Dynamic Investment with Adverse Selection and Moral Hazard

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March 28, 2012

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Abstract

This paper develops a dynamic model of capital structure and investment. In a world with low and high ability managers, the former mask as the latter, but to do so have to overstate both earnings and investment. Debt is a mechanism that eventually separates investors’ abilities, at the cost of intervening unlucky high productivity managers. Immediate separation is counterproductive, as it generates costs and no expected payoff. The security design that asymptotically implements optimal investment includes the use of excess non-operating cash, of proportional cash flow compensation, and of "golden parachutes". Relative to a first best case, high ability managers will underinvest. Low ability managers will generally overinvest, except when their firm is close to bankruptcy, in which case they will loot the company by underinvesting and overstating their earnings.

Keywords: Bayesian updating, reorganizations, bankruptcy, security design.

This paper develops a model of dynamic investment with moral hazard and adverse selection. In the model, a manager of unknown type must invest every period and report corporate earnings and payoffs. In this model, financial assets are costlessly verifiable while physical assets require a costly intervention.
The capital structure that emerges in this setting is as follows. First, rising long term debt is a mechanism that effectively distinguishes managerial ability. The second result is that the company will hold excess cash to plug in temporary operational cash flow shortfalls. If excess cash is ruled out, then the firm would use credit lines for the same purpose, with the drawback that there will be an additional probability of bankruptcy beyond what is caused by long term debt. This setting also generates outside equity.

From a contracting perspective and under linear contracts (i.e., the manager receives a proportion of the cash flows) we will observe the following results regarding investment: first, optimal investment by high ability managers will be achieved asymptotically, although high ability managers will underinvest during normal times. When close to bankruptcy, these managers will either invest optimally or underinvest, depending on the contract offered to them. The contract that implements optimal investment during financial stress requires the use of golden parachutes. Low ability managers overinvest during normal times, but if they come close to bankruptcy, they will underinvest. Moreover, low ability managers will mis-report earnings and investments to mask as their high ability counterparts. Hence, if there is intervention, investors will find in some cases significantly deteriorated physical assets and a set of coherent lies that justified the observed cash flows. I show that compensation with options may implement the first best investment levels for high ability managers but will also worsen over-investment for the low ability managers. Deferred compensation will worsen things relative to linear contracts. This is because under deferred compensation, low ability managers will simply copy their high ability counterpart’s investment policy, grossly overinvesting relative to their first best solution.

The literature on security design and investment in multiple periods is vast, and has undergone important changes in recent years. In these models, there is usually a moral hazard problem that can be resolved with a specific security. These problems can also be interpreted as an analysis of managerial compensation schemes. The first models by Townsend (1979) and Gale and Hellwig (1985) where static and assumed that corporate earnings could be verified at a cost. The moral hazard problem in both papers is that the
managers can divert cash flows and report "low earnings". The solution to the problem is to have risky debt, i.e., a security with fixed payoffs for high reported states, and intervention for low reported states. Another way of stating the result is that the optimal executive compensation is leveraged inside equity. Although truly a breakthrough, these findings were subject to three criticisms: the first was that they lacked dynamics. The second objection was that these security design papers could explain the existence of debt but not that of outside equity. The third criticism was that these instruments were not renegotiation proof, i.e., that at intervention it may be in the investors’ and managers best interest not to verify or punish the executives.

The first multi-period security design problems date to Harris and Raviv (1995), Hart and Moore (1998), Fluck (1998), Gromb (1999), and Myers (2000). These models relaxed the assumption on cash flows, in that they were observable but not contractible. In these setups, it was possible to get outside equity, excess cash, and debt with different maturities. Hart and Moore (1998) derive a renegotiation proof contract in three periods, solving a deficiency of static models.

Newer papers by Quadrini (2004), Clementi and Hopenhayn (2006), and De Marzo and Fishman (2007a) have focused on mutiperiod firm investment and growth under moral hazard. Papers by DeMarzo and Sannikov (2006) and by Biais et al. (2007) have looked at optimal contracts in continuous time. DeMarzo and Sannikov (2006) find that it may be optimal to have compensating balances, i.e., to borrow and hold excess cash at the same time. Biais et al (2007) study the optimal mechanism design with binomial cash flows. Finally, DeMarzo and Fishman (2007b) study the optimal security design when cash flows can be diverted at a cost by managers. They find that there is scope for outside equity, and that managers’ optimal compensation is a type of levered equity plus a credit line. They show that adding a renegotiation proof constraint worsens the contract significantly.

This paper takes a dynamic perspective, and uses adverse selection to avoid renegotiation. A fourth more marginal criticism was that these models did not consider stochastic mechanisms, which would change the optimal security design. Boyd and Smith (1994) show that such stochastic mechanisms barely raise welfare, and given their inherent complexity, they are probably not worthwhile relative to a simpler debt instrument.
ation problems when designing securities. The adverse selection naturally generates debt, i.e., an instrument that specifies a minimum cash flow below which there is intervention. The debt level grows over time to better distinguish managerial types. The paper introduces a relatively mild moral hazard problem, where managers invest and cash flows are perfectly observed and are contractible, but their specific components are not freely observable. This moral hazard generates under- and over-investment during normal times, depending on the manager type. Moral hazard becomes an acute problem when the firm is close to bankruptcy, i.e., when average cash flows are close to the cutoff debt repayment level. In these situations, both types of managers would naturally tend to underinvest and to lie coherently, i.e., overstate earnings and investment.

The paper’s structure is as follows: section 1 sets up the model. Section 2 discusses environments that deliver the first best equilibrium, which is characterized. Section 3 derives the results under adverse selection, showing investors’ optimal intervention rule and managers’ investment policy under linear contracts. Section 3 shows that some popular deviations from linear contracts (options and deferred pay) may worsen the outcome.

1 The Basic Model

The economy has infinite discrete periods and no taxes. Agents are risk neutral with a discount rate \( r \) per period\(^2 \). One type of agents are capitalists who can only invest and consume. The other type of agents have intrinsic management abilities \( a_{jt} \), which are either low \( a_{lt} \), with probability \( p_0 \) or high \( a_{ht} > a_{lt} \) with a probability \( 1 - p_0 \). The available investments are risky (explained below) or safe. Safe investments yield a return \( r \) per period. Capitalists are assumed to have no managerial talent, so their productivity is zero, i.e., \( a_{jt} = 0 \) for them.

There is a fixed number \( N \) of firms, which can be thought of as proprietary investment ideas. There are more potential managers than firms, so there will always be some willing to work. The outside opportunity for managers is \( \omega > 0 \) per period starting at \( t = 1 \),

\(^2\)Alternatively, agents could be risk averse, and we would be dealing with equivalent martingale measures. For more on this, see Duffie (1992), for example
so its present value at \( t = 0 \) is \( \omega^3 \). A representative firm \( n \) (for \( n = 1, \ldots, N \)) requires physical capital \( k_t \) in period \( t \) to operate next period. Capital depreciates at a rate \( \delta \), \( 0 < \delta < 1 \) per period. New capital can be added by investing \( i_t \) at the end of period \( t \), so the capital’s law of motion for any given firm is:

\[
k_t = (1 - \delta)k_{t-1} + i_t = \sum_{d=0}^{t} (1 - \delta)^d i_{t-d}
\]

I will consider the case of a representative firm, whose only a priori distinction is the manager type (denoted by subscript \( j \)) who runs it. Capital \( k_{t-1} \) generates an operating cash flow \( x_t \) in period \( t \) given by:

\[
x_{jt} = a_{jt-1} f(k_{t-1}) s_t - i_t
\]

where \( a_{jt-1} \) is managerial productivity, and \( f(k) \) is a production function with decreasing returns to scale, and \( i_t \) is the new investment. \( s_t \in (0, \infty) \) is a firm specific random \( iid \) variable with a density function \( g(s) \) and \( E(s) = 1 \). An \( s_t \) with no lower bound would not change the paper’s results substantially, but it makes some sense to put a lower bound of 0 in that \( a_{jt-1} f(k_{t-1}) s_t \) is akin to the firm’s EBITDA, which should normally be positive. Although I will mostly assume that managerial productivity is constant, I will state the results with a time varying productivity \( a_{jt} \), as this will be useful later in the paper.

At the very end of period \( t \) the firm will give a payout \( p_{0jt} \) to its investors. This payout is observable. I will define \( p_{0jt} > 0 \) to mean a payment from the firm to investors, and \( p_{0jt} < 0 \) to mean a capital injection from investors to the firm.

Apart from physical capital, the firm may also hold excess cash \( m_{jt} \) in period \( t \). Throughout the paper I will assume that investors can costlessly verify \( m_{jt} \) and write enforceable contracts based on it. The idea is that financial capital is easier to ascertain.

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3. \( \omega \) is small but positive, so as to get well behaved contracts later on. It also makes intuitive sense that it is positive - managers can do something else - and small - they are willing to work-
4. Thus, \( i_0 = k_0 \)
5. In particular, I assume that \( f'(0) > 0, f''(0) < 0, f'(0) = +\infty, f'(+\infty) = 0 \), to assure a unique well behaved equilibrium. From now onwards I will not use any subscript on \( s \) given its i.i.d. nature. The manager can underutilize capital, i.e., to produce as if \( \tilde{k}_{jt} \leq k_{jt} \) without investors knowing this. This last condition will be used to rule out non-sensical Bayesian equilibria later on.
than physical capital or investment. The excess cash is invested in the riskless security bearing \( r \) interest per period. The excess cash at the beginning of period \( t + 1 \) is given by:

\[
    m_{jt+1} = (1 + r)m_{jt} + x_{jt} - po_{jt}
\]

\[
    x_{jt} = m_{jt+1} - (1 + r)m_{jt} - po_{jt}
\]

Given the assumption about the observability of \( m_{jt} \) and \( po_{jt} \), investors can always infer the cash flow \( x_{jt} \) at \( t + 1 \) but not their individual components. Similarly, investors can write enforceable contracts based on operating cash flows \( x_{jt} \). Figure 1 summarizes the actions within each time period.

Figure 1: Model Time-line

2 Environments that Deliver First Best Outcomes

2.1 Full Manager Ownership

Suppose that each firm \( j \) has a manager-owner with enough wealth \( w_{j0} \) to invest by himself. In period \( t = 0 \), this executive decides how much to hold in safe bonds \( b_{j0} \) for one period (without loss of generality), and how much to physically invest \( i_{j0} \) in the company. Thereafter, the manager decides how much to invest at the end of period \( t \), both in the firm’s capital \( i_{jt} \) and in safe bonds \( b_{jt} \) to be held from period \( t \) to \( t + 1 \). \(^6\) The manager’s expected utility is:

\(^6\)It can be seen that this framework allows the investor to keep the same bond for longer periods by setting \( b_t = b \) for some \( t \in [s, s + d] \), where \( d \) is the holding period.
$$E(U_j) = c_{j0} + \sum_{t=1}^{\infty} \frac{E(c_{jt})}{(1+r)^t}$$

Where $c_{j0} = w_{j0} - b_{j0} - i_{j0}$, and $c_{jt} = (1 + r)b_{jt-1} + a_{jt-1}f(k_{jt-1})s_{jt} - i_{jt} - b_{jt}$. Hence $E(c_{jt}) = (1 + r)b_{jt-1} + a_{jt-1}f(k_{jt-1}) - i_{jt} - b_{jt}$. The manager’s expected utility is:

$$E(U_j) = (w_{j0} - b_{j0} - i_{j0}) + \sum_{t=1}^{\infty} \frac{(1 + r)b_{jt-1} + a_{jt-1}f(k_{jt-1}) - i_{jt} - b_{jt}}{(1+r)^t}$$

$$= (w_{j0} - i_{j0}) + \sum_{t=1}^{\infty} \frac{a_{jt-1}f(k_{jt-1}) - i_{jt}}{(1+r)^t}$$  \hspace{1cm} (1)

Notice that the executive is indifferent about how much to invest in the riskless asset since it always cancels out with its return, as $-b_{jt-1} + \frac{(1+r)b_{jt-1}}{1+r} = 0$ for all $t$. Theorem 1 derives the first best solution to this problem.

**Theorem 1.** The unique solution to this problem is:

$$f'(k^*_jt) = \frac{r + \delta}{a_{jt}}$$

Proof: See Appendix. The envelope theorem defines how the optimal capital changes with variations in the model’s primitive variables:

$$\frac{\partial k^*_jt}{\partial r} = \frac{\partial k^*_jt}{\partial \delta} = \frac{1}{a_{jt}f''(k^*_jt)} < 0$$

$$\frac{\partial k^*_jt}{\partial a_{jt}} = -\frac{r + \delta}{a^2_{jt}f''(k^*_jt)} > 0$$

In other words, higher real interest or depreciation rates lower the optimal capital, while greater managerial talent raises the optimal capital used. The capitalist-entrepreneur invests the balance of his wealth in the safe asset, i.e., $b_{jt} = w_{jt} - k^*_jt$.

If managerial productivity is constant, i.e., $a_{jt} = a_j$, then so will be the optimal capital $k_{jt} = k_{js} \equiv k^*_j$ for any $t, s$. Optimal investment policy will be $i_{j0} = k^*_j$ and $i^*_jt = \delta k^*_j$ for $t > 0$. The firm’s operating cash flows are:
\[ x_{jt} = a_j f(k_j^*) s_t - \delta k_j^* \] (2)

The expected value of the firm’s operations at \( t > 0 \) is the present value of the expected operating cash flows \( x_{jt} \), i.e.,

\[ V_{jt}^* = \frac{[a_j f(k_j^*) - \delta k_j^*]}{r} \]

To find the firm’s value, we would add any excess cash \( m_t \) that it may have. The expected utility for the manager-owners at \( t = 0 \) is:

\[ E[U_j^*] = w_{0j} - k_j^* + \frac{[a_j f(k_j^*) - \delta k_j^*]}{r} \]

The net present value of the firm’s operations, \( V_j^* - k_j^* \) is positive, as proven in the appendix. For the high ability manager agent to become manager, it is necessary that \( E[U_h^*] > w_{0j} + \frac{\omega}{r} \Rightarrow V_h^* - k_h^* > \frac{\omega}{r} \), i.e., that the firm’s NPV be greater than the manager’s outside opportunity, which I assume to be the case. For the low ability manager a similar condition must hold, \( V_l^* - k_l^* > \frac{\omega}{r} \).

If managers cannot have negative consumption in any period, then they must hold sufficient amounts of the riskless asset. In particular, they must inject capital when \( x_{jt} \equiv a_j f(k_j^*) s_t - \delta k_j^* < 0 \). To do this on their own, managers’ initial wealth should then be high enough so that consumption is positive at all times, i.e. \( c_{jt} = rb_t + x_{jt} > rb_t - \delta k^* > 0 \), so \( w_{j0} > (1 + \frac{\delta}{r})k_j^* \). In other words, manager owners need to invest \( b_t \geq \frac{\delta}{r}k_j^{*T} \) in the riskless asset. In a corporation, this is equivalent to holding excess cash, a result that will re-appear in other environments. Notice that the wealth constraints are higher for high ability managers than for their low ability counterparts. This means that firms managed by high ability executives need more excess cash than companies with low ability executives.

What conditions could produce an executive who is also a sole owner? In a constant

\[ \text{For time varying managerial ability this necessary excess cash will be } b_{t-1} \geq \frac{k_j^* - (1 - \delta)k_j^{*T}}{r} \]
ability setup, the manager must have high initial wealth. This could also happen if efficient scale can be reached even by modestly endowed entrepreneurs, i.e., for small and medium sized businesses.

For manager owned companies that have a large size $k_t$ we would need a time-varying $a_t$, a relatively small $k_0$, extraordinary productivity $f'(\cdot)$, exceptional improvement in managerial abilities $a_{jt}$, and mild or positive initial state variables $s_t$. This combination of skill and luck is rare.

At time $t = 0$, a low ability manager who initially owns a firm will find it profitable to sell the company, and a high ability manager to buy it, for any price $P$ as long as $P \in (V^*_l - k^*_l, V^*_h - k^*_h)$. Given the assumption that there are many unemployed and wealthy high ability managers, the owner could make a take it or leave it offer for $P = V^*_h - k^*_h - \omega/r$. This offer will be accepted if there are more than one unemployed high ability manager without any proprietary idea but such level of wealth. In more general settings, the price $P$ would be determined by the bargaining setup between well endowed buyers and sellers.

2.2 Wealth Constrained Managers and Outside Capitalists with Perfect Information

The first best is also obtained in a perfect information world, even if managers have no initial wealth. Suppose that investors can observe everything perfectly. The optimal contract depends on who has the firm’s original property rights, but the outcome is identical. This is a straightforward application of Coase’s Theorem (1960).

If a manager originally owns the firm, and has no capital for the initial investment, he would offer a stake $\alpha(x_{jt} + rm_{jt-1})$ to outside stakeholders in exchange for the initial investment $(k^*_j + m_{j0})$ in physical capital and excess cash, where $k^*_j$ is the amount found

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8 an example of this is the case of William Randolph Hearst
9 Technically, because $|f''(k)|$ is large in absolute value
10 some examples of these large scale manager owners are Andrew Carnegie, John D. Rockefeller, Henry Ford, Bill Gates and their associates.
11 The transfer price for $t > 0$ is $P^* \geq V^*_l < P < V^*_h - \omega/r$, and this range exists as long as $\omega/r < V^*_h - V^*_l$, i.e., the high ability manager outside opportunity is small
in theorem (1). Any cash shortfalls for \( t > 0 \) are met by capitalists who put \( \alpha \) and managers who put \( 1 - \alpha \) percentage of the shortfall. With this contract, managers face the following maximization problem:

\[
E(U_j) = -b_{j0} + \sum_{t=1}^{\infty} \frac{(1 + r)[b_{jt-1} + (1 - \alpha)m_{jt-1}] + [a_{jt-1}f(k_{jt-1}) - i_{jt}] - [b_{jt} + (1 - \alpha)m_{jt}]}{(1 + r)^t}
\]

\[
= (1 - \alpha)m_{j0} + (1 - \alpha)\sum_{t=1}^{\infty} \frac{a_{jt}f(k_{t-1}) - i_{jt}}{(1 + r)^t}
\]

Where the manager chooses \( i_{jt} \) for \( t > 1 \) (remember that the initial capital was set by outside investors at \( k^*_j \)). The first order conditions are the same as those for equation (1), where managers had enough wealth to invest on their own. The solution is \( k_{jt} = k^*_j \) for \( t > 1 \).

The stake \( \alpha \) has to be such that the outside shareholders are willing to invest, i.e., that their returns are \( r \). The appendix\(^{12}\) shows that there exists \( \alpha \) such that \( \alpha(V_j + m) - (k^* + m) \geq 0 \(^{13}\). This setup looks like a hedge fund contract, where the manager retains a stake \( \alpha \) even though he does not contribute any physical capital.

If capitalists are the initial owners, they would hire high ability managers\(^{14}\), and offer them a stake \( \beta \) of the operating and financial cash flows, \( x_{jt} + rm_{jt-1} \) for \( t > 0 \) such that \( \beta V_h + \beta m_0 = \frac{\omega}{r} \). There exists such a \( \beta > 0 \) as long as \( V_h > \frac{\omega}{r} \). In this contract, managers would inject capital whenever there is a shortfall, i.e. when \( a_h f(k_h^*)s_t + rm_{ht-1} < \delta k_h^* \).

Notice that in both contracts, managers have to put in money in bad states of the world \( s_{jt} < \frac{\delta k^*_j - rm_{jt-1}}{a_j f(k^*_j)} \). Theorem (2) considers how much a firm can borrow at any given point, and studies the effect of excess cash. It shows that a company can borrow on a contingent credit line up to the present value of its expected cash flows. This credit line accumulates interest if the firm is unable to pay the principal or interest in any given

\(^{12}\)Theorem 2, Lemma 1

\(^{13}\)which of course would of course also satisfy the interim cash infusion condition that \( \alpha V_j - \alpha \delta k^* > 0 \)

\(^{14}\)This is intuitively obvious. At time \( t=0 \) the present value of cash flows is \( V_j = \frac{a_j f(k^*) - \delta k^*}{r} \), and \( \frac{\partial V}{\partial m} = \frac{f(k^*)}{r} - \frac{f'(k^*)}{r} > 0 \) because \( f''(k^*) < 0 \). This derivative uses the optimality conditions found in section 2.
period. It concludes that if \( m_t < \frac{\delta r_k}{r} k^*_j \), the company will eventually go bankrupt with a 100% probability.

A low ability manager with initial ownership rights would find it advantageous to fire himself and hire a high ability manager, offering outside investors a participation \( \alpha \), and the high ability manager a stake \( \beta \) such that satisfy their participation constraints\(^{15}\).

**Theorem 2.** At time \( t \) The firm can borrow up to an amount \( V_j^* + m_t \) on a contingent credit line. The firm will go bankrupt with a 100% likelihood if \( m_t < \frac{\delta r_k}{r} k^*_j \).

**Proof:** See appendix

**Example:**

Suppose that \( a f(k) = 2\sqrt{k} \), that \( r = \delta = 0.26 \), that the company holds no excess cash so \( m_t = 0 \), and that the random variable \( s_t \) takes the following values:

\[
 s_t = \begin{cases} 
 0 & \text{w.p. } \frac{1}{2} \\
 2 & \text{w.p. } \frac{1}{2} 
\end{cases}
\]

The optimal capital level is \( \frac{1}{\sqrt{k^*}} = 0.52 \Rightarrow k^* = 3.70 \). The operating cash flows and enterprise values are:

\[
 x_t = 3.85s_t - 0.96 \\
 V = \frac{3.85 - 0.96}{0.26} = 11.09
\]

In this situation, the company requires 5 consecutive bad draws to run up its borrowing plus accrued interest to the full value \( V \). The probability of failure at \( t < 5 \) is 0. The probability of failure at \( t = 5 \) is \( \frac{1}{2^5} = \frac{1}{32} \). The probability of failure at \( t = 6 \) is \( \frac{31}{32} \times \frac{1}{32} \), i.e., one good draw at \( t = 1 \) and five consecutive bad draws for \( t = 2, \ldots, 6 \). And so on. If the company had excess cash equal to \( m_t = 0.765Vt \), i.e. approximately 20% of the physical

\(^{15}\)To see this, note that the low ability manager owner would get a payo ff of \( [1 - \beta - \alpha]V_h + \frac{\omega}{r} \) by delegating management and enjoying the outside option, while he would get \( [1 - \alpha^2]V_l \) by running the firm alone. The first expression reduces to \( V_h - k_h \) using the participation constraints of outside capitalists and managers, while the second expression yields \( V_l - k_l \) again using the outside capitalists' participation constraint. Notice that \( \frac{\partial V}{\partial k} - \frac{\partial k}{\partial a} = \frac{\partial V}{\partial a} = \frac{f(k)}{r} - \frac{r + \delta}{r \sqrt{f(k)}} \).
capital, then the earliest default would be at $t = 6$, and the probability of such default would be $q = \frac{1}{2^6} = \frac{1}{64}$. In this example, if $m_t \geq 3.70$ then the company would never go bankrupt.

In this context, bankruptcy is defined as a change of control from old shareholders to the credit line holders, assuring managers a position in the firm. If this control event was costly, the firm would maximize its value by holding $m_t = \frac{\delta k^*}{c}$ at all times. The idea is that such excess cash allows the firm to replace its depreciated value even in the case that $s_t = 0$. Biass et al. (2007) find a similar result in their continuous time model.

3 Equilibrium with Hidden Information

In this section I will abandon the assumptions that generated the first best results, and study the following complications:

- Adverse selection: investors cannot freely distinguish between high and low ability managers. Their initial belief is that they have low ability managers with probability $p_0$ (i.e., the proportion in the general population). Their assessment is updated using Bayes’ theorem.

- Moral Hazard: Investors are unable to costlessly verify a firm’s physical capital and investment $k_{jt}$ and $i_{jt}$. However, they are able to observe the firm’s financial assets at no cost at the beginning of each period. At the beginning of each period, investors can pay a fixed cost $c$ that allows them to verify physical capital $k_{jt}$ and to change managers. The control verification event will also degrade current managerial ability by a factor $\theta < 1$ such that $\theta a_h < (1 - p_0)a_h + p_0a_l$. This means that after the control event has shown different capital levels that allow me to figure out managerial abilities for sure, even high ability managers will be at a disadvantage relative to re-hiring outsiders from a mixed pool in the market. Another consequence of this assumption is that both old managers, good or bad, would be unable to re-enter the labor market. Managers are aware of the consequences of the control verification event and must be offered a contract that delivers their ex-ante outside option. For
investors, the control verification event acts as a "fresh start", in the sense that they will be faced with new management, and in that it is as if they are at \( t = 0 \) again. I assume that if investors wanted only to intervene but not verify the capital, they would still have to pay \( c \).

This section is divided in three parts. The first subsection develops the investors’ intervention rule based on managers’ general investment policies. The intervention rule will depend on a likelihood ratio \( \Lambda_t \) of the average operating cash flows \( \pi_t \). The likelihood ratio \( \Lambda_t \) will be constructed both exactly and asymptotically, and these two will be compared. In the second subsection I will show managers’ specific investment policy when they know the intervention rule. The first result is that managers will distort their investment relative to the first best case. The second result is that managers, good and bad, will lie when the average cash flow comes close to the intervention level. The set of lies will be to overstate earnings and investment in physical capital. These lies will worsen the situation over time, and take the company to a deteriorated capital level when investors intervene. The lies affect both good and bad managers, but the latter will be more likely to suffer them. I construct the optimal mechanism (optimal here means the one that asymptotically delivers the best results net of intervention costs \( c \)). Low ability managers will always lie about unverifiable information. The third subsection calculates the equilibrium, i.e, a specific intervention rule and managers’ investment policy that are coherent with each other.

### 3.1 The Intervention Control Rule

At the beginning of period \( t + 1 \) investors can infer the firm’s operational cash flow:

\[
x_t = a_j f(k_{jt-1})s_t - i_{jt}
\]

Investors, however, are unable to see the individual investments and level of physical capital except if they trigger a control verification event. Investors have a prior so that \( Prob(a_j = a_l) = p_0 \).\(^{16}\) Given a sequence of cash flows \( X_t = \{x_1, x_2, \ldots, x_t\} \), their posterior

\(^{16}\)More exactly, they believe that the probability \( p \) can take a value of 0 (a high ability manager) or 1
probability $p_t$ is given by equation (3):

$$p_t = \frac{p_0}{p_0 + (1 - p_0)\Lambda_t} \quad (3)$$

Where $\Lambda_t = \frac{L(a_h|X_t)}{L(a_l|X_t)} = \frac{P(X_t|a_h)}{P(X_t|a_l)}$ is a likelihood ratio that compares the alternative hypotheses\textsuperscript{17}. Theorem (3) states that there exists a unique probability $p_i \in (p_0, 1)$ such that for all $p_t > p_i$ investors intervene the firm and force a change in control (the subscript $i$ stands for intervention). Notice that the posterior probability is constructed at the beginning of period $t+1$, so the soonest that investors can change managers is period $t+1$ for events that happened at $t$.

The first assumption needed for theorem (3) is that intervention costs $c$ are relatively low. High intervention costs $c$ would prevent investors from changing managers, who would in effect become entrenched. The second condition for Theorem (3) is that if the firm’s value depends on $p_t$, then it goes down as $p_t$ rises. Properly speaking, this last condition is not an assumption, but a result that I will verify in section 3.3 when I solve for $V_j(p)$.

**Theorem 3.** Suppose that

1. $V_{jt}$ depends on the probability $p_t$ so that $V_{jt} = V_j(p_t)$, with $\frac{\partial V_j}{\partial p} \leq 0$.

2. $c < p_0V_l(p_0) + (1 - p_0)V_h(p_0) - V_l(1)$

There exists a unique $p_i \in (p_0, 1)$ so $\forall p > p_i$ investors will intervene the firm and change managers.

An intervention $p_i$ implies a value for the likelihood ratio, called $\Lambda_i$, such that if $\Lambda_t < \Lambda_i$, then investors will intervene. I assume that investors will also intervene if they believe they are being defrauded, i.e., if they are almost sure that manager $j$, good or bad, is stating a $s_t$ and $i_{jt}$ different from reality or is destroying physical capital.

The likelihood ratio is a fundamental tool in probability and statistics, so I can use some well known results. The first is that there is a closed form approximation using the

\textsuperscript{17}If investors had diffuse priors then $p_0 = \frac{1}{2}$ and the formula simplifies to $p_t = \frac{1}{1 + \Lambda_t}$.
central limit theorem for $x_t = a_j f(k_{jt-1}) s_t - i_{jt}$.

**Theorem 4.** Suppose that $s_t$ are independent, $\text{var}(s_t) \equiv \sigma^2_s$, and $E(s_t^{2+\eta}) < \Delta < \infty$ for some $\eta > 0$. The likelihood ratio can be approximated by

$$\Lambda^A_t = \exp \left\{ \frac{t}{2} \left[ \left( \frac{x_t - \mu_{ht}}{\sigma_{ht}} \right)^2 - \left( \frac{x_t - \mu_{lt}}{\sigma_{lt}} \right)^2 \right] \right\}$$

(4)

where $\mu_{jt} = a_j f(k_{jt-1}) - i_{jt}$ and $\sigma_{jt} = a_j f(k_{jt-1}) \sigma_s$

Proof: see Appendix

This approximation makes intuitive sense. For example, if $x_t = \mu_{ht}$, then $\Lambda_t \to +\infty$ as $t \to \infty$, so $p_t \to 0$, i.e., agents become certain that they are dealing with a high ability manager. Similarly, if $x_t = \mu_{lt}$, then $\Lambda_t \to 0$ as $t \to \infty$, so $p_t \to 1$, i.e., investors will believe that they are dealing with a low ability manager. The disadvantage of equation (4) as opposed to an exact formulation is that it is not efficient. For instance, if $s_t$ had an upper bound, there would be some high realizations that are impossible for low ability managers, so $L(X_t|a_l) = 0$ and $L(X_t|a_h) > 0$. This would imply that $\Lambda_t = +\infty$ and that $p_t = 0$ immediately. This criticism notwithstanding, equation (4) has two very useful advantages. The first is that it yields a tractable closed formula instead of an unwieldy joint distribution. The second benefit is that investors may be unsure about the underlying specification of $f(s)$ apart from the first two moments. In that case, any mis-specification would produce incorrect decisions. In other words, the asymptotic likelihood ratio is a more robust intervention mechanism.

### 3.1.1 A Numerical Calibration

Suppose that the firm’s annual production function is

$$a_j f(k_{jt-1}) = \frac{a_j}{\gamma} k_{jt-1}^\gamma$$

---

18 The assumption that $x_t$ are independent is not necessary, as discussed in White (1984), V. 4. The conditions for a central limit theorem to apply are well known and not particularly strong. In this case, since I assume that $s_t$ are i.i.d, they should easily be met.
where \( \gamma = \frac{2}{3}, a_h = 1.924, a_l = 1.786 \) and \( \delta = 6.77\% \). The optimality condition in theorem 1 implies that the invested capital is:

\[
k_j^* = \left( \frac{a_j}{r + \delta} \right)^3
\]

This numerical calibration matches the profitabilities and sizes of publicly traded firms and tells us that the firm size will grow to the cube of manager’s ability. Hence, although Exxon was 322 times larger than Western Refining Inc., this could be explained by the same underlying technology and by an Exxon management team being ‘only’ 6.85 times more productive than Western Refining Inc.’s\(^\text{20}\). Similarly, an annual growth of 1\% in managerial productivity would translate into a 3\% growth in capital, cash flows, etc.

Suppose that the shocks \( s_t \) occurred every quarter, so quarterly cash flows were:

\[
x_{j,t} = \frac{a_j f(k_{j,t-1})s_t}{4} - \frac{i_{j,t-1}}{4}
\]

\[
s_t = \begin{cases} 
0.2956 & w.p = 0.5 \\
1.7044 & w.p = 0.5 
\end{cases}
\]

Note that \( E(s) = 1 \) and \( \sigma_s = 0.7044 \). Table (1) shows the results of this economy using parameter values meant to capture the main known facts of all publicly traded U.S. non-financial firms in 2010\(^\text{21}\).

If the verification control cost is \( c = 37.98 \) and the prior probability of the low type is \( p_0 = 0.20 \), then we can find the posterior probability \( p_i \) that prompts investors to verify and change control:

\[
p_i = p_0 + \frac{c}{V_h - V_l} = 0.20 + \frac{37.98}{759.53} = 0.25
\]

\(^{19}\) These numbers are chosen to match the profitabilities, cost of capital, and size of all publicly traded firms in the United States in 2010.

\(^{20}\) This uses stock market data from March 8, 2011. Although a productivity differential of 7 times seems large, such have been found in knowledge workers such as software developers. See Albrecht (1979) and Curtis (1981), for studies on programmer productivity.

\(^{21}\) 5810 firms, with sales totaling 16 trillion dollars and invested capital equal of 10 trillion
Table 1: Numerical Example

<table>
<thead>
<tr>
<th></th>
<th>$a_h = 1.924$</th>
<th>$a_l = 1.786$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production function $\gamma$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>Annual discount rate $r$</td>
<td>8.50%</td>
<td>8.50%</td>
</tr>
<tr>
<td>Annual depreciation rate $\delta$</td>
<td>6.77%</td>
<td>6.77%</td>
</tr>
<tr>
<td>Optimal Capital $k_j^*$</td>
<td>2000</td>
<td>1600</td>
</tr>
<tr>
<td>Expected Annual Cash Flow</td>
<td>322.72</td>
<td>258.12</td>
</tr>
<tr>
<td>Enterprise Value $V_j$</td>
<td>3796.71</td>
<td>3037.18</td>
</tr>
<tr>
<td>Prior Probability</td>
<td>0.80</td>
<td>0.20</td>
</tr>
<tr>
<td>Expected Quarterly Cash Flow</td>
<td>$\mu_j = 80.68$</td>
<td>64.54</td>
</tr>
<tr>
<td>Quarterly Cash Flow volatility $\sigma_{jt}$</td>
<td>$80.68$</td>
<td>64.54</td>
</tr>
</tbody>
</table>

The assumptions are meant to match the 2010 averages for non-financial companies in the United States (consult http://pages.stern.nyu.edu/~adamodar/ for more on these numbers). The $r$ equals the average corporate WACC. These parameters imply that $ROIC = \frac{\mu_j}{k_j} \approx 16.95\%$. The resulting size of $k_h^* = 2000$ matches the approximate invested capital of the average 2010 U.S. non-financial firm in the United States.

I will now compare the exact and asymptotic likelihood functions. Suppose we want to know how quickly investors would throw out managers who generate quarterly cash flows of $x_{tq} = 64.54$. This value is equivalent to the low ability manager expected cash flow values.

Given the binomial nature of $s_t$, we can calculate the operational cash flows and probability distributions conditional on the manager type:

$$x_{tq|a_h} = \begin{cases} 0 & w.p. = 0.5 \\ 161.36 & w.p. = 0.5 \end{cases}$$

$$\bar{x}_{tq|a_h} = \frac{g_h}{t} \times 161.36 \quad w.p. = \frac{t!}{2^t(t-g_h)!g_h!}$$

where $g_h$ are the number of times that $s_t = 1.7044$ is drawn. We can do the same for the low ability managers:

$$x_{tq|a_l} = \begin{cases} 0 & w.p. = 0.5 \\ 129.08 & w.p. = 0.5 \end{cases}$$

$$\bar{x}_{tq|a_l} = \frac{g_l}{t} \times 129.08 \quad w.p. = \frac{t!}{2^t(t-g_l)!g_l!}$$

If we observe a given $\bar{x}_t$, we can justify it by a number good draws $g_h$ under $a_h$ and
of good draws $g_l$ under $a_l$, with the constraint that $g_l = 1.25g_h$. The exact likelihood function for any given $\pi_t$ is

$$\Lambda_t(\pi_t) = \frac{(t - g_l)!g_l!}{(t - g_h)!g_h!}$$

Where $g_h \equiv \frac{t_{161.36}}{161.36}$ and $g_l = 1.25g_h$. The asymptotic likelihood function using Theorem 4 yields a significantly simpler formula:

$$\Lambda_t^A(\pi_t) = \exp \{-0.02t\}$$

Figures 2 and 3 compare the results of these likelihood functions. Since the differences are minuscule, Figure 2 shows the error of the asymptotic distribution versus the exact one, defined as:

$$\varepsilon_t = \frac{\Lambda_t^A - \Lambda_t}{\Lambda_t}$$

As can be seen, even with only 10 quarters, the error is less than 1.8%.

Figure 3 reports the posterior probabilities using the different likelihood ratio formulas. As can be seen, the asymptotic likelihood ratio yield a good enough decision, as it will force a management change as if we used an exact likelihood function, i.e., between periods 10 and 20. Here, the asymptotic likelihood ratio will tell us that managers will change in quarter 15\textsuperscript{23}, that is after 3.5 years at the company. The management change is faster if the prior probability of low ability managers was higher than 20%. For example, if investors held diffuse priors about the managerial types, and used the asymptotic likelihood ratio test, they would change management in 10 quarters, i.e., 2.5 years.

It is important to stress that likelihood ratio tests are naive in many respects. In the first place, for very low realizations of $\pi_t$ it will say that the high type $a_h$ is more likely. This result is independent of using an exact or an asymptotic likelihood ratio, and has

\textsuperscript{22}This is because high ability managers are 25% more productive than low ability managers, so to justify a given observation, low ability managers must have been 25% luckier than high ability managers to compensate for the productivity disadvantage

\textsuperscript{23}t=14.38 in fact
Figure 2: Error or Asymptotic Likelihood Ratio

to do with the relative likelihood of improbable events. Mathematically, it is not true in this model that $x_t|a_h$ will first order stochastically dominate $x_t|a_l$. The reason is that high ability managers will generate better average returns, but will also require bigger investments to replenish the depreciated capital. Figure 4 plots $p_t$ as a function of $\pi_t$ for $t = 10, 30, 50$ using the asymptotic likelihood ratio and the numerical example:

There are a some notable features in Figure 4. The first is that $p_t$ is non-monotonic function of $\pi_t$. The change of sign in the numerical example happens at $\pi_t = 35$, and in the general formula is at $\pi_t^0 \equiv \frac{\overline{\mu}_l \sigma_h^2 - \overline{\mu}_h \sigma_l^2}{\sigma_h^2 - \sigma_l^2} < \overline{\mu}_l$. Furthermore, there will be some posterior probabilities $p_t > p_i$ for very low cash flows (in this example, $\pi_{10} = 13, \pi_{30} = 4$ and $\pi_{50} = 2$)\textsuperscript{24}. These cash flows and posterior probabilities for $\pi_t \ll \pi_t^0$ will be ruled out by the simple argument that they are easily falsifiable by the low type. Consider the numerical example with 10 draws. If the low ability manager had an average cash flow of 60, investors will infer a posterior likelihood of 25.3% and would change management.\textsuperscript{24}

\textsuperscript{24}To see that this is not an artifact of the asymptotic likelihood ratio approximation, consider a situation where $\pi_t = -\delta k^*_h$. Investors know that this can only happen to high ability managers, so $L(-\delta k^*_h|a_l) = 0$ and $p_t(-\delta k^*_h) = 0$.\n
19
If investors used the likelihood function naively, the low ability manager would find it advantageous to under-utilize capital and generate cash flows of 10, so investors would infer $p_t = 24.61\%$ and keep old management. Hence I will assume that if investors observe $\bar{x}_t < \bar{x}_t^0$ they will believe that they are being defrauded, and should change managers.

Looking now at the region where $\frac{dp_t}{dx_t} < 0$, there is minimal cash flow $\bar{x}_{ti}$ such that if $\bar{x}_t < \bar{x}_{ti}$ then investors will change managers. In the numerical example this minimal average cash flow is $\bar{x}_{10} = 60.79$, $\bar{x}_{30} = 68.49$, and $\bar{x}_{50} = 69.82$. As the number of quarters goes up, this intervention value approaches 71.72. Conceptually speaking, $\bar{x}_{ti}$ is like a debt payment. If $\bar{x}_{ti}$ is not met, investors intervene the firm and change managers. The proof in the appendix of theorem 4 derives this average debt payments, given by

$$\bar{x}_{it} = a_1 + a_2 \sqrt{a_3 - \frac{a_4}{t}}$$

The incremental change in debt cash flows will be
\[ \Delta d_t \equiv t \pi_{it} - (t - 1) \pi_{it-1} = a_1 + a_2 \left[ \sqrt{a_3 t^2 - a_4 t} - \sqrt{a_3 (t - 1)^2 - a_4 (t - 1)} \right] \]

\( \pi_{it} \) rises at a decreasing rate and tends toward \( a_1 + a_2 \sqrt{a_3} \), a weighted average of the low and high expected cash flows. Another way of understanding these results is that debt is a security that can be justified as the only credible mechanism for managers to reveal their true ability. The fact that a bankruptcy automatically alters property rights may be a way to lower the verification-intervention costs \( c \), and to credibly pre-commit to intervening even for extremely low cash flows \( x_t < x_0 \) when the naive use of the likelihood ratio would counsel patience.

One obvious precaution for investors is that they should account for the fact that managers know the intervention rule, and they will change their behavior (subsection 3.2. shows that they will underinvest relative to the first best equilibrium) and that around...
the verification cash flow point, they will engage in fraud: that is, they will deviate from the optimal investment policy and starve the firm of capital, inflating earnings and investment. This is a manifestation of the classic under-investment and gambling for resurrection problems in financial distress. This means that we need to find the equilibrium $\mu_{jt}$ that is a rational expectation from the investors’ point of view.

### 3.2 The manager’s investment policy

Suppose that managers are wealth constrained, have limited liability, and need positive consumption every period. Then, the contract would have:

1. Since managers are wealth constrained and need positive consumption, the firm needs a cash level that covers depreciation for the high type, i.e. $m_0 = \frac{\delta}{r}k^*_h$. Given this excess cash, the low type will always have a strictly positive payoff.

2. At the beginning of period $t$, suppose that investors decide not to intervene. Managers know the investor intervention rule, so that when they observe $s_t$ they choose their investment policy $i_t$, which will imply an $x_t$ and an $\pi_t$, and possible intervention next period. After the investment is made, managers get a payout $\beta(x_t + rm_{t-1})$ and investors a payout $(1 - \beta)(x_t + rm_{t-1})$ that is publicly observable. Managers consume their payout immediately.

3. Executives justify the payout by stating a state variable $s'_t$ and investment policy $i'_t$. With these reports, investors can get an idea of $x'_t$ that they can verify at $t + 1$. Notice that the low ability manager will always pool with the high type since otherwise investors can readily tell the two types apart. I will need to show that lying is more profitable than truth telling.

---

25Criminal penalties can be thought of as another way to punish a wealth constrained manager. We will show that in general these penalties will have counterproductive effects on the firm’s investment policy

26The assumption is that current wages are a binding contract that cannot be voided. This will considerably clean our results, but in no way changes the intuition of what follows. This will be discussed in Theorem 5. If this contract were not binding, managers would simply take this into account in period $t - 1$ when they would underinvest in a similar way to what I will develop in this setting.
4. Since $m_{t+1}$ is observable at $t+1$, investors will then infer $x_t$ and $\bar{x}_t$, and compare it to the announced results. Investors intervene the firm whenever $\bar{x}_t < \bar{x}_{it}$, or when $x'_t \neq x_t$, i.e. they were lied to. Investors pay the control verification cost and set the optimal capital level at $k^*_h$ defined in theorem 1 and hire new managers from the competitive market, who will be compensated starting in period $t+2$.

5. In period $t+1$ investors may or may not share the payoff or extra ‘golden parachutes’ with old managers. In particular, I assume that they cannot credibly commit to reward a manager who has been falsifying reports. This means that the low ability manager will never get paid in $t+1$.

3.2.1 The manager’s problem

The manager’s investment policy is described in Theorem 5

Theorem 5. Define

$$\bar{x}_t = \bar{x}_t|\bar{x}_{t-1} \equiv \frac{(t-1)\bar{x}_{t-1} + a_{jt}(k_{jt-1})s_t - \delta k^*_j}{t}$$

Then, for

1. Case R: $\bar{x}_t^* \leq \bar{x}_{ti} - \delta k^*_j$, the high ability manager invests optimally while the low ability manager invests nothing at all.

2. Case Y: $\bar{x}_{ti} > \bar{x}_t^* > \bar{x}_{ti} - \delta k^*_j$, the high and low ability managers will underinvest

$$i_{ht} = \delta k^*_j - t (\bar{x}_{ti} - \bar{x}_t^*) < \delta k^*_j$$

3. Case G: $\bar{x}_t^* \geq \bar{x}_{ti}$, then the high ability managers will underinvest while the low ability managers will overinvest.

Figure 5

In words, theorem 5 explains the logical reaction given the intervention rule.
Case R refers to the situation where no matter how little it invest, the manager will be intervened in period $t+1$. For low ability managers it is best to invest nothing, consume the complete cash flow at time $t$ and then be intervened. The high ability manager will invest optimally if he is offered a golden parachute equal to a stake $\beta$ of the cash flow at $t+1$. The denial of golden parachutes will be counterproductive because managers will look at period $t$ as the final one, and 'loot' the company as much as they can. In this case, it would even be optimal to offer a golden parachute to the low ability manager.

Case Y refers to the situation where intervention will be triggered if they invest optimally, but not if they underinvest a little and so boost cash flows. In this situation, both high and low ability managers will underinvest, basically 'gambling for their resurrection' in period $t+1$. From the investors point of view, it is unclear what design is optimal for high ability managers. The cost of inducing perfect investment is that it triggers a change of control, with a cost of $c$. The benefit of forcing perfect investment is the difference between optimal and suboptimal cash flows in period $t+1$. If no special inducements are given in case Y, the high ability manager will simply underinvest.
• Case G refers to the case where optimal investment does not trigger intervention. In this situation high ability managers underinvest while low ability managers underinvest. Holmström (1999) finds a parallel result when analyzing managerial effort in an adverse selection model. To understand this result, notice that there is a signaling effect whereby managers want to increase investment to separate themselves. There is also a volatility effect, whereby a larger investment may cause cash flows with fatter tails, putting the manager in danger of breaching the intervention level next period. Theorem 5 shows that for high ability managers the volatility effect is stronger than the signaling incentive, so they will underinvest. The net effect gets weaker as time goes by, as the average cash flow become more stable, and hence the capital level moves toward \( k^*_h \). For low ability managers, the signaling effect is stronger than the volatility effect, so they overinvest to be closer to high ability managers. Again, as time goes by, the low ability manager tends toward \( k^*_l \), and eventually falls into a case Y or R, i.e., is eventually caught.

3.3 Other possible managerial compensation schemes

Compensation with call options mitigates and may even eliminate the underinvestment problem for high ability managers. Notice that such option incentives should be stronger at the beginning of the firm’s life, and lessen in importance as the company matures. The problem with call option incentives in this case is that will worsen the overinvestment problem for low ability managers, making it specially acute at the beginning of a firm’s life.

Another possible compensation ‘improvement’ would be to defer pay until intervention time. If the capital is less than \( k^*_h \), managers are paid nothing. There are several problems with this compensation scheme. First, managers may require positive consumption levels at all times. The second problem (not in this model) is that managers have a higher discount rate than \( r \), in which case they need to be compensated for their patience. The third problem is that deferred compensation would actually make high ability managers want to bankrupt the company, for example in case Y in theorem 5, a counterintuitive
result. The final and most important problem with deferred compensation is that low
ability managers will simply set capital to $k^*_h$, and when intervention comes, they will
receive pay based on their cash flows. Relative to Theorem 5’s result, this will create a
very large overinvestment problem relative to linear pay.

4 Conclusions

This paper has developed a dynamic investment policy under moral hazard and adverse
selection. The moral hazard problem is how to incentivize managers to invest optimally.
The adverse selection problem is to distinguish the high from the low ability managers.
The adverse selection part of the problem generates a debt level that increases with time.
This security is renegotiation proof, and would make investors intervene and change
managers whenever cash flows are below this debt interest.

To solve the moral hazard part of the problem, linear contracts are used. I show
that these linear contracts (a fixed proportion of a firm’s cash flows) will induce high
ability managers to underinvest. Low ability managers will overinvest in normal times,
and severely underinvest when they are close to bankruptcy. I show that linear contracts
asymptotically implement first best investment for high ability managers. I also discuss
the benefits and drawbacks of some popular alternative rules, i.e. option compensation
and deferred pay. I argue that linear contracts are a relatively good and simple answer,
and that the alternatives will generate important complications that cannot be easily
resolved.

5 References

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6 Appendix: Proofs

Theorem (1): The unique solution to this problem is:

\[ f'(k_{jt}) = \frac{r + \delta}{a_{jt+1}} \]

Proof: We will take the derivatives of \( i_t \) and \( i_{t+1} \):

The first order conditions to the investment problem are:

\[
\frac{\partial E(U_j)}{\partial i_{jt}} = \sum_{s=t+1}^{\infty} \frac{a_{js-1}f'(k_{js-1})}{(1+r)^s} \frac{\partial k_{js-1}}{\partial i_{jt}} - \frac{1}{(1+r)^t} = 0
\]

\[
\frac{\partial E(U_j)}{\partial i_{jt}} = 0 \Rightarrow \sum_{s=t+1}^{\infty} \frac{a_{js-1}f'(k_{js-1})}{(1+r)^s-t} \frac{\partial k_{js-1}}{\partial i_{jt}} = \sum_{s=t+1}^{\infty} \frac{a_{js-1}f'(k_{js-1})}{(1+r)^s-t} (1 - \delta)^{s-t-1} = 1
\]

\[
\frac{\partial E(U_j)}{\partial i_{jt}} = 0 \Rightarrow \sum_{s=t+1}^{\infty} \frac{a_{js-1}f'(k_{js-1})}{(1+r)^s-t} (1 - \delta)^{s-t-1} - 1 = 0
\]

\[
\frac{\partial E(U_j)}{\partial i_{jt}} = 0 \Rightarrow \Psi_{jt} \equiv \frac{a_j f'(k_j)}{1 + r} + \frac{a_{jt+1} f'(k_{jt+1})(1 - \delta)}{(1 + r)^2} + \sum_{s=t+1}^{\infty} \frac{a_{js-1}f'(k_{js-1})}{(1+r)^s-t} (1 - \delta)^{s-t-1} - 1 = 0
\]
\[
\frac{\partial E(U_j)}{\partial i_{jt+1}} = 0 \Rightarrow \Psi_{jt+1} = \frac{a_{jt+1}f'(k_{j,t+1})}{1+r} + \sum_{s=t+3}^{\infty} \frac{a_{js-1}f'(k_{js-1})}{(1+r)^{s-t-1}} (1-\delta)^{s-t-2} - 1 = 0
\]

\[
\Psi_{jt} - \frac{1-\delta}{1+r} \Psi_{jt+1} = \frac{a_{jt}f'(k_{jt})}{1+r} - 1 + \frac{1-\delta}{1+r} = 0 \Rightarrow f'(k_{jt}) = \frac{1+r-1+\delta}{a_{jt}} = \frac{r+\delta}{a_{jt}}
\]

QED.

**Theorem** (2) The firm can borrow on its operating income up to an amount \(V_j^* + m_{jt}\) on a contingent credit line. The firm will go bankrupt with a 100% likelihood if \(m_{jt} < \frac{2}{r} k_j^*\).

**Proof**

At any time \(t > 0\) and given the optimal investment strategy defined in Theorem (1), the actual present value of the firm is given by

\[
V_{jt} = \sum_{\tau=t+1}^{\infty} \frac{a_j f(k_j^*) s_{\tau} - \delta k_j^*}{(1+r)^{\tau-t}}
\]

We can change variables such that \(t' = \tau - t\) and the present value of the firm reduces to:

\[
V_{jt} = \sum_{t'=1}^{\infty} \frac{a_j f(k_j^*) s_{t'+t} - \delta k_j^*}{(1+r)^{t'}}
\]

Define

\[
Z_{t'} = \frac{a_j f(k_j^*) s_{t'+t} - \delta k_j^*}{(1+r)^{t'}}
\]

We need to prove that \(E[Z_{t'}] < \infty\). Define \(s_0 = \frac{\delta k_j^*}{a_j f(k_j^*)}\) as the state below which the operating income cannot cover the required investment. Then \(E[Z_{t'}]\) is:

\[
(1+r)^{t'} E[Z_{t'}] = \int_0^{s_0} [\delta k_j^* - a_j f(k_j^*) s] g(s) ds + \int_{s_0}^{\infty} [a_j f(k_j^*) s - \delta k_j^*] g(s) ds
\]

\[
(1+r)^{t'} E[Z_{t'}] = 2 \int_0^{s_0} [\delta k_j^* - a_j f(k_j^*) s] g(s) ds + \int_0^{s_0} [a_j f(k_j^*) s - \delta k_j^*] g(s) ds
\]

\[
(1+r)^{t'} E[Z_{t'}] = 2a_j f(k_j^*) \int_0^{s_0} G(s) ds + [a_j f(k_j^*) - \delta k_j^*] < 2a_j f(k_j^*) s_0 G(s_0) + a_j f(k_j^*) - \delta k_j^*
\]

\[
E[Z_{t}] < \frac{[2G(s_0) - 1] \delta k_j^* + a_j f(k_j^*)}{(1+r)^{t'}}
\]

\[
\sum_{t=1}^{\infty} E[Z_{t}] < \frac{[2G(s_0) - 1] \delta k_j^* + a_j f(k_j^*)}{r} < \infty
\]
The third line uses integration by parts and exploits the definition of $s_0$. The fourth line again exploits the fact that $a_j f(k_j^*) s_0 \equiv \delta k_j^*$. We can use the Proposition 3.52 in White (1984) so that $V_{jt}$ converges almost surely (i.e., with probability 1) to $E (\sum_{t=1}^{\infty} Z_t) = \sum_{t=1}^{\infty} E(Z_t) = \frac{a_j f(k_j^*) - \delta k_j^*}{r}$.

Suppose that the company has the policy of keeping excess cash $m_t = m$ for $t \geq 0$. At time $t$ a risk neutral outside capitalist with perfect information would therefore be willing to lend up to a total of $\frac{a_j f(k_j^*) - \delta k_j^*}{r} + m$, and to be repaid with a required return per period of $r$. The company’s cash flow $c_{fjt}$ is given by:

$$c_{fjt} = a_j f(k_j^*) s_t - \delta k_j^* + rm$$

The credit line $L_{jt}$ is total amount owed to capitalist, as long as $L_{jt} \leq \frac{a_j f(k_j^*) - \delta k_j^*}{r} + m$. The credit line has the following law of motion:

$$L_{jt} = \min(0, (1 + r)L_{jt-1} - c_{fjt})$$

As long as $c_{fjt}$ can take a negative number the company can always suffer a number of consecutive shocks so that it will go bankrupt. Define $\tau$ as the minimum number of periods of consecutive negative shocks that yields a bankruptcy i.e., that $L_t \geq \frac{a_j f(k_j^*) - \delta k_j^*}{r} + m$ for some $\tau > 0$. The minimum $\tau'$ would be the case where $s_i = 0$ for $\tau'$ periods in a row, and is given by $\tau' = \frac{\ln(a_j f(k)) - \ln(\delta k_j - mr)}{\ln(1 + r)} - 1$. In a continuous density function, the probability of this is zero. We will take the more likely case that the cash flow is negative. Define $s_0 \equiv a f(k_j^*) s_0 - \delta k_j^* + mr \equiv 0$. The average number of periods $\tau$ with a negative shocks $s_i < s_0$ for $\forall i \in [t - \tau, t]$ is given by:

$$\tau = \frac{\ln(a_j f(k_i) - \delta k_j + mr + cf_0) - \ln(cf_0)}{\ln(1 + r)} - 1$$

$$cf_0 \equiv E(cf | s \leq s_0) = \frac{a_j f(k_j) \int_{s_0}^{s_0} G(s) ds}{G(s_0)}$$

Where the average cash flow in bad states $cf_0$ uses integration by parts to simplify the equation. The probability of such a default is $q \equiv G(S_0)^\tau$. At period $\tau + 1$ the probability of default would be $(1 - q)^q$, i.e., an initial good shock followed by the consecutive bad shocks. The cumulative probability of default is given by:

$$\text{Prob}(L_\tau > \frac{a_j f(k_j^*) - \delta k_j^*}{r} + m) = \sum_{t=\tau}^{\infty} (1 - q)^{t - \tau}q = q + \sum_{t' = 1}^{\infty} (1 - q)^{t'}q = q + \frac{1 - q}{q} \times q = 1$$

This means is that if the company arrives at a default point, the outside capitalists will be unwilling to hand any more money beyond the barrier point, and that old shareholders will have to give up their shares to the lenders. The new shareholders can then turn around and borrow on this all-equity firm against new lenders. The expected time between bankruptcies is $\tau + \frac{1}{q}$. For bankruptcy never to happen, we would need $c_{fjt} \geq 0 \Rightarrow rm \geq \delta k_j^* \Rightarrow m \geq \frac{\delta}{r} k_j^*$. QED.

Theorem 2, Lemma 1: There exists a $\alpha > 0$ such that outside capitalists are willing to invest $k^*$ initially and $\alpha \delta k^*$ in exchange for the $\alpha$ of the future cash flows need to

\[27\text{Notice that if } m > \frac{\delta}{r} k_j^* \text{ then } s_0 = 0.\]
The likelihood ratio can be approximated by theorem establishing the result.

The first equality follows from the definition of the optimal capital rate \( k^* \). The first inequality follows from the concavity of \( f() \). The second equality follows from the optimality condition of \( f'(k^*) \). Notice then that \( \pi(\alpha = 1) > 0 \). Also, \( \pi(\alpha = 0) = -k^* < 0 \). Further, \( \pi(\alpha) \) is monotonic. Therefore, by the intermediate value theorem, there exists a \( \alpha' \) s.t. \( \pi(\alpha') = 0 \).

Managers and capitalists are willing to inject temporary cash flows because for an interim cash flow injection, the most that is needed to put is \( \alpha' \delta k^* \) in exchange for the \( \alpha \) of the future cash flows, with present value of \( \alpha V_j \). Therefore, the interim cash injection makes sense as long as \( \alpha' V_j - \alpha' \delta k^* > \alpha' V_j - k^* \geq 0 \). Therefore at \( \alpha' \) the capitalist is willing to commit the initial capital \( k^* \) and any future capital injections, up to in fact \( k^* \) again. Cash constrained managers can borrow a PIK amount contingent on a good payoff, this has an NPV = 0

Theorem 3.
Suppose that

1. \( V_{jt} \) depends on the probability \( p_t \) so that \( V_{jt} = V_j(p_t) \), with \( \frac{\partial V_j}{\partial p} \leq 0 \).
2. \( c < p_0 V_l(p_0) + (1 - p_0) V_h(p_0) - V_l(1) \)

There exists a unique \( p^* \) s.t. \( \forall p > p^* \) investors will intervene the firm and change managers.

Proof:
Let us define

\[
\Psi(p) \equiv p V_l(p) + (1 - p) V_h(p) - p_0 V_l(p_0) - (1 - p_0) V_h(p_0) + c
\]

We can verify that

\[
\Psi(p_0) \equiv c > 0
\]

\[
\Psi(1) \equiv c - [p_0 V_l(p_0) - (1 - p_0) V_h(p_0) - V_l(1)] < 0
\]

The second inequality follows from the condition on verification costs. The derivative of \( \Psi(p) \) is

\[
\Psi'(p) = -[V_h(p) - V_l(p)] + p V'_l + (1 - p) V'_h < 0
\]

These three facts, \( \Psi(p_0) > 0 \), \( \Psi(1) < 0 \) and \( \Psi'(p) < 0 \) and the intermediate value theorem establish the result.

Theorem 4

If cash flows \( x_t \) are independent, \( var(s_t) \equiv \sigma_s^2 \), and \( E(s_t^{2+\delta}) < \Delta < \infty \) for some \( \delta > 0 \). The likelihood ratio can be approximated by

\[
\Lambda_t \approx \exp \left\{ \frac{1}{2} \left[ \left( \frac{\sqrt{\tau_t} - \mu_t}{\sigma_t} \right)^2 - \left( \frac{\sqrt{\tau_t} - \mu_{ht}}{\sigma_{ht}} \right)^2 \right] \right\}
\]
where \( \mu_{jt} = a_j f(k_{jt-1}) - i_{jt} \) and \( \sigma_{jt} = a_j f(k_{jt-1}) \sigma_s \).

Proof: First, define \( X_t = \{x_1, x_2, \ldots, x_t\} \) as a sequence of past realizations, and \( \pi_t \equiv \frac{1}{t} \sum_{j=1}^{t} x_j \). Then

\[
\Lambda_t \equiv \frac{L(X_t|a_h)}{L(X_t|a_i)} = \frac{L(t\pi_t|a_h)}{L(t\pi_t|a_i)}
\]

This is true because \( x_t \) are assumed to be independent, and the result (26.12) in Billingsley (1979) that uses the characteristic function formulation of the density function \( f(s) \) to derive this result.

We now need to derive the central limit theorem for \( \pi_t \). We know that for \( x_t = a_j f(k_{jt-1}) s_t - i_{jt} \) we know that \( E(x_t|a_j) = \mu_{jt}, \text{var}(x_t|a_j) = [a_j f(k_{jt-1}) \sigma_s]^2 > \delta' > 0 \) and that \( E\left((x_t - \mu_{jt})^{2+\delta} | a_j\right) = [a_j f(k_{jt-1})]^{2+\delta} E(s_t^{2+\delta}) < [a_j f(k_{jt-1})]^{2+\delta} \Delta < \infty \), where \( k^*_h \) was the optimal capital found in theorem 1. We can apply Liapounov’s central limit theorem (see White 1984, p. 112) which states that

\[
z_t \equiv \left[ \frac{\sqrt{t} [\pi_t - \mu_{jt}]}{\sigma_{jt}} \right] \sim N(0,1)
\]

where \( N(0,1) \) is a normal distribution with mean 0 and variance of 1. To obtain \( L(X_t|a_j) = \text{Prob}(t\pi_t|a_j) = \text{Prob}(z_t) \), i.e.,

\[
L(X_t|a_j) = \text{Prob}(z_t|a_j) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{\sqrt{t} [\pi_t - \mu_{jt}]}{\sigma_{jt}} \right)^2 \right\}
\]

so the likelihood ratio reduces to

\[
\Lambda_t = \frac{L(X_t|a_h)}{L(X_t|a_i)} = \exp \left\{ \frac{1}{2} \left[ \left( \frac{\sqrt{t} [\pi_t - \mu_{ht}]}{\sigma_{ht}} \right)^2 - \left( \frac{\sqrt{t} [\pi_t - \mu_{lt}]}{\sigma_{lt}} \right)^2 \right] \right\}
\]

\[
= \exp \left\{ \frac{t}{2} \left[ \left( \frac{\pi_t - \mu_{ht}}{\sigma_{ht}} \right)^2 - \left( \frac{\pi_t - \mu_{lt}}{\sigma_{lt}} \right)^2 \right] \right\}
\]

Which proves the theorem. QED

\( \Lambda^A_t \) is not monotonic as a function of \( \pi_t \), as shown by its derivative

\[
\frac{d\Lambda^A_t}{d\pi_t} = \Lambda^A_t \left\{ \frac{t}{\sigma^2_{lt}\sigma^2_{ht}} \left( \sigma^2_{ht} (\pi_t - \mu_{ht}) - \sigma^2_{lt} (\pi_t - \mu_{lt}) \right) \right\} = \Lambda^A_t \left\{ \frac{t}{\sigma^2_{lt}\sigma^2_{ht}} \left[ \pi_t (\sigma^2_{ht} - \sigma^2_{lt}) - (\mu_{ht}\sigma^2_{ht} - \mu_{lt}\sigma^2_{lt}) \right] \right\}
\]

The lowest point of \( \Lambda^A_t \) is reached at

\[
\bar{\pi}_t = \frac{\mu_{ht}\sigma^2_{ht} - \mu_{lt}\sigma^2_{lt}}{\sigma^2_{ht} - \sigma^2_{lt}} < \frac{\mu_{lt}\sigma^2_{ht} - \mu_{ht}\sigma^2_{lt}}{\sigma^2_{ht} - \sigma^2_{lt}} = \bar{\pi}_t
\]

To find \( \pi_{jt} \), i.e., the cash flow below which investors intervene, we must find the highest root of \( x \) that solves

\[
p_t = \frac{p_0}{p_0 + (1 - p_0)\Lambda^A_t}
\]

32
\[ \Lambda_t^4 = \frac{p_0}{p_i} - p_0 = \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) < 1 \]

This follows because \( p_i > p_0 \) and \( (1-p_0) > (1-p_i) \)

\[
\frac{t}{2} \left[ \left( \frac{x_t - \bar{\mu}_{lt}}{\sigma_{lt}} \right)^2 - \left( \frac{x_t - \bar{\mu}_{ht}}{\sigma_{ht}} \right)^2 \right] = \ln \left[ \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) \right]
\]

\[
\sigma_{ht}^2 \left( x_t - \bar{\mu}_{lt} \right)^2 - \sigma_{lt}^2 \left( x_t - \bar{\mu}_{ht} \right)^2 - \frac{2\sigma_{ht}^2 \sigma_{lt}^2}{t} \ln \left[ \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) \right] = 0
\]

\[
\sigma_{ht}^2 \left( x_t^2 - 2\bar{x}_t \bar{\mu}_{lt} + \bar{\mu}_{lt}^2 \right) - \sigma_{lt}^2 \left( x_t^2 - 2\bar{x}_t \bar{\mu}_{ht} + \bar{\mu}_{ht}^2 \right) - \frac{2\sigma_{ht}^2 \sigma_{lt}^2}{t} \ln \left[ \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) \right] = 0
\]

This is simply a quadratic equation whose determinant is

\[
\Delta = 4 \left\{ \left( \sigma_{ht}^2 \bar{\mu}_{lt} - \sigma_{lt}^2 \bar{\mu}_{ht} \right)^2 - \left( \sigma_{ht}^2 - \sigma_{lt}^2 \right) \left[ \sigma_{ht}^2 \bar{\mu}_{lt}^2 - \sigma_{lt}^2 \bar{\mu}_{ht}^2 \right] - \frac{2\sigma_{ht}^2 \sigma_{lt}^2}{t} \ln \left[ \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) \right] \right\}
\]

\[
4\sigma_{ht}^2 \sigma_{lt}^2 \left\{ (\bar{\mu}_{ht} - \bar{\mu}_{lt})^2 + \frac{2(\sigma_{ht}^2 - \sigma_{lt}^2)}{t} \ln \left[ \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) \right] \right\}
\]

\[
\bar{x}_{lt} = \frac{(\sigma_{ht}^2 \bar{\mu}_{lt} - \sigma_{lt}^2 \bar{\mu}_{ht}) + \sigma_{ht} \sigma_{lt} \sqrt{(\bar{\mu}_{ht} - \bar{\mu}_{lt})^2 + \frac{2(\sigma_{ht}^2 - \sigma_{lt}^2)}{t} \ln \left[ \left( \frac{p_0}{1-p_0} \right) \left( \frac{1-p_i}{p_i} \right) \right]}}{\sigma_{ht}^2 - \sigma_{lt}^2}
\]

For the numerical example

\[
= 35.86 + 2.22 \sqrt{261 - \frac{1348.65}{t}}
\]

As \( t \to \infty \), this simplifies to

\[
\bar{x}_{lt} = \frac{\sigma_{ht} \bar{\mu}_{ht} + \sigma_{lt} \bar{\mu}_{lt}}{\sigma_{ht} + \sigma_{lt}}
\]

**Theorem 5**

Define

\[
\bar{x}_t^*|a_j = \bar{x}_t|\bar{x}_{t-1} = \frac{(t-1)\bar{x}_{t-1} + a_j f(k_{jt-1})s_t - \delta k_j^*}{t}
\]

Then, for
1. Case R: \( \pi_t < \pi_{ti} - \frac{\delta k^*_j}{t} \) the high ability manager invests optimally while the low ability manager invests nothing at all.

2. Case Y: \( \pi_{ti} > \pi_t \geq \pi_{ti} - \frac{\delta k^*_j}{t} \), then the high and low ability managers underinvest, so that
   \[
   i_{jt} = \delta k^*_j - t (\pi_{ti} - \pi_t) < \delta k^*_j
   \]

3. Case G: \( \pi_t \geq \pi_{ti} \), then the high and low ability managers attain the first best investment rule.

Proof

At the beginning of period \( t \) the manager knows \( \pi_{t-1} \), \( s_t \), and \( \pi_{ti} \). He has to choose the investment \( i_{jt} \geq 0 \) that generates the operating cash flow

\[
x_t = a_{jf}(k_{jt-1})s_t - i_{jt}
\]

\[
\pi_t = \frac{(t-1)\pi_{t-1} + a_{jf}(k_{jt-1})s_t - i_{jt}}{t}
\]

The manager will get a share \( \beta \) of this cash flow. The manager faces three possibilities:

**Case R**: The firm will be intervened no matter what investment the manager chooses, i.e.,

\[
\frac{(t-1)\pi_{t-1} + a_{jf}(k_{jt-1})s_t}{t} < \pi_{ti}
\]

In this situation, the manager’s investment decision depends on what payoff he will get in period \( t + 1 \). Low ability managers will underinvest, given that investors will find out that they have been masking as high ability managers and cannot commit to pay managers at \( t + 1 \). Hence the low ability executive will set \( i_{lt} = 0 \), report operating cash flows, consumes his share, and is fired at \( t + 1 \).

For the high ability managers the situation is different. They are entitled to a share \( \beta \) of next period’s operating cash flows. Note that it is investors’ best interest to guarantee this cash flow payment. In period \( t + 1 \), investors will set an investment \( i_{jt+1} \) that restores the optimal capital at \( k^*_h \) defined in theorem 1 and install new management. Hence:

\[
i_{jt+1} = k^*_h - (1 - \delta)k_{ht} = k^*_h - (1 - \delta)^2k_{jt-1} - (1 - \delta)i_{ht}
\]

I will assume that investors will bear all the verification-change of control cost \( c \). The high ability manager then faces this local maximization problem for what will be his last period in the company:

\[
\tilde{u}_{ht} = \frac{E(u_{ht})}{\beta} = -i_{ht} + \frac{a_{hf}((1 - \delta)k_{jt-1} + i_{ht}) - [k^*_h - (1 - \delta)^2k_{jt-1} - (1 - \delta)i_{ht}]}{1 + r}
\]

\[
\frac{d\tilde{u}_{ht}}{di_{ht}} = -1 + \frac{a_{hf}(k_{ht}) + (1 - \delta)}{1 + r} = 0 \rightarrow f'(k_{ht}) = \frac{r + \delta}{a_{hf}} \rightarrow k_{ht} = k^*_h
\]
This is the optimality condition under the first best solution. To obtain this result it was crucial that investors guarantee a payment to the high ability manager even if the firm is intervened. To "go after" managers in bankruptcy is counterproductive as it makes them underinvest in anticipation to such persecution. This is what happens to low ability managers. To avoid this fate, investors would like to precommit to paying a stake $\beta$ of cash flows $x_t$ to all managers, whether good or bad.

**Case Y:** In this case, if managers invest optimally they will trigger intervention, but there is an investment level that triggers no intervention.

In the case of the low ability manager, it is clear that he will set cash flows so that he goes to the border and no less, i.e.,

$$\frac{(t-1)x_t^{-1} + a_j f(k_{jt-1})s_t - i_{jt}}{t} \Rightarrow i_{jt} = \delta k^*_j - t(x_{ti} - x^*_t)$$

The low ability manager does not invest less than this because it would lower future $x_t$ unduly, and this would make intervention more likely. In other words, doing this is allows the low ability manager to extend his tenure by at least one more period. Additionally, if the low ability manager is lucky in period $t + 1$, he may be off the hook for more periods.

For the high ability manager (and the mechanism designer) the question is whether this suboptimal investment cost is greater than the intervention cost. Suppose that the worst case

$$x^*_t = x_{ti} - \frac{\delta k^*_j}{t}$$

In this case, the high ability manager invests nothing. This generates a cost in lost profits. If these lost profits are less than the intervention cost, then the mechanism designer would find it advantageous to let this 'slide'. For the high ability manager, the payoff of underinvestment is that he will have a next period to operate.

**Case G:** This is when managers will be not be intervened even when they invest optimally, i.e.,

$$\frac{(t-1)x_t^{-1} + a_j f(k_{jt-1})s_t - \delta k^*_j}{t} \Rightarrow x_t > x_{ti}$$

In this case, the local maximization, i.e., comparing this period’s investment with next period’s payoff will yield and optimal capital investment for both the low and high ability managers.

To see the optimal investment rule, define a variable

$$q_{jt+1} = \text{Prob}(x_t \geq x_{ti} | x_{t-1}, a_j, s_t)$$

This variable tells whether the manager will get his pay on period $t + 1$. As we have seen, in period $t$ this $q_{jt+1}$ will either be zero (we are hopelessly below the intervention cash flow even if we invest nothing, i.e. case R) or it is one (cases Y and G).

In the case that $q_{jt+1} = 1$, the manager’s utility (net of all fixed payments) is given by:

$$\tilde{u}_{jt} = \frac{E(u_{jt})}{\beta} = -i_{jt} + \frac{a_j f((1-\delta)k_{jt-1} + i_{jt}) - i_{jt+1}}{1 + r} + \sum_{s=t+2}^{\infty} q_{js} \frac{a_j f(k_{s-1}) - i_{js}}{(1 + r)^{s-t}}$$
The optimal investment is found by:

\[
\frac{\partial \tilde{u}_{jt}}{\partial i_{jt}} = -1 + a_j f((k_{jt}) - i_{jt}) + \sum_{\tau=t+2}^{\infty} q^t_{jr}[a_j f(k_{\tau-1}) - i_{jr}] + \sum_{\tau=t+2}^{\infty} q^t_{jr}[a_j f(k_{\tau-1})(1 - \delta)^{\tau-t-1}] = 0
\]

Where \( q^t_{hr} = \frac{\partial q_{hr}}{\partial i_{ht}} < 0 \) and \( q^t_{lr} = \frac{\partial q_{lr}}{\partial i_{lt}} > 0 \) given the information set at \( t \), so \( \xi_{ht} > 0 \) and \( \xi_{lt} < 0 \), proven in Lemma 6. Given the conditions of Theorem (4) the average cash flow conditional on the knowledge of \( a_j \) (since it is part of the manager’s information set) is given by

Solution

\[
f'(k_{\tau-1}) = \frac{(r + \delta)(1 + \xi_{jt})}{a_j r}
\]

Hence \( \tilde{k}_r > k^*_r \) found in theorem 1.

Notice that given the information set, at \( t - 1 \) this probability will either be one or zero. One can use the law of iterated expectations to derive the probability of intervention given less information.

Lemma 6. \( q^t_{hr} < 0 \) and \( q^t_{lr} > 0 \)

Proof

Define \( y_j = x_j - \mu_j \), with the \( t \) subscript eliminated. Note that \( y_{it} \equiv x_i - \mu_i > 0 \) and for \( y_{ih} \equiv x_i - \mu_h < 0 \) \( x_i \) defined in Theorem 4 by

\[
\Psi = \Lambda_i - \kappa = \exp \left\{ \frac{1}{2} \left( \frac{y_{it}^2}{\sigma_{ii}^2} - \frac{y_{ih}^2}{\sigma_{hh}^2} \right) \right\} - \kappa \equiv 0
\]

\[
x_i[a_j] = a_j f(k_{i-1})s_t - i_{jt}
\]

\[
k_t = \sum_{d=0}^{t} (1 - \delta)^d t_d
\]

\[
\bar{x}_t[a_j] = \frac{1}{t} \sum_{\tau=1}^{t} [a_j f(k_{\tau-1})s_{\tau} - i_{jr}]
\]

\[
\bar{m}_{jt} = E(\bar{x}_t[a_j]) = \frac{1}{t} \sum_{\tau=1}^{t} [a_j f(k_{\tau-1})] - i_{jr}
\]

\[
\bar{\sigma}_{jt}^2 = E([\bar{x}_t - \bar{m}_{jt}]^2[a_j]) = \frac{1}{t} \sum_{\tau=1}^{t} [a_j f(k_{\tau-1})\sigma^2]
\]

Notice that by rational expectations \( \frac{\partial \bar{x}_t}{\partial i_{jt}} = 0 \) since any change in \( \bar{x}_t[a_j] \) is neutralized by a change in \( \bar{m}_{jt} \). We will only look at changes in volatility induced by changes in investment, so that
We can define \( q|a_j \), that as the distribution knowing the type. The conditional asymptotic distribution

\[ y_j|a_j \sim N(0, \sigma_j^2) \]

\[ g(y) = \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{y^2}{2\sigma_j^2}} \]

since this distribution is symmetric

\[ \int_{-\infty}^{0} y^2 g(y) dy = \int_{0}^{\infty} y^2 g(y) dy = \frac{\sigma_j^2}{2} \]

The probability of intervention is given by

\[ q|a_j = \int_{y_{ij}}^{\infty} g(y) dy = 1 - \int_{-\infty}^{y_{ij}} g(y) dy = 1 - G(y_{ij}) \]

\[ \frac{\partial q}{\partial \sigma_j} = -g(y_i) \frac{\partial x_i}{\partial \sigma_j} - \int_{-\infty}^{y_{ij}} g_\sigma(y) dy \]

The first term in this equation is the signaling effect of increasing the volatility, the second term is the effect of decreasing the second order stochastic dominance parameter. We use the definition of \( x_i \) and the implicit function Theorem to sign \( \frac{\partial x_i}{\partial \sigma_j} \) as

\[ \frac{\partial x_i}{\partial \sigma_j} = -\Psi_{x_i} \frac{\sigma_j}{\Psi_{x_i}} \]

\[ \Psi_{\sigma_h} = \Lambda_i \frac{y_{ih}^2}{\sigma_h^3} > 0 \]

\[ \Psi_{\sigma_l} = -\Lambda_i \frac{y_{il}^2}{\sigma_l^3} < 0 \]

\[ \Psi_{x_i} = \Lambda_i \left( \frac{y_{il}}{\sigma_l^2} - \frac{y_{ih}}{\sigma_h^2} \right) > 0 \]

\[ \frac{\partial x_i}{\partial \sigma_h} = -\frac{y_{ih}}{\sigma_h^3} \left( \frac{y_{il}}{\sigma_l^2} - \frac{y_{ih}}{\sigma_h^2} \right) > \frac{y_{ih}^2}{\sigma_h^2} = y_{ih} < 0 \]

\[ \frac{\partial x_i}{\partial \sigma_l} = \frac{y_{il}}{\sigma_l^3} < \frac{y_{il}^2}{\sigma_l^2} = \frac{y_{il}}{\sigma_l} > 0 \]

\[ g_\sigma(y) = \frac{g(y)}{\sigma_j} \left[ \frac{y_j^2}{\sigma_j^2} - 1 \right] \]

The last formula uses the normal distribution.
\[
\frac{\partial q}{\partial \sigma_h} = -g(y_i) \frac{\partial x_i}{\partial \sigma_h} - \int_{-\infty}^{y_h} \frac{g(y)}{\sigma_h} \left[ \frac{y^2}{\sigma_h^2} - 1 \right] dy < -\frac{1}{\sigma_h} \left[ y_h g(y_h) + \int_{-\infty}^{y_h} \left[ \frac{y^2}{\sigma_h^2} - 1 \right] g(y) dy \right] = 0
\]

\[
\frac{\partial q}{\partial \sigma_l} = -g(y_i) \frac{\partial x_i}{\partial \sigma_l} - \int_{-\infty}^{y_l} \frac{g(y)}{\sigma_l} \left[ \frac{y^2}{\sigma_l^2} - 1 \right] dy > -\frac{1}{\sigma_l} \left[ y_l g(y_l) + \int_{-\infty}^{y_l} \left[ \frac{y^2}{\sigma_l^2} - 1 \right] g(y) dy \right] = 0
\]

To prove the last two equations, define

\[
\theta_j \equiv y_{ij} g(y_{ij}) + \int_{-\infty}^{y_{ij}} \left[ \frac{y^2}{\sigma_j^2} - 1 \right] g(y) dy
\]

\[
\frac{\partial \theta_j}{\partial y_{ij}} = g(y_{ij}) + y_{ij} g(y_{ij}) \left[ -\frac{y_{ij}^2}{\sigma_j^2} + \frac{y_{ij}^2}{\sigma_j^2} - 1 \right] g(y_{ij}) = 0
\]

\[
\theta_j = \theta_j(y_{ij} = 0) = \int_{-\infty}^{0} \left[ \frac{y^2}{\sigma_j^2} - 1 \right] g(y) dy = \frac{1}{2} - \frac{1}{2} = 0
\]

The last is by symmetry of \( g(y) \). Now, the final result of the lemma is

\[
\frac{\partial q_h}{\partial \sigma_h} = \frac{\partial q}{\partial \sigma_h} \times \frac{\partial \sigma_h}{\partial \sigma_h} < 0
\]

\[
\frac{\partial q_l}{\partial \sigma_l} = \frac{\partial q}{\partial \sigma_l} \times \frac{\partial \sigma_l}{\partial \sigma_l} > 0
\]

Which is what needed to be proved. QED.