Imperfect Competition in Securities Markets with Diversely Informed Traders

by

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1 Introduction

A trader with long-lived information faces a dilemma as to whether to trade heavily in the short run in the hope of getting a good price before the information leaks out, or whether to spread his trades to minimize the price impact. Kyle (1985) develops a model in which a single privately informed trader with long-lived information optimally exploits his monopoly power over time. The informed trader, and noise traders who trade randomly, submit orders to a risk-neutral market maker who sets the price equal to his expectation of the risky asset payoff. In equilibrium, the informed trader trades in a gradual manner so that his information is incorporated into the price at a linear rate, and in a continuous auction economy the expected profit of the informed trader is twice that in the single auction economy. The financial market in this model is semi-strong form efficient but not strong-form efficient.

When there is more than one informed trader who trades strategically, competition between them causes information to be revealed more rapidly. Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) (HS-FV) have found that when there is more than one perfectly informed trader who trades strategically, there exists a unique linear equilibrium in which the informed traders trade very aggressively. The entry of even one additional informed trader causes nearly all of the private information to be revealed to the market maker extremely rapidly so that the depth of the market becomes extremely large almost immediately. Consequently, a market with more than one informed trader is close to perfectly strong-form efficient, in that the security prices reflect virtually all available information.

In this paper, we generalize the HS-FV models to the case in which the private signals of the informed traders are noisy and less than perfectly correlated. While models of informed investors with diverse signals have been extensively studied in the literature on competitive noisy rational expectations equilibrium, their use in non-competitive dynamic trading models has been limited by the recognition of an infinite regression problem. We develop a fixed point method for solving the infinite regression problem and use it to analyze a model in which the information structure is symmetric, but informed traders receive different signals.

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1Spiegel and Subrahmanyam (1992) replace the noise traders in the Kyle model with strategic, utility maximizing hedgers who trade to hedge endowment shocks.
2HS assumed normality of the risky asset value while FV allowed for elliptically contoured distributions. FV went on to show that the expected profits of the informed traders vanish as trading occurs more frequently.
3Even if the informed traders buy information from the same information seller, the information seller may want to add diversified noise to the signal before selling it to the information buyers. For a discussion of diversified signals and value of information, see Admati and Pfleiderer (1986).
5After the completion of this paper, we become aware of a working paper by Foster and Viswanathan (1993) with similar results using a slightly different parameterization.
In a dynamic multiperiod auction model, each informed trader learns about the private information of other informed traders through the time series of prices and revise his expectation of the value of the risky asset accordingly. It is natural then to suppose that informed traders will restrict their trading in the early rounds in order to conserve their private information advantage, as in Kyle (1985). This intuition is confirmed in a linear symmetric equilibrium. We find that as long as the private signals of the informed traders are not perfectly correlated, they trade conservatively in order to limit the leakage of their private information into the price. The expected profits of the informed traders do not vanish as the market opens more frequently as happens when the private signals are perfectly correlated. Market depth initially increases over time but decreases towards the final auction. When auctions occur continuously, all private information is revealed by the end.

The intuition for our results is that when traders have diverse information, the idiosyncratic error in each informed trader’s private signal makes him act like a monopolist of his own private signal and causes him to trade more conservatively. For example, suppose that there are two informed traders who observe imperfectly correlated noisy signals and that there exists an equilibrium in which traders trade so aggressively in the first round that virtually all private information is incorporated into the price. We show in the following arguments that informed traders will deviate from such a proposed equilibrium. Since the first informed trader trades very aggressively in the first trading session, his information will be almost fully incorporated into the price after the first trading session. Then the second informed trader will be better off not trading in the first session but waiting until the later rounds where he can trade as an information monopolist. By not trading in the first round, the second trader manipulates the beliefs of both the first informed trader and the market maker. The market maker, believing that the price aggregates all private information almost perfectly (which is false when the second trader deviates from the equilibrium), will make the price insensitive to the order flow in later trading rounds. The first informed trader, drawing the wrong inference from the price, has incorrect expectations about the value of the risky asset and believes that the price is very close to the conditional expectation given all private information. As a result, the first informed trader will trade a very small amount while the second informed trader will earn large profits from trading in later rounds in the off-equilibrium strategy, which breaks the proposed equilibrium. Consequently, informed traders trade conservatively so that prices incorporate the private information gradually.

However, if the two informed traders both observe a perfectly informative signal as in HS-FV,⁶ i.e., the value of the risky asset, the case will be different. In their perfect revealing equilibrium, if the second informed trader deviates by not trading in the first round, the second informed trader can manipulate the belief of the market maker but not the belief of the first informed trader. In the subsequent trading rounds the first informed trader again competes with the second informed trader on the same signal and drives both traders’ profits

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⁶The same argument goes through when the two informed traders observe the same noisy signal. Both traders will trade very aggressively on the difference between price and the expectation of the risky asset value given the private signal.
from later rounds close to 0. Therefore, the proposed equilibrium is robust to this kind of deviation.\(^7\)

The paper is organized as follows: Section 2 presents the economic structure and discusses the infinite regression problem. Section 3 analyzes the equilibrium in a multiple-period setting. Section 4 discusses a limiting result when the number of auctions goes to infinity. Section 5 presents the numerical examples for both the discrete time model and the continuous auction model. Section 6 concludes the paper.

### 2 The Economic Structure and the Infinite Regression Problem

Following Kyle (1985), a single risky asset is traded by three types of traders: risk neutral informed traders who possess private information about the liquidation value of the risky asset, \(v\), liquidity traders whose demands are exogenous, and a competitive risk neutral market maker. The market maker absorbs the net demands that others trade and sets the price equal to the expected liquidation value of the risky asset given the order flow. We assume that each of the \(M\) informed traders, \(i, i = 1, \ldots, M\), observes a signal of the form

\[
z_i = v + \delta + \epsilon_i.
\]  

The total random demand by the noise traders is denoted \(u\) and \(v, \delta, \epsilon_i, u\) are normally and independently distributed with mean 0.\(^8\). The variances of \(v, \delta, \epsilon_i, u\) are \(\sigma_v^2, \sigma_\delta^2, \sigma_{\epsilon_i}^2, \sigma_u^2\) respectively. The informed traders are assumed to be risk neutral.

In this model, informed traders have different but correlated private signals. This gives rise to a potential infinite regression problem, which may be thought of as follows. Consider the case of two informed traders. Let \(x_i\) denote the optimal trading strategy of informed trader \(i\) and \(E_i[\cdot]\) denote his expectation. The first informed trader’s optimal trading strategy, \(x_1\), depends on his own private signal, \(z_1\), and his expectation of market maker’s price which depends on the total order flow. Therefore, \(x_1\) depends on \(z_1\), and the the first informed trader’s expectation of the second investors’ optimal trading strategy, \(E_1[x_2]\). Similarly, the second informed trader’s optimal trading strategy also depends on his own private signal, \(z_2\) and his expectation about the first informed trader’s optimal strategy, \(E_2[x_1]\). This implies that the first informed trader’s optimal trading strategy depends on

\(^7\)However, the market maker will incur very large costs if the second trader deviates from the equilibrium by mistake.

\(^8\)The model can be easily extended to include more general correlation structure among \(v, \delta\), and \(\epsilon_i\) as long as the information structure among the informed traders is symmetric in which case the average of their private signals is a sufficient statistic of all private information.
his private signal, \( z_1 \), his expectation of the second informed trader’s private signal, \( E_1[z_2] \), and his expectation about the second informed trader’s expectation of his optimal strategy, \( E_1[E_2[x_1]] \). Returning to the second informed trader, his optimal trading strategy now depends on \( z_2, E_2[z_1], E_2[E_1[z_2]], E_2[E_1[E_2[x_1]]] \). In this way, there appears to be a problem of infinite regression.

This problem of infinite regression can be solved if we can show that there exists an equilibrium in which the form of each informed trader’s demand function, expressed in his private information and public information, is common knowledge. Conforming to the literature on non-competitive trading, we restrict our analysis to linear Cournot equilibria in which each informed trader’s equilibrium demand is optimal given others’ equilibrium demand. We then show that the infinite regression problem reduces to a fixed point problem.

We conjecture that in equilibrium the optimal order of informed trader \( i \), \( x_i \), is given by

\[
x_i = \beta_i z_i.
\]  

(2)

The aggregate order of the informed traders is denoted by \( x = \sum_{i=1}^{M} x_i \) and in the conjectured equilibrium the market maker sets the price, \( p \), according to the linear rule,

\[
p = \lambda (X + u).
\]  

(3)

The vector \( \beta_1, \ldots, \beta_M, \lambda \) is common knowledge and, given the assumption that other informed traders follow the conjectured equilibrium strategy, informed trader \( i \)’s optimization problem is to maximize his expected profit given his private signal \( z_i \). Let \( \pi_i \) denote the profit from trading for informed trader \( i \), his expected profit from trading, \( E[\pi_i | z_i] \), is given by

\[
E[\pi_i | z_i] = E[x_i(v - p) | z_i] = E[x_i(v - \lambda \{x_i + \sum_{j \neq i} \beta_j z_j\}) | z_i].
\]

The first order condition of the problem is

\[
E[v - \lambda (x_i + \sum_{j \neq i} \beta_j z_j) | z_i] - \lambda x_i = 0.
\]

Since the prior means of \( v, \delta, \epsilon_i \) are 0, multivariate normality implies that both \( E[v | z_i] \) and \( E[z_j | z_i, j \neq i] \) are proportional to \( z_i \) as conjectured, therefore, the optimal demand of

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9This approach is similar to that in Townsend (1983), p569-575.
10There are potentially many Stackelberg equilibria, with multiple leaders and followers and there could also exist many non-linear equilibria.
11We are implicitly using the condition that the prior mean of \( v \) is 0. If the prior mean is \( p_0 \), the demand function should be \( x_i = \beta_i (z_i - p_0) \), where \( p_0 \) is the prior mean of \( v \).
informed trader $i$ is of the form given by (2) and $\beta_i$ is a function of $\lambda, \beta_j, j \neq i$ written as

$$\beta_i = f_i(\lambda, \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_M).$$

The market efficiency condition, that the market maker sets price equal to the expectation of the liquidation value of the risky asset given the order flow, gives (3) where $\lambda$ is also a function of $\beta_1, \ldots, \beta_M$ written as

$$\lambda = g(\beta_1, \ldots, \beta_M).$$

The infinite regression problem then reduces to the problem of finding a fixed point of the mapping $(f_1, \ldots, f_M, g)$ that maps the Euclid space $\mathbb{R}^{M+1}$ into itself.

A similar analysis carries through in a dynamic model with $N$ auctions. We assume that the liquidity traders trade $\Delta u_n$ at the $n$th auction and conjecture a linear equilibrium in which the optimal order of informed trader $i, \Delta x_{ni}, \forall i$ at period $n, n = 1, \ldots, N$ is given by

$$\Delta x_{ni} = L_{ni}(z_i, p_1, \ldots, p_{n-1}).$$

The total order flow $\Delta x_n$ at the $n$th auction is

$$\Delta x_n = \sum_{i=1}^{M} \Delta x_{ni} + \Delta u$$

and the market maker sets the price according to

$$p_n = L_{nm}(\Delta x_1, \ldots, \Delta x_n)$$

where $L_{ni}(\cdot), L_{nm}(\cdot)$ are linear functions in $\mathbb{R}^n$. What we need to show is that given the price function $L_{nm}(\cdot)$ and other informed traders' equilibrium demand functions, $L_{nj}(\cdot), j \neq i, L_{ni}(\cdot)$ is the optimal trading strategy for trader $i$ at $n$th auction. Then, given the informed traders' equilibrium demand functions, $p_n = L_{nm}(\cdot)$ is the expected value of the risky asset for the market maker at the $n$th auction. The infinite regression problem again reduces to a fixed point problem.

In the next section, we use this approach to solve the infinite regression problem in the case where the information structure is symmetric. When the informed traders have a symmetric information structure, the $(M + 1)N(N + 1)/2$-dimensional fixed point problem can be reduced to a $2N$-dimensional fixed point problem and the fixed point can be found using a simple recursive procedure. Whether the fixed point problem can be solved when the information structure is asymmetric, and more generally, the necessary and sufficient conditions for the existence of the fixed point, are beyond the scope of this simple paper.
3 The Dynamic Auction Model with Symmetric Diverse Signals

Throughout the analysis we set the variance $\sigma_{\epsilon i}^2 = M \sigma_{\epsilon}^2$ so that the precision of the private signals is identical for each informed trader. To derive the equilibrium, it is convenient to define a sufficient statistic of the informed traders' private information, $Y$, which is equal to the average of the private signals. $Y$ is related to the liquidation value of the risky asset $v$ and investor $i$'s private signal, $z_i$, as follows,

$$Y = \frac{\sum_{i=1}^{M} z_i}{M} = v + \delta + \frac{\sum_{i=1}^{M} \epsilon_i}{M}$$

$$E[v|Y] = \frac{\sigma_v^2 Y}{\Sigma_0} = kY$$

$$E[Y|z_i] = \frac{\Sigma_0}{\Sigma_0 + (M-1)\sigma_{\epsilon}^2} z_i = s_0 z_i,$$

where

$$\Sigma_0 = \text{var}[Y] = \sigma_v^2 + \sigma_{\delta}^2 + \sigma_{\epsilon}^2$$

$$k = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{\delta}^2 + \sigma_{\epsilon}^2}$$

$$s_0 = \frac{\Sigma_0}{\Sigma_0 + (M-1)\sigma_{\epsilon}^2}.$$

We set the variance of $\epsilon_i$ to be proportional to $M$ so that the variance of the sufficient statistic of the private information does not change with $M$.

We assume that there are $N$ auctions starting at time 0 and ending at time 1. Following Kyle (1985), let $\Delta t_n$ denote the time interval between the $n$th auction and the previous auction. Let $\Delta u_n$ be the aggregate order submitted by noise traders at the $n$th auction. We assume that $\Delta u_n$ is serially uncorrelated and normally distributed with mean zero and variance of $\sigma_u^2 \Delta t_n$. Let $\Delta x_n$ denote the total order submitted by the informed traders and $\Delta x_{ni}$ denote the order submitted by informed trader $i$. Finally, let $\pi_n$ denote the total profits of the $i$th informed trader from positions acquired at all future auctions $n, \cdots, N$. Trading takes place through a competitive risk neutral market maker who observes the combined order flow $\Delta x_n + \Delta u_n$ and sets the price equal to his expectation given the order flow. The market price of the risky asset at the $n$th auction is denoted $p_n$. Since the prior mean of $v$ is 0, we set the price before the auctions start, $p_0$, to be 0.

Let informed trader $i$ conjecture that all other informed traders submit orders of the form $\beta_n \Delta t_n (kz_j - p_{n-1}), j \neq i$. Given this conjecture, we derive the optimal strategy of
informed trader $i$, and show that he also submits an order of the form $\beta_n \Delta t_n (kz_i - p_{n-1})$. Since $\beta_n$ is independent of $i$, the price which depends on the aggregate order flow is a noisy signal of the sufficient statistic of the private information. Due to the symmetric structure of the informed traders' private signals, each informed trader's estimation of the aggregate trades by the informed traders is reduced to the estimation of the sufficient statistic of the private signals in the market. Therefore, the optimal trading strategy of informed trader $i$ depends only on the series of prices and his private signal. This permits a solution of the infinite regression problem. We show in the next theorem that a linear equilibrium exists in this model.

**Theorem 1** There exists a unique recursive linear symmetric equilibrium in which the demands for the risky asset and the market price are as described below,

$$\Delta x_{ni} = \beta_n (kz_i - p_{n-1}) \Delta t_n, \quad \Delta x_n = \sum_{i=1}^{M} \Delta x_{ni}, \quad \Delta p_n = \lambda_n (\Delta x_n + \Delta u_n) \quad (4)$$

$$\Sigma_n \equiv \text{var}[Y|\Delta x_1 + \Delta u_1, \cdots, \Delta x_n + \Delta u_n] \quad (5)$$

$$E[\pi_{ni}|z_i, p_1, \cdots, p_{n-1}] = \alpha_{n-1} (kz_i - p_{n-1})^2 + \delta_{n-1} \quad (6)$$

$$s_{n-1} = \frac{\Sigma_{n-1}}{\Sigma_{n-1} + (M - 1)\sigma_u^2} \quad (7)$$

$$\beta_n \Delta t_n = \frac{s_{n-1} - \gamma_n \lambda_n}{\lambda_n [1 + (1 - \gamma_n \lambda_n) s_{n-1} M]} \quad (8)$$

$$\alpha_{n-1} = \beta_n s_{n-1} \Delta t_n (1 - \lambda_n M \beta_n \Delta t_n) + \alpha_n (1 - \lambda_n M \beta_n s_{n-1} \Delta t_n)^2 \quad (9)$$

$$\gamma_{n-1} = [\gamma_n + \beta_n \Delta t_n (1 + (1 - \gamma_n \lambda_n) s_{n-1} M)] [1 - \beta_n \Delta t_n \lambda_n (M - 1)] (1 - M \beta_n \Delta t_n \lambda_n) \quad (10)$$

$$\eta_{n-1} = \frac{[1 - (M - 1) \beta_n \Delta t_n \lambda_n]^2}{4 \lambda_n (1 - \eta_n \lambda_n)} \quad (11)$$

$$\lambda_n = M k^2 \Sigma_n \beta_n / \sigma_u^2 \quad (12)$$

$$\Sigma_n = (1 - M \beta_n \Delta t_n \lambda_n) \Sigma_{n-1} \quad (13)$$

$$\delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \Delta t_n \sigma_u^2 + \alpha_n \lambda_n^2 M^2 (M - 1) \beta_n^2 k^2 s_{n-1} \sigma_u^2 \Delta t_n^2 \quad (14)$$

for all $n = 1, \cdots, N$ and for all informed traders $i = 1, \cdots, M$. subject to the boundary conditions

$$\alpha_N = 0, \quad \gamma_N = 0, \quad \eta_N = 0, \quad \delta_N = 0 \quad (15)$$

$$\Sigma_N = \frac{\Sigma_{N-1}}{1 + M s_{N-1}}, \quad \lambda_N = \frac{k \sqrt{s_{N-1} \Sigma_{N-1} M}}{\sigma_u \sqrt{\Delta t N (1 + M s_{N-1})}}, \quad \beta_N = \sqrt{\frac{s_{N-1} \sigma_u}{M \Delta t N \Sigma_{N-1} k}} \quad (16)$$

and the second order condition$^{12}$

$$\lambda_n (1 - \eta_n \lambda_n) > 0 \quad (17)$$

$^{12}$The parameters $\gamma_n, \eta_n$ relate to the off-equilibrium expected profits for the representative informed trader and is discussed in the following proof.
Proof: Consider a symmetric linear equilibrium, let $I_{ni}$ denote the information set of informed trader $i$ before the $n$th auction, and $F_n$ denote the information set of the market maker at the $n$th auction. First notice that in the proposed equilibrium, $\Delta x_n + \Delta u_n = M\beta_\Delta t_n (kY - p_{n-1}) + \Delta u_n$, so that the aggregate order flow is a noisy signal of the sufficient statistic of the private signals $Y$. Given the trading strategies of the informed traders, the market maker sets the price $p_n = E[v|F_n] = E[E[v|Y]|F_n]$, which is linear in the aggregate order flow due to the assumption of multivariate normality. From the $n$th auction price, informed trader $i$ can estimate the total expected trading by other informed traders in the next trading session through his updated estimate of $Y$. Since the only variable informed trader $i$ needs to estimate is $Y$, the infinite regression problem is easily solved in this model.

Given the proposed equilibrium described in the theorem, each informed trader uses price to update his estimate of $Y$. Informed trader $i$ may have an incentive to deviate from the proposed equilibrium so that other traders estimate the wrong $Y$ from the price, while trader $i$ who estimates $Y$ correctly gains an advantage over other traders. In addition, when trader $i$ deviates from the proposed equilibrium strategy, his expected profits in future tradings may be different from equation (6). Therefore it remains to be shown that informed traders have no incentive to deviate from the equilibrium. This is shown in Appendix A.

By the market efficiency assumption of the market maker, $\lambda_n$ is a regression coefficient of $v$ on $\Delta x_n + \Delta u_n$, given $F_n$. Normality assumption implies that

$$\lambda_n = \frac{M\beta_n k^2 \Sigma_{n-1}}{M^2 k^2 \beta_n^2 \Sigma_{n-1} \Delta t_n + \sigma_u^2},$$  \hspace{1cm} (18)

and

$$\Sigma_n = \frac{\sigma_u^2 \Sigma_{n-1}}{M^2 \beta_n^2 k^2 \Sigma_{n-1} \Delta t_n + \sigma_u^2}. \hspace{1cm} (19)$$

(12), (13) can be derived from equations (18), (19). Boundary condition (15) essentially means that there will be no more profits for the informed traders after the auction is completed. Boundary condition (16) can be derived in a single auction model analyzed in Admati and Pfleiderer (1988). The second order condition (17) is derived in Appendix A.

Q.E.D.

In the next proposition, we provide a procedure to solve the recursive system described above. The technique is similar to the recursive method used in Holden and Subrahmanyam (1992). Let $q_n \equiv \gamma_n \lambda_n$. we have

**Proposition 1** The recursive system of the equilibrium described in Theorem 1 can be solved starting from $q_N = 0$ and a conjecture of $s_{N-1}$ and iterating backward for $q_{N-1}, \ldots, q_1, s_{N-2}, \ldots, s_0$ using the following equations,

$$M(s_{n-1} - q_{n-1})s_{n-2} + [1 - M(s_{n-1} - q_{n-1})]s_{n-2} - s_{n-1} = 0 \hspace{1cm} (20)$$
\[ M s_{n-2} \left( \frac{\Delta t_{n-1}}{\Delta t_n} \right) q_{n-1}^3 - (1 + M s_{n-2}) \left( \frac{\Delta t_{n-1}}{\Delta t_n} \right) q_{n-1}^2 - C_n q_{n-1} + C_n s_{n-2} = 0, \]  

where

\[ C_n = \frac{s_{n-1}^2 [1 + M q_n (1 - s_n)] [1 + s_{n-1} + (M - 1 - M s_{n-1}) q_n]^2}{(s_{n-1} - q_n) [1 + (1 - q_n) M s_{n-1}]^2}. \]

If the resulting \( s_0 \) derived from the recursive solutions is different from the initial condition, \( s_{N-1} \) is revised until the \( s_0 \) derived from the recursive solutions is close to the initial condition.

Given the solution of \( q_n, s_{n-1}, n = 1, \cdots, N \), each of the following variables can be obtained by the following expressions,

\[ \Sigma_n = \frac{(M - 1) \sigma_u^2 s_n}{1 - s_n} \]  

\[ \lambda_n = \left( \frac{M k^2 \Sigma_n (s_{n-1} - q_n)}{\Delta t_n \sigma_u^2 [1 + (1 - q_n) M s_{n-1}]} \right)^{\frac{1}{2}} \]  

\[ \beta_n = \frac{\lambda_n \sigma_u^2}{M k^2 \Sigma_n}. \]

4 A Limiting Result

When informed traders have perfect information about the liquidation value of the risky asset \( v \), HS-FV have shown that the market approaches strong-form efficiency as trading occurs frequently. We have argued informally in the introduction that when informed traders have idiosyncratic errors in their signals, they would prefer to trade smoothly so that price gradually aggregates private information. For each informed trader, the idiosyncratic error in his signal causes him to act like a monopolist of his own signal as in Kyle (1985). As shown by the numerical results in the next section, when \( N \to \infty \), \( \lambda_n \) and \( \Sigma_n \) converge to smooth functions of calendar time \( t \). We now examine the analytic limits of the above recursive system when the number of auctions goes to infinity. We set \( t_n = n/N \). Let \( \Sigma_n, \lambda_n, \beta_n, \gamma_n, \delta_n, \alpha_n, s_{n-1} \) be defined as continuous function \( \Sigma(t), \lambda(t), \alpha(t), \delta(t) \), etc., by the convention \( \lambda(t) = \lambda_{n-1} \) for all \( t \in [t_{n-1}, t_n) \), etc. We then have the following assertion:

**Assertion 1** Consider a sequence of sequential equilibria such that \( \max_n |\Delta t_n| \to 0 \). If \( \lambda(t), \beta(t), \Sigma(t), \alpha(t), \delta(t) \) converge to continuous and differentiable functions of time \( t \) in any closed interval in \([0,1)\), then the limit functions are given by

\[ \lambda(t) = \frac{k}{\sigma_u} \sqrt{\alpha \Sigma(t)^{2-2/M} e^{(1-1/M)\alpha^2/\Sigma(t)}} \]  

10
\[ \beta(t) = \frac{\sigma_u}{Mk} \sqrt{a} \Sigma(t)^{1-2/M} e^{(1-1/M)\sigma^2_t/\Sigma(t)} \]  

(26)

\[ \int_{\Sigma(t)}^{\Sigma_0} \sigma^{(4/M-4)} e^{2(1/M-1)\sigma^2_t/\sigma} d\sigma = at \]  

(27)

\[ \alpha(t) = \frac{\sigma_u}{Mk\sqrt{a}} \int_0^{\Sigma(t)} \frac{[\sigma + (M - 1)\sigma^2_t] e^{(1/M-1)\sigma^2_t/\sigma}}{[\Sigma(t) + (M - 1)\sigma^2_t]^2 \sigma^{2-2/M}} d\sigma \]  

(28)

\[ \delta(t) = k^2 \int_0^{\Sigma(t)} \alpha(\sigma) d\sigma = \frac{k \sigma_u}{M \sqrt{a}} \int_0^{\Sigma(t)} \frac{[\Sigma(t) - \sigma] e^{(1/M-1)\sigma^2_t/\sigma}}{[\Sigma(t) + (M - 1)\sigma^2_t]^2 \sigma^{2-2/M}} d\sigma, \]  

(29)

where

\[ a = \int_0^{\Sigma_0} \sigma^{4/M-4} e^{2(1/M-1)\sigma^2_t/\sigma} d\sigma. \]

Justification for Assertion 1: From equations (8)-(14) we obtain

\[ s_{n-1} - \gamma_n \lambda_n = O(\Delta t_n) \]  

(30)

\[ \frac{\alpha_n - \alpha_{n-1}}{\Delta t_n} = \beta_n s_{n-1} (2\alpha_n \lambda_n M - 1) + O(\Delta t_n) \]  

(31)

\[ \frac{\gamma_n - \gamma_{n-1}}{\Delta t_n} = \beta_n [Ms_{n-1}(\gamma_n \lambda_n - 1) - 1 + (2M - 1)\gamma_n \lambda_n + O(\Delta t_n) \]  

(32)

\[ 1 - 2\eta_n \lambda_n = O(\sqrt{\Delta t_n}) \]  

(33)

\[ \frac{\epsilon_n - \epsilon_{n-1}}{\Delta t_n} = -\alpha_n \lambda_n^2 \sigma_u^2 \]  

(34)

\[ \frac{\Sigma_n - \Sigma_{n-1}}{\Delta t_n} = -M \beta_n \lambda_n \Sigma_{n-1} \]  

(35)

Standard convergence results for converting difference equations into differential equations allow us to conclude that the solution for the continuous time limit functions should satisfy\(^\text{13}\)

\[ s(t) = \gamma(t) \lambda(t) \]  

(36)

\[ \alpha'(t) = \beta(t) s(t) [2\alpha(t) \lambda(t) M - 1] \]  

(37)

\[ \gamma'(t) = \beta(t) [Ms(t) \gamma(t) \lambda(t) + (2M - 1) \gamma(t) \lambda(t) - Ms(t) - 1] \]  

(38)

\(^\text{13}\)For all \( \epsilon > 0 \), the assumption that \( \Sigma(t) \), etc., uniformly converges in \( [0, 1 - \epsilon] \) implies that the limit functions must satisfy the differential equations (36)-(43). We then take \( \epsilon \) to 0 and obtain the differential equations in the limit for the interval \( [0,1] \). Since \( \Sigma(t), \lambda(t), \beta(t) \) are unbounded at \( t = 1 \), the difference equations are not stable near \( t = 1 \) and it is difficult to prove rigorously that the solutions from the difference equations converge to the solutions of the corresponding differential equations. Nevertheless, our assumption that the solutions of the difference equations converge to the solutions of the corresponding differential equations is supported by the numerical solutions presented later.
\[ 2\eta(t)\lambda(t) = 1 \]  
\[ \delta'(t) = -\alpha(t)\lambda(t)^2\sigma_u^2 \]  
\[ \Sigma'(t) = -M\beta(t)\lambda(t)\Sigma(t). \]  

The equations (7), (12) in the limit become

\[ s(t) = \frac{\Sigma(t)}{\Sigma(t) + (M - 1)\sigma_i^2} \]
\[ \lambda(t) = M k^2 \Sigma(t) \beta(t)/\sigma_u^2. \]

As shown in appendix C, the system of differential equations (36)-(43) can also be derived heuristically in a continuous-time auction model. In appendix D, we solve for the differential equations to yield

\[ \Sigma'(t) = -a \Sigma(t)^{4-4/M} e^{2(M-1)\sigma_i^2/M\Sigma(t)}. \]

Using the boundary condition \( \Sigma(1) = 0 \),\(^{14}\) equation (50) can be solved directly, yielding a closed form solution (27).

\( \lambda(t), \beta(t) \) can then be obtained by solving the simultaneous equations (41), (43), which yields (25), (26), (37) now becomes the standard first order Bernoulli equation, and we get (28). Finally, integration of (40) and a little algebra gives the expression for \( \delta(t) \) in (29).

Integration of \( \lambda(t)\sigma_u^2 \) over \( t \in [0,1] \) gives the expected loss of the liquidity traders. Since the market maker makes zero profits, the total expected profits of the informed traders is the same as the expected loss of the liquidity traders. This implies that

\[ \Pi \equiv \mathbb{E}[\sum_{i=1}^{M} \pi_i(0)] = \int_0^1 \lambda(t)\sigma_u^2 dt = \frac{k\sigma_u}{\sqrt{a}} \int_0^{\Sigma_0} e^{2(M-2)e^{(1/M-1)\sigma_i^2/\sigma}\, d\sigma}. \]

Notice that from the expression of the integration coefficient \( a \) in Assertion 1, it is clear that when \( M > 1, \sigma_i^2 = 0, a \) goes to infinity and the solution is not well defined. This corresponds to HS-FV’s results.

When \( M = 2 \), (25), (26) and (27) can be simplified to obtain explicit expressions for \( \lambda(t), \beta(t), \) and \( \Sigma(t) \),

\[ \lambda(t) = \frac{\sigma_0^2}{\sigma_u \sqrt{1 - t}} \frac{\sigma_e}{\Sigma_0 \ln(1 - t)} \]

\(^{14}\)For \( M \geq 2 \), the rational is that only the boundary condition \( \Sigma(1) = 0 \) is consistent with the boundary condition \( \gamma(1) = 0 \). The case where \( M = 1 \) is described in Kyle (1985).
\[
\beta(t) = \frac{\sigma_u \Sigma_0}{2\sigma_e^2 \sigma_e \sqrt{1 - t}}
\]

\[
\Sigma(t) = \frac{\Sigma_0 \sigma_e^2}{\sigma_e^2 - \Sigma_0 \ln(1 - t)}.
\]

Notice that \(\beta(t)\) is inversely proportional to \(\sigma_e\). \(\beta(t)\) measures the aggressiveness of the trading strategy of the informed traders. When \(\sigma_e\) is small, the informed traders trade very aggressively all the time. As a result, when \(\sigma_e \rightarrow 0\), price aggregates private information very efficiently. Therefore \(\Sigma(t)\) goes to zero and market depth goes to infinity for all \(t > 0\).

5 Characterization of the Equilibrium

In this section we present numerical results for the multi-period auction model in Figures 1-6 and numerical results of the continuous auction model in Figures 7-15.

As in Kyle (1985), the parameters \(\Sigma_n\) and \(\lambda_n\) are inverse measures of price efficiency and market depth, respectively. To compare the case with diverse signals and the case with perfectly correlated signals we present a series of numerical examples. We assume that \(\Sigma_0 = 1, \sigma_e^2 = 0.5, \sigma_u^2 = 1, \sigma_e^2 = 0\) and \(\Delta t_n = 1/N, n = 1, \ldots, N\) unless otherwise stated.

Figure 1 plots \(\lambda_n\) for the cases of \(N = 2, 4, 8, 16, 64, \infty\). \(M\) is set at 2 in Figure 1. As can be seen from Figure 1, \(\lambda_n\) initially decreases with time but then increases with time at the end of the auction. The curve at \(N = \infty\) is plotted using the limiting expression of \(\lambda_n\) in the last section. It is clear that the discrete time solution quickly converges to the continuous time solution.

Figure 2 plots \(\Sigma_n\) for the cases of \(N = 2, 4, 8, 16, \infty\). Notice that \(\Sigma_n\) quickly converges to the continuous auction solution as \(N\) increases. Moreover, at \(N = \infty\), the \(\Sigma_n\) starts as a convex curve but becomes concave near the end of the auction. This is due to the fact that near the end of the auction, each informed trader's private signal becomes highly correlated with the private signals of other informed traders. Competition between the informed traders then causes the traders to trade very aggressively and increases the rate of reduction of \(\Sigma_n\).

Figures 3 and 4 plot \(\lambda_n\) and \(\Sigma_n\) respectively for the cases of \(M = 1, 2, 4, 20\) fixing the number of auctions \(N\) at 256. As \(M\) increases, both \(\lambda_n\) and \(\Sigma_n\) initially decrease faster. However, \(\lambda_1\) increases with \(M\) while \(\lambda_{N/2}\) decreases with \(M\). Notice that \(\Sigma_n\) at \(M = 4\) is not much different from that at \(M = 20\). As will be shown later, in the continuous auction model, when \(M \rightarrow \infty\), both \(\lambda(t)\) and \(\Sigma(t)\) will converge to a smooth curve.

Figures 5 and 6 demonstrate the effect of changing \(\sigma_e^2\) on \(\lambda_n\) and \(\Sigma_n\). We set \(N = 256, \Sigma_0 = 1, \sigma_e^2 = 0, 0.02, 0.1, 0.5\) respectively. Notice that \(\lambda_1\) decreases with \(\sigma_e^2\) while \(\lambda_{N/2}\)
increases with $\sigma_0^2$. At $\sigma_0^2 = 0.02$, $\lambda_n$ starts very high but quickly decreases to zero. Similarly at $\sigma_0^2 = 0.02$, $\Sigma_n$ goes to zero very quickly. These results indicate that when $\sigma_0^2 \to 0$, our results converge to HS-FV's results at $\sigma_0^2 = 0$.

Figures 7 and 8 give the graph of $\Sigma(t), \lambda(t)$, for different $M$. Imperfect competition causes $\lambda(t)$ to rise above the monopolistic case near the beginning and the end of the auction and fall below the monopolistic case in the middle of the auction. Since $\lambda(t)\sigma_0^2 dt$ also measures the expected loss of liquidity traders between time $t$ and $t + dt$, Figure 7 indicates that the expected loss of liquidity traders is higher in the two ends of the auction and lower in the middle of the auction when private information spreads evenly among more than one informed trader. Notice that when $M \to \infty$, due to the assumption that the total private information remains constant, the liquidity parameter $\lambda(t)$ and the variance of the remaining private information $\Sigma(t)$ decrease smoothly over time.

Figures 9 and 10 give the graphs of $\Sigma(t)$ and $\lambda(t)$ when $M = 2$ with different $\sigma_0^2$. In these figures, we set $\Sigma_0 = 1, \sigma_0^2 = 0.02, 0.1, 0.5$. Notice that $\Sigma(t)$ goes down faster when $\sigma_0^2$ is smaller. Moreover, $\lambda(0)$ is proportional to the inverse of $1/\sigma_0$ while for $t$ sufficiently large $\lambda(t)$ decrease with $\sigma_0^2$. When $t = 0$, From (52), (54), we have $\Sigma(0) = \Sigma_0$, $\lambda(0) = \sigma_0^2/\sigma\sigma_u$. However, for any $t$ strictly positive, when $\sigma_0^2 \to 0$, $\Sigma(t) \to 0$, $\lambda(t) \to 0$. Our result is consistent with the results of HS-FV, where they showed in a discrete time model, $\lambda_1 \to \infty, \lambda_n \to 0, \Sigma_n \to 0$, when $N \to \infty$, and $n/N > \tau$ for any $\tau > 0$.

The effects of $\sigma_0^2$ on $\lambda(0), \lambda(t), \Sigma(t)$ is illustrated in Figures 11 and 12. In Figures 11 and 12, we set $M = 2, \sigma_0^2 = 1$, and draw the graph of $\lambda(0), \lambda(0.5)$ with respect to $\sigma_0^2$. Since at $\sigma_0^2 = 0$ the solution is not well defined, we set the value at $\sigma_0 = 0$ to be the limit of our solutions when $\sigma_0^2 \to 0$. Clearly, $\lambda(0)$ decreases with $\sigma_0^2$, and $\lambda(0.5)$ first increases with $\sigma_0^2$, and then decreases with $\sigma_0^2$. On the contrary, as shown in Figure 12, $\Sigma(0.5)/\Sigma_0$ starts at 0 and increases monotonically with $\sigma_0^2$.

In Figures 13-15, we set $\sigma_0^2 = 1$ and examine the effects of $\sigma_\epsilon^2$ on the total expected profits of the informed traders from trading. In Figure 13, we compare the total expected profits of the informed traders for $M = 1, 2, 4, \infty$. For $M = 1$, $\Pi(0)$ always decreases with $\sigma_\epsilon^2$, while for $M > 1$, $\Pi(0)$ starts at 0 and initially increases with $\sigma_\epsilon^2$ and then decreases with $\sigma_\epsilon^2$. Notice that $\Pi(0)$ decreases with $M$ but even at $M = \infty$, $\Pi(0)$ is finite for all strictly positive $\sigma_\epsilon^2$. The total expected profits for the informed traders decrease with $M$. It would be better for the informed traders to form a "mutual fund" and trade as a group with all of their private information if the informed traders agree not to trade on their own. When $\sigma_\epsilon$ is large, the total expected profits of the informed traders for the case with $M = 1$ are

\begin{footnote}{More generally, for $M > 1, t > 0$, it can be shown that When $\sigma_\epsilon^2 \to 0$, $\lambda(0) = O(\sigma_\epsilon^4/M^3)$, $\lambda(t) = O(\sigma_\epsilon)$, $\Sigma(t) = O(\sigma_\epsilon^2)$. Moreover, $\beta(0) = O(\sigma_\epsilon^4/M^3), \beta(t) = O(\sigma_\epsilon^{-1})$. This implies that, when $\sigma_\epsilon$ is small, each informed trader will trade very aggressively on the difference between the expected value of the risky asset conditional only on his own signal and the expected value of the risky asset conditional only on the public signal extracted from the price all the time.}

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very close to the case when \( M = \infty \). Therefore, when the informed traders have information of low quality, the incentive for them to form a mutual fund is very weak. Let \( \Pi^I \) denote their profits when they trade individually and \( \Pi^M \) denote their profits when they pool their information together and trade through a mutual fund. Figure 14 illustrates the ratio of \( \Pi^I \) over \( \Pi^M \) with respect to \( \sigma^2 \), for \( M = \infty \). This ratio starts at zero and quickly increases to 0.91 at \( \sigma^2 = 2 \) and stabilizes for larger \( \sigma^2 \).

Finally, we compare the expected profits of the informed traders in the single auction model and the continuous auction model at \( M = 2, \sigma^2 = 1 \). Let \( \Pi^c \) and \( \Pi^s \) denote the total expected profits from trading in the continuous auction and the single auction economy respectively. As illustrated in Figure 15, when \( \sigma^2 \) is small, the informed traders earn higher expected profits in the single auction economy. However, for \( \sigma^2 \) sufficiently large, the informed traders’ expected profits are higher in the continuous auction economy. This is different from the monopolist case in which in the continuous auction economy the informed trader always earns twice of the expected profit he gets in the single auction economy.

6 Conclusion

We have shown that when informed traders’ private signals are not perfectly correlated, their total expected profits will not vanish when market opens frequently. Furthermore, informed traders trade smoothly in the beginning of the auctions and the private information is incorporated into prices gradually. Market depth initially increases but decreases dramatically in the end. This is in substantial contrast to the results of HS-FV, who assume that information signals are perfectly correlated. When the variance of the uncorrelated element \( \epsilon_t, M \sigma^2 \), goes to zero, our results converge to HS-FV’s results.\(^{16}\)

There are many extensions of the current work. We have assumed that the informed traders receive private signals only in the first period and there are no public signals besides the price. It is straightforward to extend our model to the case in which public signals are released before trading starts and informed traders receive private signals of the same precision in every period.\(^{17}\) An alternative model structure is to let informed traders have perfect information of different elements of the risky asset.\(^{18}\) Another extension of the current study is...

\(^{16}\) The results in this paper are also sensitive to the assumption that each informed trader is the only recipient of his private signal. Consider a model in which there are \( M \) private signals as described in this paper. But there are \( MI \) informed traders \((I > 1)\) and each of the \( M \) private signals is shared by \( I \) different informed traders. Then it can be shown numerically that when trading occurs frequently, all private information is revealed to the market almost immediately. This indicates that when there is competition on exactly the same signal, the information content in that signal will be revealed very rapidly.

\(^{17}\) He and Wang (1993), and Brennan and Cao (1994) have developed such models in competitive noisy rational expectations frameworks.

\(^{18}\) For example, we can assume that the asset payoff is given by \( v = \sum_{i=1}^{M} v_i \), where \( v_i \) is identical and...
paper is to let the informed traders be risk averse. In our solution of the continuous trading model, the price becomes extremely sensitive to the order flow at the end of the auction. This is due to the risk-neutrality of the informed traders, which implies that they may take very large positions in the risky asset. Near the end of the auction, the private signals of the informed traders become highly correlated with the signals of other informed traders and they will compete very aggressively to extract profits from the remaining information. When informed traders are risk averse, they will not take very large positions in the risky asset. As a result, when trading occurs frequently, the price will not reveal all the private information and the price will not be very volatile near the end of the auction. Moreover, the current work may be extended to more general distributions of the risky asset value and to include limit orders in the informed traders' strategy.

In this paper, as in most existing literature, we assume that the informed traders receive private signals at the same time. It would be interesting to examine how the timing of private information acquisition affects the trading strategy of the informed traders. In the continuous trading economy, consider the case of two informed traders in which the early informed trader acquires a private signal at time 0 and the late informed trader acquires a private signal at time $1/2$. When the two traders observe the same signal, clearly the optimal strategy is for the early informed trader to reveal the signal at time $1/2$ since the late informed trader will compete with the early informed trader to reveal the private signal instantaneously in later trading rounds. However, if the two informed traders observe different signals, it is likely that the early informed trader will not trade so aggressively as to reveal his signal at time $1/2$ since he wants to earn additional profits from the noisy traders in later trading rounds. The intuition that the informed traders will trade less competitively when their signals are less correlated still applies in this case.

Independent normally distributed and each $v_i$ is observed by informed trader $i$. However, we can write $M v_i = v + (M-1) v_i - \sum_{j \neq i} v_j = v + \epsilon_i$, where $\epsilon_i \equiv (M-1) v_i - \sum_{j \neq i} v_j$. Notice that $\epsilon_i$ is independent of $v$. Therefore, this extension is a variant of the model presented here except that $\epsilon_i$s are correlated among the informed traders. The results obtained in the two models are essentially the same.

In the continuous auction economy, let $dz$ denote the total quantity of informed trading and $E[dz/dt|Y]$ denote the total expected rate of informed trading. Then it can be shown that $E[dX/dt|Y] = \sigma \sqrt{\Sigma(t)}^{1-1/M} e^{(1-1/M)\sigma^2/2} \Sigma(t)^{1/M}$. For $M = 1$, the expected rate of informed trading is a constant. However, for $M > 1$, the total expected rate of informed trading initially decreases and then increases to infinity near the end of auction.

The intuition that market depth initially increases due to the reduction of adverse selection, as private information is revealed through the time series of prices, and decreases near the end of the auctions due to more intensive competition between the informed traders still applies to the model with risk averse informed investors. It is likely that for informed traders with small risk aversion, $\lambda(t)$ is non-monotonic in $t$.

Foster and Vishwanathan (1993) have extended the model to the class of elliptically contoured distributions and Back (1992) has extended the continuous time version of the Kyle (1985) model to general distributions of the risky asset value. Rochet and Vila (1994) discussed an extension of the Kyle (1985) model to general distributions of the risky asset value when the informed trader can observe the demands of the noise traders. Rochet and Vila's model is equivalent to the Kyle (1989) limit order model when there are free entry of uninformed traders in Kyle (1989).

An exception is provided by Hirshleifer, Subrahmanyam, and Titman (1994).
In order to make the analysis of our paper tractable, we have made the assumption that all the informed traders’ signals have the same precision. It would be worthwhile to analyze the model in which there is asymmetry in the structure of informed traders’ private signals. In principle, the fixed point technique described in Section 2 may offer a general solution to the infinite regression problem. However, even if the solution to the fixed point problem exists, it would probably be difficult to find the fixed point in multiple trading sessions or extend the result to continuous time trading.
Appendix A

Suppose that informed trader $i$ follows a strategy different from the proposed equilibrium strategy. In this case the off-equilibrium derived profit function will differ from equation (6) in the proposed equilibrium. Suppose that trader $i$ trades $\bar{x}_n$, $\forall n$, and denote the resulting prices by $\bar{p}_n$, $\forall n$. Since every trader agrees on the prior mean of the risky asset, we set $\bar{p}_0 \equiv p_0 = 0$. Other informed traders still follow the equilibrium strategy and their demand functions are denoted $\Delta \bar{x}_n = \beta_n \Delta t_n(kz_j - \bar{p}_{n-1})$. We conjecture that the off-equilibrium derived profit function for informed trader $i$ after the $n$th auction is described by the following equation,

$$E\{\tilde{\pi}_{(n+1)i} | I_{ni}\} = \alpha_n(kz_i - p_n)^2 + \gamma_n(kz_i - p_n)(p_n - \bar{p}_n) + \eta_n(p_n - \bar{p}_n)^2 + \delta_n. \quad (49)$$

Before we proceed further, we need to discuss how informed trader $i$ extracts other informed traders’ information from the price if he deviates from the proposed equilibrium strategy. In our setup, even if informed trader $i$ deviates from the proposed equilibrium strategy, his deviation cannot be detected by other participants in the market since the state space is continuous and there is no off equilibrium outcome here. When informed trader $i$ deviates in the $(n - 1)$th auction, other participants in the market are not aware of his true action. In the next auction, informed traders other than $i$ still put the same weight on their private signals in their orders. Similarly, the market maker also uses the same rule to form his conditional expectation of $v$ given the new order flow. As a result, what informed trader $i$ learns from the price is independent of his deviation. This indicates that informed trader $i$’s conditional expectation of the risky asset value after the $n$th auction is independent of his orders in previous auctions.

However, when informed trader $i$ deviates from the proposed equilibrium, he causes the market maker and other informed traders to form wrong expectations from the order flow. The market maker believes that the expectation of $v$ given the order flow should be $\bar{p}_n$ at auction $n$, and informed trader $i$ knows that it should be $p_n$ instead. When another informed trader $j$ calculates his conditional expectation of $v$, he uses the precision-weighted average of the expectation of $v$ given only his private signal and the expectation of $v$ given only the public signal, $\bar{p}_n$. By deviating from the proposed equilibrium strategy, informed trader $i$ can mislead both the market maker and other informed traders and gains an informational advantage. Consequently, for informed trader $i$, his derived profits from future tradings at auction $n+1, \cdots, N$ must also be a function of $\bar{p}_n - p_n$ which measures how much informed trader $i$ misleads others.

To prove that the conjectured derived profit function is correct for all $n$, we proceed by backward induction. First, after the last auction is completed, there will be no more trading, the derived profit function is clearly 0. This is consistent with the conjectured derived profit function by setting $\alpha_N = \gamma_N = \eta_N = \delta_N = 0$. We assume that the proposed profit function
holds at auction $n$ and prove that it also holds at auction $n-1$,

\[
E[\pi_{ni}|I_{(n-1)i}] = \max_{\Delta x_{ni}} \left\{ (v - \bar{v}_n) \Delta x_{ni} + \alpha_n(kz_i - p_n)^2 
+ \gamma_n(kz_i - p_n)(p_n - \bar{p}_n) + \eta_n(p_n - \bar{p}_n)^2 + \delta_n|I_{(n-1)i}) \right\}.
\]

From the proposed equilibrium, we have

\[
\Delta \bar{p}_n \equiv \bar{p}_n - \bar{p}_{n-1} = \lambda_n(\Delta \bar{x}_{ni} + \sum_{j \neq i} \Delta \bar{x}_{nj} + \Delta u_n).
\]

In order to form optimal demands, the deviating trader need to estimate the random variables $v$, $\Delta p_n$, and $\Delta \bar{p}_n$ assuming that others follow the equilibrium strategy,

\[
E[\Delta p_n|I_{(n-1)i}] = \lambda_n \sum_{i=1}^{M} \Delta x_{ni} = E[\lambda_n \beta_n \Delta t_n (MkY - Mp_{n-1})|I_{(n-1)i})]
= \lambda_n \beta_n \Delta t_n Ms_{n-1}(kz_i - p_{n-1})
\]

\[
\text{var}[\Delta p_n|I_{(n-1)i}] = \lambda_n^2 \sigma_u^2 \Delta t_n + \lambda_n^2 \beta_n^2 M^2 k^2 \Sigma_{n-1} \Delta t_n^2
\]

\[
E[\Delta \bar{p}_n|I_{(n-1)i}] = \lambda_n(\Delta \bar{x}_{ni} + \sum_{j \neq i} \Delta \bar{x}_{nj})
= \lambda_n(\Delta \bar{x}_{ni} + E[\beta_n \Delta t_n (MkY - kz_i - (M - 1)\bar{p}_{n-1})|I_{(n-1)i})]
= \lambda_n[\Delta \bar{x}_{ni} + \beta_n(Ms_{n-1} - 1)(kz_i - p_{n-1}) + \beta_n \Delta t_n(M - 1)(p_{n-1} - \bar{p}_{n-1})]
\]

\[
E[\pi_n|I_{(n-1)i}] = \max_{\Delta \bar{x}_{ni}} \left\{ (s_{n-1}(kz_i - p_{n-1}) + p_{n-1} - \bar{p}_{n-1}
- \lambda_n[\Delta \bar{x}_{ni} + \beta_n \Delta t_n[(Ms_{n-1} - 1)(kz_i - p_{n-1}) + (M - 1)(p_{n-1} - \bar{p}_{n-1})]\\
+ \alpha_n[(1 - \lambda_n \beta_n \Delta t_n(Ms_{n-1})]^2(kz_i - p_{n-1})^2 + \lambda_n^2 \beta_n^2 M^2 k^2 \Sigma_{n-1} \Delta t_n^2\\
+ \gamma_n(1 - M \beta_n \Delta t_n \lambda_n s_{n-1})(kz_i - p_{n-1})\\
\times \{p_{n-1} - \bar{p}_{n-1} - \lambda_n \Delta \bar{x}_{ni} + \beta_n \Delta t_n \lambda_n[kz_i - p_{n-1} - (M - 1)(p_{n-1} - \bar{p}_{n-1})]\\
+ \eta_n(p_{n-1} - \bar{p}_{n-1} - \lambda_n \Delta \bar{x}_{ni} + \beta_n \Delta t_n \lambda_n[kz_i - p_{n-1} - (M - 1)(p_{n-1} - \bar{p}_{n-1})]\}^2. (50)
\]

The first order condition for the above problem is

\[
\Delta \bar{x}_{ni} = \frac{A_n(kz_i - p_{n-1}) + B_n(p_{n-1} - \bar{p}_{n-1})}{\lambda_n[2 - 2\eta_n \lambda_n]},
\] (51)
where
\[
A_n = s_{n-1}(1 - M\beta_n\Delta t_n\lambda_n) + \lambda_n\beta_n\Delta t_n - \lambda_n\gamma_n - 2\eta_n\lambda_n^2\beta_n\Delta t_n + \gamma_n\lambda_n^2M\beta_n\Delta t_n\lambda_{n-1} \tag{52}
\]
\[
B_n = (1 - 2\eta_n\lambda_n)[1 - (M - 1)\beta_n\Delta t_n\lambda_n]. \tag{53}
\]

The optimal demand function (51) is a linear combination of \(kz_i - p_{n-1}\) and \(p_{n-1} - \bar{p}_{n-1}\). From (50), trader \(i\)'s expected profit function is a quadratic function of \(\Delta \bar{x}_{ni} \), \(kz_i - p_{n-1}\) and \(p_{n-1} - \bar{p}_{n-1}\). Consequently, trader \(i\)'s expected profit function is of the form conjectured in (49). In equilibrium, \(p_{n-1} = \bar{p}_{n-1}\), the second term in (51) drops out and (8) can be derived from (51) and (52). Substitute \(\Delta \bar{x}_{ni} \) in (50) using (51), (52), (53), recursive equations (9)-(11),(14) can be derived easily. Since \(p_0 = \bar{p}_0 = 0\), from (51), it follows that in the first auction, trader \(i\) will follow the equilibrium strategy and therefore \(\bar{p}_1 = p_1\). Proceeding inductively, informed trader \(i\) will follow his equilibrium strategy in every period and we have \(\bar{p}_n = p_n, \forall n\).

The second order derivative of (50) with respect to \(\Delta \bar{x}_{ni} \) is given by \(2\eta_n\lambda_n^2 - 2\lambda_n\), and this gives the second order condition (17). In all the numerical examples provided in the paper, the second order condition is satisfied. Existence of the equilibrium follows Proposition 1 and its proof in appendix B.

**Appendix B**

In this appendix, we derive the recursive method used to solve the discrete time equilibrium. Define \(q_n \equiv \gamma_n\lambda_n\). From (8) and (10), we have
\[
\gamma_n = s_{n-1}\lambda_n[1 - \frac{(M - 1)(s_{n-1} - q_n)}{1 + (1 - q_n)s_{n-1}M}]\left[1 - \frac{M(s_{n-1} - q_n)}{1 + (1 - q_n)s_{n-1}M}\right],
\]
or
\[
q_n = \frac{s_{n-1}\lambda_{n-1}}{\lambda_n}[1 - \frac{(M - 1)(s_{n-1} - q_n)}{1 + (1 - q_n)s_{n-1}M}]\left[1 - \frac{M(s_{n-1} - q_n)}{1 + (1 - q_n)s_{n-1}M}\right],
\]
implying that
\[
\frac{\lambda_n}{\lambda_{n-1}} = \frac{s_{n-1}}{q_{n-1}}[1 - \frac{(M - 1)(s_{n-1} - q_n)}{1 + (1 - q_n)s_{n-1}M}]\left[1 - \frac{M(s_{n-1} - q_n)}{1 + (1 - q_n)s_{n-1}M}\right]. \tag{54}
\]

Now, from (10) we also have
\[
\frac{\lambda_n}{\lambda_{n-1}} = \frac{\beta_n}{\beta_{n-1}}\frac{\Sigma_n}{\Sigma_{n-1}}
\]

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and, from (13), this is equivalent to

\[
\frac{\lambda_n}{\lambda_{n-1}} = \frac{\beta_n}{\beta_{n-1}} [1 - M \beta_n \lambda_n \Delta t_n].
\]  

(55)

Multiply both sides of (55) by \(\lambda_n/\lambda_{n-1}\) and Substituting for \(\beta_n\) from (8) to (55), we have

\[
\frac{\lambda_n^2}{\lambda_{n-1}^2} = \frac{(s_{n-1} - q_n)(1 - M q_n (1 - s_{n-1}))}{[1 + (1 - q_n) s_{n-1} M]^2} \frac{1 + (1 - q_{n-1}) s_{n-2} M}{s_{n-2} - q_{n-1}} \frac{\Delta t_{n-1}}{\Delta t_n}.
\]  

(56)

Squaring the RHS of (54), equating the resulting expression to the RHS of (56) and rearranging yields the cubic equation (21) in Proposition 2. Equation (20) can be derived from (7), (8) and (13). From (7) we have

\[
\frac{\Sigma_n}{\Sigma_{n-1}} = \frac{s_n (1 - s_{n-1})}{s_{n-1} (1 - s_n)}.
\]  

(57)

From (8) and (13) we have

\[
\frac{\Sigma_n}{\Sigma_{n-1}} = \frac{1 - M q_n (1 - s_{n-1})}{[1 + (1 - q_n) s_{n-1} M]}.
\]  

(58)

Equating the RHS of (57) and (58) we get (20). We can start from a conjectured \(q_1\) to get \(s_1\) from (20) and \(C_2\) from (21). Then we can derive \(q_2\) from the expression of \(q_2\). Repeat this process, we will derive \(q_N\). To show there exists a series such that \(q_N = 0\), we use the method of deduction. Clearly for \(N = 1\), from Admati and Pfeiderer (1988), the assertion that the equilibrium exists holds. Suppose it holds for \(N = T - 1\), we have a series of \(q_n, s_n\) satisfying (20), (21) and \(q_{T-1} = 0\). From the expression of \(C_n\), we get \(q_T = -1/[M (1 - s_{T-1})]\). Now we show that we can also find \(q_1\) such that \(q_T > 0\). If we choose \(q_1 = s_0 - O(\epsilon)\) arbitrarily close to \(s_0\), from (20) we have \(s_1 = s_0 - O(\epsilon)\). From (21) and the expression of \(C_2\), we have \(q_2 = s_1 - O(\epsilon) = s_0 - O(\epsilon)\). Therefore, we can find \(q_T > 0\) arbitrarily close to \(s_0\). Consequently, there must exists a \(q_1\) such that \(q_T = 0\).

**Appendix C**

In this section we discuss a model in which trading takes place continuously. Using a similar technique as that in Kyle (1985), we assume that a linear equilibrium with a structure analogous to the discrete time model exists as described below,

\[
dx_i(t) = \beta(t)(k z_i - p(t))dt
\]  

(59)
\[ dx(t) = \sum_{i=1}^{M} dx_i(t) \]
\[ dp(t) = \lambda(t)(dx(t) + du). \]

As discussed in the case of discrete time model, we need to define the optimal strategy of informed trader \( i \), when he deviates from the proposed equilibrium trading. Let \( \bar{x}(t) \) denote the optimal strategy when trader \( i \)’s trade deviates from proposed in the equilibrium due to an error or price manipulation in early trading rounds. we restrict the strategy for the informed trader in the market to the diffusion processes. Specifically we assume that
\[ d\bar{x}_i(t) = dx_i(t) + (p(t) - \bar{p}(t))dy, \]
where \( dy = \mu(t)dt + \sigma(t)dw \) follows a diffusion process. This particular form of \( d\bar{x}_i \) is chosen so that when \( \bar{p}(t) = p(t) \), it is optimal form informed trader \( i \) to follow the trading strategy in the proposed equilibrium. Let \( I_{ui} \) be the information filtration for trader \( i \). We assume that the off equilibrium maximized profit function have the following form analogous to the discrete time model,
\[ E[\pi(t)|I_{ui}] = \alpha(t)(kz_i - p(t))^2 + \gamma(t)(kz_i - p(t))(p(t) - \bar{p}(t)) + \eta(t)(p(t) - \bar{p}(t))^2 + \delta(t). \]

Since \( \bar{x}_i(t) \) is the optimal strategy of informed trader \( i \), \( \bar{x}_i(t) \) must be chosen so that the following the Bellman equation is satisfied.
\[ E[\pi_t dt + \pi_p dp + \pi_{\bar{p}} d\bar{p} - \frac{1}{2} \pi_{pp}(dp)^2 + (v - \bar{p}) d\bar{x}_i + \pi_{p\bar{p}} dp d\bar{p} - \frac{1}{2} \pi_{pp\bar{p}} (d\bar{p})^2 |I_{ui}] = 0. \]

This basically means that the instantaneous profit is exactly offset by the expected change in \( \pi \) when an optimal policy is followed.

Since
\[ \pi_t = \alpha'(t)(kz_i - p(t))^2 + \gamma'(t)(kz_i - p(t))(p(t) - \bar{p}(t)) + \eta'(t)(p(t) - \bar{p}(t))^2 + \delta'(t) \]
\[ \pi_p dp = -2\alpha(t)(kz_i - p(t)) + \gamma(t)(p(t) - \bar{p}(t) - kz_i - p(t)) + 2\eta(t)(p(t) - \bar{p}(t)) \]
\[ \pi_{\bar{p}} d\bar{p} = -\gamma(t)(kz_i - p(t)) - 2\eta(t)(p(t) - \bar{p}(t)) \]
\[ \pi_{pp} = 2\alpha(t) \]
\[ \pi_{p\bar{p}} = \gamma(t) - 2\eta(t) \]
\[ \pi_{pp\bar{p}} = 2\eta(t) \]
\[ E[dp|I_{ui}] = M\beta(t)\lambda(t) s(t)[kz_i - p(t)] \]
\[ E[d\bar{p}|I_{ui}] = \lambda(t)\{dx_i + \beta(t)(Ms(t) - 1)[kz_i - p(t)] + \beta(t)(M-1)[p(t) - \bar{p}(t)]\} \]

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\[ E[(dp)^2|I_{tt}] = E[(dpd\bar{p})|I_{tt}] = E[(dp)^2|I_{tt}] = \lambda(t)^2 \sigma_u^2 dt \]

\[ E[v - \bar{p}(t)|I_{tt}] = s(t)(kz_i - p(t)) + p(t) - \bar{p}(t). \]

Plug these expressions back to the Bellman equation (63) and collect the terms, we have

\[
\{ [\alpha'(t) - \beta(t)s(t)(2\alpha(t)\lambda(t)M - 1)][kz_i - p(t)]^2 + [\gamma'(t) - \beta(t)[M\gamma(t)\lambda(t)s(t) + (2M - 1)\gamma(t)\lambda(t) - Ms(t) - 1][kz_i - p(t)][p(t) - \bar{p}(t)] + [\eta'(t) - \eta(t)\lambda(t)(\gamma(t)M - 1) + \lambda(t)\sigma(t)^2](p(t) - \bar{p}(t))^2 + \delta'(t) + \alpha(t)\lambda(t)^2\sigma_u^2 \} dt
\]

\[-{(\gamma(t)\lambda(t) - s(t))[kz_i - p(t)] + [1 - 2\eta(t)\lambda(t)][p(t) - \bar{p}(t)]} d\bar{x}_i(t) = 0. \tag{64}

Set the coefficients of the (64) to be 0 gives the following system of differential equations.

\[ \alpha'(t) = \beta(t)s(t)(2\alpha(t)\lambda(t)M - 1) \tag{65} \]

\[ \gamma'(t) = \beta(t)[M\gamma(t)\lambda(t)s(t) + (2M - 1)\gamma(t)\lambda(t) - Ms(t) - 1] \tag{66} \]

\[ \eta'(t) = 2\eta(t)\lambda(t)\beta(t)(M - 1) - \eta(t)\lambda(t)^2\sigma(t)^2 \tag{67} \]

\[ \delta'(t) = -\alpha(t)\lambda(t)^2\sigma_u^2 \tag{68} \]

\[ \gamma(t)\lambda(t) = s(t) \tag{69} \]

\[ 2\eta(t)\lambda(t) = 1. \tag{70} \]

The system of differential equations (65), (66), (68)-(70) is identical to (37)-(40), (36), (41) and (43) can be derived using the appropriate Kalman filtering. (42) obtains by definition. Therefore the solutions derived from differential equations (36)-(43) and \( \sigma(t) \) obtained from (67) satisfy the Bellman equation (63) and give the continuous time trading equilibrium.

**Appendix D**

In this appendix, we solve for the differential equation described in Section 4.

From (36), (38), (43) we can get

\[
\left( \frac{s(t)^2}{\lambda(t)^2} \right)' = \frac{2\sigma_u^2}{Mk^2\Sigma(t)}[Ms(t)^2 + (M - 1)s(t) - 1]. \tag{71}
\]

From (41) and (43) we have

\[ \lambda(t)^2 = -k^2\Sigma'(t)/\sigma_u^2. \tag{72} \]
Substitute \( \lambda(t)^2 \) in (72) to (71) and substitute \( s(t) \) from (41), we get

\[
\frac{\Sigma''(t)}{\Sigma'(t)} = \left[ \frac{4M - 4}{M \Sigma(t)} - \frac{2(M - 1)\sigma^2}{M \Sigma(t)^2} \right] \Sigma'(t),
\]

which implies that

\[
[\ln(-\Sigma'(t))]' = \frac{4M - 4}{M} \ln[\Sigma(t)] + \frac{2(M - 1)\sigma^2}{M \Sigma(t)} + c.
\]

It is immediate that (44) follows.
References


Figure 1
Figure 2
Figure 5
Figure 7
Figure 8
Figure 9
Figure 10
Figure 11
Figure 12
Figure 14
Figure 15