho^2}} w_2^*(t)$$

Then, $(B_1, B_2)$ is a standard (uncorrelated) two-dimensional Brownian motion. Hence we get for the density function $f_{+0}$,

$$f_{+0}(x_1, x_2, t) \, dx_1 \, dx_2$$

$$= P(X_1(t) \in dx_1; X_2(t) \in dx_2)$$
\[ P(\max_{0 \leq s \leq t} (\alpha_1 s + \sigma_1 w_1(s)) \in dx_1; \alpha_2 t + \sigma_2 w_2(t) \in dx_2) \]
\[ = P(\max_{0 \leq s \leq t} (\alpha_1 s + \sigma_1 B_1(s)) \in dx_1; \alpha_2 t + \sigma_2 (\rho B_1(t) + \sqrt{1 - \rho^2} B_2(t)) \in dx_2) \]
\[ = P(\max_{0 \leq s \leq t} (\alpha_1 s + \frac{\sigma_1}{\sigma_2} B_1(s)) \in dx_1; \frac{\alpha_2}{\sigma_2} t + B_1(t) \in dx_2 - \sqrt{1 - \rho^2} \rho B_2(t)) \]
\[ = \int_{-\infty}^{\infty} P(\max_{0 \leq s \leq t} (\alpha_1 s + \frac{\sigma_1}{\sigma_2} B_1(s)) \in dx_1; \frac{\alpha_2}{\sigma_2} t + B_1(t) \in dx_2 - \sqrt{1 - \rho^2} \rho B_2(t)) \times P(B_2(t) \in dx) \]
\[ = \int_{-\infty}^{\infty} P(\max_{0 \leq s \leq t} (\alpha_1 s + \frac{\sigma_1}{\sigma_2} B_1(s)) \in dx_1; \frac{\alpha_2}{\sigma_2} t + B_1(t) \in dx_2 - \sqrt{1 - \rho^2} \rho x + \left(\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2}\right) t) \times P(B_2(t) \in dx) \]
\[ = \left[ \int_{a}^{\infty} \frac{2(\tilde{y} - \bar{x})}{\sqrt{2\pi t^3}} \exp\left(-\frac{((\tilde{y} - \bar{x})^2}{2t}\right) \exp\left(\frac{\alpha_1}{\sigma_1} \tilde{y} - \frac{1}{2} \frac{(\alpha_1^2)}{\sigma_1^2} t\right) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \right] \frac{dx_1 dx_2}{\rho \sigma_1 \sigma_2} \]

where we have set \( \tilde{y} = \frac{x}{\sigma_1} \) and \( \bar{x} = \frac{x}{\sigma_2} - \sqrt{1 - \rho^2} \rho x + \left(\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2}\right) t \). In the last equation, we used the density corresponding to the distribution given in Lemma 1. The lower integral bound \( a \) is defined by the condition \( \tilde{y} \geq \bar{x} \) which is equivalent to
\[ x \geq \frac{\rho}{\sqrt{1 - \rho^2}} \left(\frac{x_2}{\sigma_2} - \frac{x_1}{\sigma_1} + \left(\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2}\right) t\right) \equiv a \]

Evaluating the integral yields the result given. The result for \( f_{-0}(x_1, x_2, t) \) follows by symmetry. 

In Figure 4, we plot the semi-lookback density \( f_{+0} \) for a specific choice of parameter values. As an example of the use of this density, we consider the evaluation of the following class of semi-lookback options. We define the call options as
\[ V_{S_{\max}}(x_1, x_2) = \max\left[0, S_1(0)e^{\max(M_1, x_1)} - S_2(0)e^{x_2} - K\right] \]
\[ V_{S_{\min}}(x_1, x_2) = \max\left[0, S_1(0)e^{\max(M_2, x_2)} - S_2(0)e^{x_1} - S_2(0)e^{\min(M_2, x_2)} - K\right] \]

with corresponding formulas for the put options. The values of these options are given by the following theorem which makes use of the density derived in Lemma 5.

**Theorem 4** Let \( C_{S_{\max}} \) and \( C_{S_{\min}} \) be the prices at time 0 of a semi-lookback call options with payoffs \([S_1(T) - S_2(T) - K]^+\) and \([S_1(T) - S_2(T) - K]^+\), respectively. Then,
\[ C_{S_{\max}} = e^{-rT} \int_0^\infty dx_1 \int_{-\infty}^\infty dx_2 V_{S_{\max}}(x_1, x_2) f_{+0}(x_1, x_2, T) \quad (8) \]
\[ C_{S_{\min}} = e^{-rT} \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{0} dx_2 V_{S_{\min}}(x_1, x_2) f_{-0}(x_1, x_2, T). \quad (9) \]
In Table 3, we list values of semi-lookback spread options for various parameter values, obtained by numerical quadrature of the integrals in Theorem 4. Comparing Tables 2 and 3, we find that for a fixed set of volatilities, correlation, and strike parameters, the call premiums for semi-lookback spread options are cheaper than those of the double lookback options. However, the put premiums should be more expensive. As the correlation between two assets increases, the option premiums decrease for both calls and puts.

6 Double Knockouts

The analysis presented in the previous sections can be readily used to value knockout options based on two assets. In a standard one-asset knockout call, the option pays $\max(S_1(T) - K, 0)$, subject to a condition that the stock price $S_1(t)$ never hits a fixed boundary (which could either be larger or smaller than the initial stock price). Merton (1973) was the first to provide a solution to value this option. We refer the reader to Rubinstein (1992) for a complete list of one-asset knockout options.

In the case with two traded assets, we consider a general contingent claim which pays, at the maturity date,

$$V(T) = f(S_1(T), S_2(T))$$

for some function $f$, subject to the conditions that $S_1(t)$ and/or $S_2(t)$ never hit some pre-determined boundaries. We shall call such options as double knockouts. Obviously, the keys to valuing double knockouts are the following probability density functions, which are derived in Lemma 3:

$$P(X_1(T) \in dx_1, X_2(T) \in dx_2, X_1 \leq K_1, X_2 \leq K_2)$$

$$P(X_1(T) \in dx_1, X_2(T) \in dx_2, X_1 \geq K_1, X_2 \leq K_2)$$

$$P(X_1(T) \in dx_1, X_2(T) \in dx_2, X_1 \geq K_1, X_2 \geq K_2)$$

Integrating these functions over the payoff $V(T)$ gives rise to the desired option premium.

In Figures 5 and 6, we present plots of the knockout density functions corresponding to the distributions derived in Lemmas 2 and 3. In both cases, only a few terms in the infinite series are typically needed to obtain good convergence.

Similar to the semi-lookbacks considered in the previous section, we can also evaluate a special class of knockout options which are European calls or puts written on one asset, subject to a condition that the value of another asset never hits a pre-determined boundary. The probability densities necessary for this type of options are

$$P(X_1(T) \in dx_1, X_2 \leq K_2)$$
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Table 3: Values for call and put semi-lookback spread options with terminal payouts of, respectively, $(S_1 - S_2 - K)^+$ and $(K - S_1 + S_2)^+$. We assume that $S_1(0) = S_2(0) = S_1(0) = 100$, $r = 0.05 \text{ yr}^{-1}$, $q_1 = q_2 = 0$, $t = 1 \text{ yr}$.

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\[ P(X_1(T) \in dx_1, X_2 \geq K_2) \]

which can be obtained by integrating the densities in Lemma 5.\(^2\)

7 Conclusions

We have presented a technique for pricing lookback options on two assets following lognormal distribution. The essential part of this technique is to derive the probability distribution function of the extreme values of two correlated Brownian motions. With this technique, the prices of many kinds of lookback and knockout options can be calculated efficiently. We hope that our pricing technology will be useful for future research that involves the extreme values of two correlated geometric Brownian motions.

\(^2\)Heynen and Kat (1994) have derived the same density function and obtained closed-form solutions for semi-knockout options.
References


Joint Density of the Maximum and Minimum of One Brownian Motion

FIGURE 1
\[ \rho = -\frac{1}{2} \]
\[ \beta = \frac{\pi}{3} \]
Joint Density of the Maximum and Minimum of Two Correlated Brownian Motions

FIGURE 3
Joint Density of the Maximum and End Point of Two Correlated Brownian Motions
Knockout Density (Min = -0.2, Max = 0.2)
Double Knockout Density (Barriers at -0.1, Rho = 0.5)