One-Way Essential Complements

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Abstract

We study the behavior of firms in markets with one-way essential complements. These are markets in which one good is essential to the use of another but not vice versa, as arises with an operating system and application software. Our interest is in the division of surplus between the two markets and the related incentive for firms to create complements to an essential good.

Formally, we study a model where consumers value two goods (A and B) and can consume A alone, but can only enjoy B if they also purchase A. When one firm sells A and another firm sells B, the firm that sells B earns a majority of the profits in the B market. However, if the value of the B product is small relative to A, then were the two firms to merge, the optimal price to charge for B is zero. This implies that absent strong antitrust or intellectual property protections, the firm that manufactures A can become a monopolist over A and B costlessly, by producing a competing version of B and giving it away. For example, Microsoft introduced the free Internet Explorer as competition for Netscape; this may have brought Microsoft joint monopoly profits over both the operating and browsers markets. Furthermore, Microsoft has no incentive to raise prices, even after all competition exits.

There are other means for the essential A monopolist to capture surplus from B. We consider the value of such strategies as adding a surcharge to the price of B, degrading quality of rival B products, or acting as a Stackelberg leader. The potential for A to capture B’s surplus highlights the challenges facing a firm whose product depends on an essential good.

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1 Introduction

We generally think of competition as being between two substitute products. But competition also arises between two complementary products. In this paper, we study the competition between two complements where one is essential and the other is not.

An operating system and a microprocessor are both essential as neither works without the other. Here we look at the case where one good ($A$) is essential and the other good ($B$) is optional. A consumer can enjoy $A$ without $B$, but not $B$ without $A$. An example that fits this rule is Windows ($A$) and a media player ($B$). The media player requires Windows to operate while Windows has utility absent a media player. A second example is a cable modem ($A$) and cable telephony ($B$). A consumer can enjoy a cable modem without using cable telephony but cannot use cable telephony without a cable modem. For these and other examples, we look at how the pie is divided between $A$ and $B$.

We consider the case where the sellers of $A$ and $B$ each have market power. We expect, all else equal, the seller of $A$ will do better than $B$. Anyone who wants to enjoy $B$ must also buy $A$, but not vice versa. Thus $A$ is in a stronger position. How much does this asymmetry hurt $B$?

It turns out that in the Nash pricing game, firm $B$ is able to earn more half the increased industry profits it creates. This ability for firm $B$ to capture surplus determines its incentive to innovate or to enter a market where its product is dependent on an incumbent firm in a complementary market.

The problem for $B$ is that there are many ways in which firm $A$ can capture the surplus created by $B$. Indeed, the best outcome for $B$ is our baseline case, the Nash equilibrium, where the two firms price simultaneously.

Firm $A$ can be more aggressive by entering the $B$ market and competing directly with $B$. Outside the case of complements, this is rarely a profitable strategy. Firm $A$ has to pay the costs of entering the $B$ market, but once in the market will find itself in Bertrand competition with the incumbent. Assuming no product differentiation, price will be driven to cost and neither firm will earn any profits. The situation is different here as firm $A$

\[ ^1 \text{If } A \text{ and } B \text{ were perfect complements, so that neither could be enjoyed without the other, then profits are split equally between the two sellers. While this is to be expected when costs are equal, the equal division of profits continues to hold even when production costs are unequal across } A \text{ and } B; \text{ see Casadesus-Masanell, Nalebuff, and Yoffie (2006)} \]
benefits from bringing down the price of \( B \). If customers know that they can get \( B \) for free, this gives them a greater incentive to buy and pay a high price for \( A \).

A second option for \( A \) is to acquire \( B \) and then set the joint profit-maximizing price. We know that this is efficient as \( A \) can eliminate the problem of double marginalization (Cournot, 1838). The surprising result is not that joint profits rise, but how this is achieved. Under a broad range of conditions, we find that \( A \) would choose to price \( B \) at zero. The joint monopolist would give \( B \) away and earn all of its profits in \( A \).

This presents a problem to a firm selling \( B \); its rival in \( A \) can earn the joint monopoly profits by driving the price of \( B \) down to zero. If firm \( A \) can enter the \( B \) market and drive the price to zero, this will be as profitable as buying \( B \); hence firm \( A \) will compare the costs of buying \( B \) to entering the \( B \) market and will only be willing to buy \( B \) to the extent this is cheaper than entering the market.\(^2\)

A third option for firm \( A \) is to be a price leader. When firm \( A \) sets its price before \( B \), this allows \( A \) to force down the price of \( B \) and thus earn more of the surplus. While this increases \( A \)’s profits, we find that the gain is relatively small.

A fourth option for \( A \) is to tack on a surcharge or a subsidy to \( B \). In the case of a subsidy, firm \( A \) could offer a coupon for \( B \) with the purchase of \( A \). In the case of a surcharge, this could be accomplished by creating two versions of \( A \), one compatible with \( B \) and one incompatible. If the incompatible version is priced at a discount, this creates a surcharge on top of the price of \( B \). We find that it is never optimal to offer a subsidy. Instead, the monopolist would always seek to impose a surcharge for the purchase of an \( A \) product compatible with \( B \).

A related problem is the incentive for an incumbent in \( A \) to influence the quality of \( B \). Instead of making an incompatible version, the \( A \) firm can either enhance or degrade the quality of the complementary product. This issue is particularly relevant to many internet-based businesses, such as the internet telephone services provided by Skype and Vonage. These services allow users to replace their land-line phone service, but they depend on the user having high-speed internet access, usually from their local cable company or telco. This had led to a policy debate over equal access. There is a concern that the cable company (or telco) will have an incentive to degrade the quality of service provided to customers

\(^2\)While it makes the situation difficult for \( B \), this is not a problem for consumers who generally come out ahead when the complements are sold by a joint monopolist.
using such a service, perhaps to increase the attractiveness of their own competing internet phone service. Our model shows that the A firm would not have an incentive to degrade the quality of B, even if A were to enter the B market itself.

There is a sense in which essential complements is similar to some aspects of the literature on tying (Bowman 1957, Stigler 1968, Adams and Yellen 1976, Posner 1976, Bork 1978, Whinston 1990) and to two-way network goods (Katz and Shapiro 1985, Economides and White 1994). Under tying, market structure asymmetries allow a monopolist in A to force consumers to buy A along with B. Here, the situation is reversed; customers only enjoy B along with A. This “tied” relationship is not imposed on the consumer by A, but is the result of consumer preferences. While the essential monopolist does not force the joint sale, it is still able to exploit the asymmetry in consumer preferences to capture surplus created in the B market.

There are several differences between one-way and two-way essential complements. When consumers must buy both A and B to enjoy either, they care only about the joint price: a subsidy on B is identical to a discount on A. With one-way complements, the individual prices are relevant as some consumers will purchase A without B. The linkage between the two markets suggests the possibility of price discrimination. We find that in one-way markets, price discrimination is advantageous when firm A does not own B, but not when A owns B.

We believe that the one-way essential complements though understudied lead to interesting and relevant market interactions. Essential complements are prevalent. They include cases of add-ons and accessories (such as iPods and their accessories). Most software runs on an operating system without which it is useless, and chains of such relationships arise; the operating system is essential to the browser, which is essential to the search engine and the media player. We turn now to explore how surplus is divided in these situations.

2 The Model and Cases

In all of our models there are two goods, A and B, where the consumption of A is essential to the enjoyment of B, but not vice-versa.

We start our analysis with the most basic case: all customers value good A at 1 and good B at $\lambda$. We further assume that both goods are produced with zero costs.
**Proposition 1** Any pair of non-negative prices \((p_a, p_b)\) is a Nash Equilibrium if and only if \(p_a + p_b = 1 + \lambda\) and \(p_b \leq \lambda\).

Observe that the division of the pie is indeterminate. As a result, if the incumbent firm A is able to move first, it would have an advantage. It could set the price of A to be \(1 + \lambda\) (or \(1 + \lambda - \varepsilon\)) and thereby capture all of the surplus from both A and B.

Next we add heterogeneity to the valuations of A. Here we assume that the valuations of good A are distributed uniformly on \([0, 1]\). As before, the value of B is the same for all consumers and equal to \(\lambda\). We are mostly interested in the case where A is more valuable than B. Hence we further assume that \(\lambda \leq 1/2\).

**Proposition 2** There is a unique Nash Equilibrium. Firm A charges \(1/2\) and firm B charges \(\lambda\).

Observe that B gets all of the value it creates. If that situation is representative, then there is no concern that a potential entrant into the complementary market would have insufficient incentives to innovate and enter.

However, it is an artificial assumption that good B is homogeneous in valuation. The results from the two base cases suggest that adding heterogeneity may change the equilibrium. Since heterogeneous valuations are the general case, we turn now to them.

### 2.1 Model with Consumer Valuations Distributed Uniformly:

Assume consumer valuations of A are distributed uniformly on \([0, 1]\), and valuations of B are distributed uniformly on \([0, \lambda]\). Further assume that consumer valuations for A and B are independent.

### 2.2 Model One: Two firms A and B, each a monopoly, and Nash Pricing

The formula for profits depend on whether \(p_a + p_b \leq \lambda\). The reason is that the geometry of market areas depends on whether the line defining the set of consumers who are indifferent about buying both A and B is truncated by the highest possible value for B or not.

As can be seen from the figures below, the demand for good A is the upper left rectangle combined with the shaded trapezoid to the right. Demand for good B is limited to the shaded trapezoid.\(^3\)

\(^3\)Note that area of the box is \(\lambda\). Thus the population density is normalized to \(1/\lambda\) so that the total
Firm A’s profits are

\[
\begin{cases}
  \frac{\lambda}{\lambda}(\lambda(1 - p_a) + \frac{1}{2}(\lambda - p_b)^2) & \text{when } p_a + p_b \geq \lambda \\
  \frac{\lambda}{\lambda}(\lambda - (p_a + p_b) - \frac{1}{2}p_b^2) & \text{when } p_a + p_b \leq \lambda
\end{cases}
\]

Firm B’s profits are

\[
\begin{cases}
  \frac{\lambda}{\lambda}((\lambda - p_b)(1 - p_a) + \frac{1}{2}(\lambda - p_b)^2) & \text{when } p_a + p_b \geq \lambda \\
  \frac{\lambda}{\lambda}(\lambda - p_b - \frac{1}{2}p_a^2) & \text{when } p_a + p_b \leq \lambda
\end{cases}
\]

Note that while we solve the general model, the case of primary interest \( \lambda \) is small, where we expect \( p_a + p_b \geq \lambda \). Maximizing profits leads to the first-order conditions. For firm A these first-order conditions are:

\[
\begin{cases}
  p_b^2 + \lambda(1 - 4p_a + \lambda) - 2p_b = 0 & \text{when } p_a + p_b \geq \lambda \\
  p_a(3p_a + 4p_b) - 2\lambda = 0 & \text{when } p_a + p_b \leq \lambda
\end{cases}
\]

population stays constant at 1.
Firm B’s first-order conditions are
\[
\begin{cases}
3p_b^2 + 4p_b(p_a - 1 - \lambda) + \lambda(2 - 2p_a + \lambda) = 0 \quad \text{when } p_a + p_b \geq \lambda \\
p_a^2 + 4p_b - 2\lambda = 0 \quad \text{when } p_a + p_b \leq \lambda
\end{cases}
\] (4)

Lemma 1 When $\lambda = 1$ there is a closed-form solution: $p_a = 2 - \sqrt{2}$ and $p_b = \sqrt{2} - 1$

Note that at $\lambda = 1$, $p_a + p_b = 1 = \lambda$. It turns out that $\lambda = 1$ is the unique crossover point where $p_a + p_b$ shifts from being bigger to being smaller than $\lambda$.

Lemma 2 $p_a + p_b \geq \lambda$ if and only if $\lambda \leq 1$.

Therefore firm A’s first-order conditions can be rewritten as
\[
\begin{align*}
p_a^2 + \lambda(1 - 4p_a + \lambda) - 2p_b &= 0 \quad \text{when } \lambda \leq 1 \\
p_a(3p_a + 4p_b) - 2\lambda &= 0 \quad \text{when } \lambda \geq 1
\end{align*}
\] (5)

Firm B’s first-order conditions become
\[
\begin{align*}
3p_b^2 + 4p_b(p_a - 1 - \lambda) + \lambda(2 - 2p_a + \lambda) &= 0 \quad \text{when } \lambda \leq 1 \\
p_a^2 + 4p_b - 2\lambda &= 0 \quad \text{when } \lambda \geq 1
\end{align*}
\] (6)

In general, the joint solution to the first-order conditions has no simple analytic form. However, we can substitute the solutions into the profit function and graph the results.

Combining first-order conditions, we find that for $\lambda \leq 1$, $p_a$ and $p_b$ solve equations:
\[
\begin{align*}
p_a &= \frac{1}{4\lambda}(p_b^2 + 2\lambda - 2p_b\lambda + \lambda^2) \\
p_b &= \frac{1}{3}(2 - 2p_a + 2\lambda - \sqrt{4 + 4p_a^2 + \lambda(2 + \lambda) - 2p_a(4 + \lambda)})
\end{align*}
\] (7) (8)

and that for $\lambda \geq 1$, $p_a$ and $p_b$ solve the equations:
\[
\begin{align*}
p_a &= \frac{1}{3}(-2p_b + \sqrt{2\sqrt{2p_b^2 + 3\lambda}}) \\
p_b &= \frac{1}{4}(-p_a^2 + 2\lambda)
\end{align*}
\] (9) (10)

As can be seen in figure two below, as $\lambda$ increases, profits rise for both firms. Thus firm A is able to capture some of the value created by B, (which it cannot do in the independent goods case.) In contrast, if the two goods were perfect complements (with two-way essentiality) then profits would be split evenly between the two firms. Here, firm B gets more than half of the surplus which it creates, but none of the surplus associated with A. As B
become increasingly valuable, eventually firm B earns more profits than A, even though A is essential for B. Finally, observe that A captures relatively more of the incremental surplus created by B when $\lambda$ is small ($\lambda < 1$) than when $\lambda$ is large.

![Figure 2:](image_url)

Given that good A is essential to consumers wishing to enjoy B, there are several ways that firm A can employ to increase the amount of B surplus it captures. First, we investigate the strategy of introducing competition into the B market, driving the price of good B down to cost, in this case 0. When Microsoft introduced its browser, Explorer, it introduced competition into the browser market dominated by Netscape. Later, we will compare this to alternative strategies such as purchasing or merging with the B monopolist, charging a different price for a version of A which is compatible with B, or purposely degrading (perhaps through barriers to compatibility) the quality of rival B products while entering the B market themselves.
2.3 Model Two: Firm A is a monopolist over A and good B is supplied competitively

In the competition to capture surplus, firm A has an incentive to lower the price of B. This might happen exogenously if there is competition in the B market. It could also happen if A were to enter the B market and drive the price of B down to 0.

To solve this case, we look at firm A’s first-order conditions where $p_b = 0$. Firm A’s first-order conditions become:

\[
\lambda (2 - 4p_A + \lambda) = \lambda \quad \text{when} \quad \lambda \leq 1 \\
p_A (3p_A) = 2\lambda \quad \text{when} \quad \lambda \geq 1
\]

Thus prices are:

\[
p_A = \begin{cases} 
\frac{1}{4}\lambda + \frac{1}{2} & \text{when} \quad \lambda \leq \frac{2}{3} \\
\sqrt{\frac{2}{3}}\sqrt{\lambda} & \text{when} \quad \lambda \geq \frac{2}{3} 
\end{cases}
\]

(12)

\[
p_b = 0
\]

(13)

Profits are:

\[
\begin{align*}
&\frac{1}{16}(2 + \lambda)^2 \quad \text{when} \quad \lambda \leq \frac{2}{3} \\
&(\frac{2}{3})^2 \lambda^{\frac{1}{2}} \quad \text{when} \quad \lambda \geq \frac{2}{3}
\end{align*}
\]

(14)

It is interesting to note that the price of A is linear in $\lambda$ up to the point where $\lambda = \frac{2}{3}$ (and $p_A = \frac{2}{3}$). At $\lambda = \frac{2}{3}$, the monopolist’s profits are $\frac{4}{9}$. Note that the maximum possible surplus in this case is $\frac{5}{6} (= (1 + \frac{2}{3})/2)$ and so the monopolist is able to capture $\frac{4}{15}$ or slightly more than half of the total.

2.4 Model Three: Firm A is a monopolist over both products.

One strategy for the A firm to capture some of the surplus generated by the B good is to buy or merge with the B firm, becoming a monopolist over both markets. This new joint monopolist will be able to increase industry profits compared to the Nash equilibrium pricing game. Total profits rise as the joint monopolist can solve the horizontal equivalent of the double-marginalization problem (Sonnenschein 1968, Nalebuff 2000).

While combined profits are higher, there remains the question of how much the A monopolist is willing to pay to acquire the B firm. The answer to that question depends on its profits as a joint monopolist compared to its other options. These other options include
entering the $B$ market, offering subsidies or surcharges for the $B$ product, degrading the quality of the $B$ good, or moving first to set price.

We have just presented the results for the case where the $B$ is competitive. $A$ benefits when the price of $B$ is as low as possible—at least when $A$ doesn’t own $B$. To the extent that a joint monopolist would do better than the outcome with a competitive $B$ market, this gives the monopolist an incentive to purchase $B$ rather than compete.

To compare all the options, we need to first understand how a joint monopolist would maximize profits and how much it would earn. Along the way, we are also able to gain insight on how the monopolist chooses to extract the surplus. Does it make money equally through $A$ and $B$ or more through one than the other?

In the case of complements, we often see one product being given away. For example, Adobe gives away a program to read pdf files and charges for the program to write them.\footnote{This may be done for several reasons, not covered in our model. The motivations include the ability to charge consumers versus firms. Also note that the case of Adobe is a two-way essentiality in that without an encoder, there is no value to a reader.} Here we find that under a range of conditions, the complement to the essential good is given away. For $\lambda$ up to $2/3$, the monopolist takes all the surplus via $A$.

To demonstrate this result, we first establish that the optimal $p_b = 0$ when the optimal $p_a$ is small, specifically for $p_a \leq \frac{2}{3}$. In that case, the previous solution for $p_a(\lambda)$ when $p_b = 0$ holds, so that $p_a = \frac{1}{4} \lambda + \frac{1}{2}$. It then follows that $p_a \leq 2/3$ holds for $\lambda \leq \frac{2}{3}$.

**Lemma 3** If the optimal $p_a < 2/3$, then the optimal $p_b$ is 0.

The intuition for the lemma is found in figure 3.

**Theorem 1** The optimal monopoly prices are given by:

\[
p_a = \begin{cases} 
\frac{1}{4} \lambda + \frac{1}{2} & \text{when } \lambda \leq \frac{2}{3} \\
\frac{2}{3} & \text{when } \lambda \geq \frac{2}{3} 
\end{cases}
\]  
\[p_a = \begin{cases} 
0 & \text{when } \lambda \leq \frac{2}{3} \\
\frac{1}{2} \lambda - \frac{1}{3} & \text{when } \lambda \geq \frac{2}{3} 
\end{cases}
\]  

(15) (16)
Consider lowering Pb by some small $\Delta$ and raising Pa by $\Delta$ (so as to hold Pa + Pb constant).

**There are three effects:**

1) Firm gains $\Delta$ on customers just buying A.
   \[ \text{Gain} = \Delta \times (1 - Pa) \times Pb \]
2) Firm loses some sales of A.
   \[ \text{Loss} = Pa \times \Delta \times Pb \]
3) Firm gains some sales of B
   \[ \text{Gain} = Pb \times \Delta \times (1 - Pa) \]

Total effect = $\Delta \times Pb \times [(1 - Pa) - Pa + (1 - Pa)] = \Delta \times Pb \times (2 - 3Pa)$

So long as $Pa = 2/3$, if Pb were positive, we would want to lower Pb and raise Pa.

These prices lead to monopoly profits of:

\[ \frac{1}{16}(2 + \lambda)^2 \quad \text{when} \quad \lambda \leq \frac{2}{3} \]
\[ \frac{1}{12}(4 + 3\lambda) - \frac{1}{2\lambda} \quad \text{when} \quad \lambda \geq \frac{2}{3} \]  \hspace{1cm} (17)

Observe that the monopolist makes its full joint monopoly profits by entering the B market when $\lambda < 2/3$ as in model two, so long as its entry drives prices down to cost. Thus there is no gain from buying firm B compared to competing with firm B. Thus the most the A monopolist would be willing to pay to purchase B is its cost of entry into the B market (assuming that entry into the B market is possible).\(^5\)

Even when $\lambda$ is bigger than 2/3rds, the monopolist does nearly as well entering the B market (and pushing $p_b$ down to 0) compared to buying B and setting $p_b$ optimally. This remains true until $\lambda$ becomes larger than 1, so that the value of B is more than A.

### 2.5 Model Four: Firm B has a positive cost of production

It is worth emphasizing the limits to our result that the joint monopolist gives away the B good. It is not the case that the B good is given away for all values of $B$ – the give away

\(^5\)These results extend to the case of general value distributions; as we show in section 2.9 below.
Figure 4:

requires that $\lambda \leq 2/3$ so that the average value of $B$ is less than $A$. When the average value of $B$ becomes large, the complement is no longer given away.

A second assumption behind the result is that marginal costs are zero. We think this is a reasonable assumption for software and many other information good industries where the essential complementarity arises. Of course, in other circumstances, production costs matter. Thus we now turn to consider how a joint monopolist would price $B$ when costs are a factor. We find that it is not the case that remains remains equal to cost over a wide range of $\lambda$s. Instead, we find that price equals cost at $\lambda = c$ and then grows very slowly. The initial derivative is zero.

The positive cost model also allows us to explore the question of whether an $A$ monopolist would want to subsidize or surcharge the $B$ good (when $A$ doesn’t own $B$). We turn to this question after looking at the joint monopoly solution with positive costs.

Suppose that the costs of production are a positive amount, $c > 0$. In this case, a $B$ firm would clearly never sell $B$ below $c$; the question remains whether a firm with a monopoly over both goods would ever chose to subsidize sales of $B$ by selling it for a price below $c$. the answer is no.
Theorem 2 A monopolist over both A and B always sets $p_b \geq c$ and $p_a \geq 1/2$.

The intuition for this result follow from figure 6. Another question is whether the joint monopoly firm’s optimal strategy of pricing $B$ at 0 for small $\lambda$ extends to this case. That is, would the firm price $B$ at $c$ for some range of $\lambda > c$? Given the presence of a mass of consumers who value the good less than $c$, the firm may have increased incentive to raise prices as $\lambda$ increases.

Theorem 3 As $\lambda$ increases from $c$, $p_b$ increases slowly: $\frac{\partial p_b}{\partial \lambda} = 0$ at $\lambda = c$.

Corollary 1 For positive $c$, as $\lambda$ increases from $c$, $p_a$ increases slowly: $\frac{\partial p_a}{\partial \lambda} = 0$ at $\lambda = c > 0$. At $c = 0$, $\frac{\partial p_a}{\partial \lambda} = 1/4$.

2.6 Model Five: Firms A sets two prices, for versions compatible and incompatible with B.

Another strategy that firm A can use to capture surplus is to directly influence the price of $B$. It can do this in either direction. The monopolist can offer a discount to its consumers who choose to buy $B$ or the monopolist can impose a surcharge on its consumers who choose
To add positive costs, note we only need to change profits from added sales of B.

**So our three effects are:**

1) Firm gains \( \Delta \) on customers just buying A:

\[
\text{Gain} = \Delta \times (1-Pa) \times Pb
\]

2) Firm loses some sales of A:

\[
\text{Loss} = Pa \times \Delta \times Pb
\]

3) Firm gains some sales of B:

\[
\text{Gain} = (Pb - c) \times \Delta \times (1-Pa)
\]

FOC implies \( Pb = c \times (1-Pa) / (2-3Pa) \)

Pa = 1/2 implies Pb = c, and increasing afterwards, so we know Pb = c, i.e. we never subsidize B.

![Figure 6:](image)

to buy B. To offer a discount, the A monopolist could provide a coupon with each purchase of A that entitles the buyer to get some amount off good B. The motivation for doing this is that if the firm were a joint monopolist, A would be giving B away (at least for small values of \( \lambda \)). From A’s perspective, the price of B is too high and the discount coupon will lower the price of B and thereby make the purchase of A more attractive.

Alternatively, the monopolist could impose a surcharge on B. In order to add a surcharge to the price of B, the A monopolist can create two versions of its good, one which is compatible with B and one which is not. By charging more for the compatible version, this has the effect of adding a surcharge to the price of B.

The results below show that A would never want to subsidize B.\(^6\) Rather, firm A always finds it optimal to add a surcharge to the price of B.

In the results below, we introduce the notation that Firm A charges \( p_a \) for the incompatible version of A and \( p_{ac} \) for the compatible versions of A. When \( p_{ac} > p_a \), the only consumers that buy the more expensive compatible version are those who also plan to buy B. In contrast, when \( p_{ac} < p_a \), we interpret this as the case of a discount coupon. Here we "force" the customers buying the compatible version to also buy B. Thus in both cases,

\(^6\)Here we show this for the case where B’s costs are zero. This continues to be true even when B has a positive cost.
the compatible version is always bought alongside good $B$ and the incompatible version is always bought alone.

**Theorem 4** The $A$ firm always charges strictly more for the compatible version.

The general intuition from the bundling literature (MacAfee et al. 1989) is that a monopolist will generally wish to sell a bundle at discount relative to the individual prices. But this intuition for a bundle discount relies on the firm owning both goods $A$ and $B$. Here the monopolist only owns $A$.

The intuition for why the monopolist wants to impose a surcharge on $B$ is based on price discrimination. Imagine that the monopolist could charge a different price for $A$ depending on the customer’s valuation for $B$. Then the $A$ monopolist would want to charge a higher price for customers with high $B$ valuations. The reason is that as the value of $B$ increases, there are more customers buying the bundle and thus more inframarginal customers. Thus the value of raising the price to these customers is larger, while the cost of raising the price (the incremental lost customer) remains the same.

In figure 7, we graph the prices that firms $A$ and $B$ would charge for $B$ and the compatible and incompatible versions of $A$, when consumer valuations are distributed uniformly.

### 2.7 Model Six: Firms $A$ and $B$ play a Stackelberg price game with $A$ moving first

Another option for the monopolist to capture surplus is to move first and preemptively set the price of $A$. The goal is to induce firm $B$ to lower its price. This is done by raising the price of $A$. While moving first does increase profits of $A$, the effect is relatively small.

The prices solve:

$$
p_a \text{ solves: } \begin{cases} 
\text{see graph} \quad & \text{when } \lambda \leq x \\
\text{see graph} \quad & \text{when } x \leq \lambda \leq \frac{10}{9} \\
p_a^3 + \lambda = \frac{3}{2}p_a^2 + p_a \lambda & \text{when } \lambda \geq \frac{10}{9}
\end{cases} \quad (18)
$$

---

$^7$The closed-form solutions for $p_a$ and $p_b$ are more than several pages in some cases. Thus we omit writing them and instead present a graph of the prices.
If firm A can create a version of A incompatible with B

\[ p_b = \begin{cases} 
\frac{1}{3} (2 - 2p_a + 2\lambda - \sqrt{4 - 8p_a + 4p_a^2 + 2\lambda - 2p_a\lambda + \lambda^2}) & \text{when } \lambda \leq \frac{10}{9} \\
\text{see graph} & \text{when } x \leq \lambda \leq \frac{10}{9} \\
\frac{1}{3} (-p_a^2 + 2\lambda) & \text{when } \lambda \geq \frac{10}{9}
\end{cases} \]

where \( x \) solves \( 2x^3 - 19x^2 + 60x - 44 = 0 \) (\( x \approx 1.036 \)).

From the graph it is clear that firm A is unable to lower the price of B very much by being a price leader. Indeed when \( \lambda < 1 \), the Stackelberg prices are nearly indistinguishable from the simultaneous-move Nash prices.

2.8 Model Seven: Firms A can degrade the value of competing B products by \( \theta \)

If moving first does not allow firm A to significantly alter prices, another strategy is to discourage the use of B goods manufactured by other parties while introducing a B good yourself. This could be accomplished by making it difficult for other firms to build compatible add-ons. An example of this is a high speed internet provider degrading the packet quality of other-party supplied internet telephone service, while providing priority to A’s
internet telephony service.

Here we assume that there are two versions of good $B$. The degraded $B$ has value discounted by $\theta < 1$ and is priced at $p_b$ and the premium $B$ product $B$ is priced at $p_b$. The main question this model addresses is whether the $A$ firm would ever purposely degrade the quality of rival $B$ products. That is, if not just prices but $\theta$ were a choice variable for firm $A$, would the firm have incentive to lower $\theta$. We first examine a situation where firm $A$ does not not produce a $B$ product, and show that if $B$ was priced above cost, firm $A$ would want to enter with an un-degraded $B$ product and drive the price of $B$ down to 0. Formally,

**Theorem 5** If $B$ is non-competitively supplied so that $p_b > 0$, then for $\lambda$ low enough (as long as $p_a < 2/3$), firm $A$ will profit from driving the price $p_b$ to 0.

Then, we show that once $p_b = 0$, for $\lambda$ low enough the profit maximizing price of the un-degraded $B$ is 0. This implies that firm $A$ has no incentive to purposely lower $\theta$ in the first place, since it will never be optimal to take any of its profits through the price of $B$ anyways. Formally:
**Theorem 6** If \( B \) is competitively supplied so that \( p_b = 0 \), then for \( \lambda \) low enough (as long as \( p_a < \frac{2}{3} \)), firm \( A \) will charge 0 for \( B \).

The intuition for this theorem is provided in figure 9.

Consider lowering \( P_b \) by some small \( \Delta \) and raising \( P_a \) by \( \Delta \) so as to hold \( P_a + P_b \) constant.

1) Firm gains \( \Delta \) on customers just buying \( A \):
\[
\Delta \cdot \left[ 1 - P_a + \left( \frac{P_b}{2(1-\theta)} \right) \right] \cdot P_b/(1-\theta)
\]

2) Firm loses some sales of \( A \):
\[
P_a \cdot \Delta \cdot P_b/(1-\theta)
\]

3) Firm gains some sales of \( B \):
\[
P_b \cdot \frac{\Delta}{(1-\theta)} \cdot \left[ 1 - P_a + \left( \frac{P_b}{1-\theta} \right) \right]
\]

This raises profits if:
\[
\Delta \cdot \frac{P_b}{(1-\theta)} \cdot \left[ 2 - 3P_a + 3(P_b)/(1-\theta) \right] > 0
\]
Hence, the optimal \( P_b = 0 \) when \( P_a < \frac{2}{3} \).

Figure 9:

### 2.9 General Distribution Box Theorems:

Consider a setting where a consumer’s value \( v_a \) for good \( A \) is described by the CDF \( F_a(\cdot) \) and \( v_b \) by \( F_b(\cdot) \). Assume consumer valuation for \( A \) and \( B \) are distributed independently, where both densities are continuous with full support on \([0,1]\) and \([0,\lambda]\), respectively. We now show that theorem 1 extends to general value distributions.

**Theorem 7** If one firm is a monopoly over both goods \( A \) and \( B \), there is a \( U > 0 \) such that if \( \lambda \in [0,U] \) then the optimal price \( p_b \) for good \( B \) is 0.

**Corollary 2** As \( \lambda \to 0 \), the optimal \( p_b \to 0 \) and \( p_a \to 1/2 \).

**Corollary 3** If the optimal \( F_a(p_a) < \frac{2}{3} \), then the optimal \( p_b = 0 \).

**Corollary 4** If \( F_a(\cdot) \) is uniform then for any \( F_b(\cdot) \), \( U > 1/3 \).

The results on subsidies and surcharges also extend to the general distribution case. Suppose firm \( A \) is a monopolist over the production of good \( A \) and firm \( B \) is a monopolist
over good $B$. Further, suppose that firm $A$ can charge two prices, $p_a$ for a baseline $A$, and $p_a + \delta$ for a version of $A$ that is compatible with $B$ (that is, charge a different price for consumers who also wish to buy good $B$). Then we have:

**Theorem 8** When the optimal $p_a > 1/2$, the optimal $\delta > 0$.

**Corollary 5** If $\frac{f_a}{1 - F_a(f)}$ increases as $p_a$ increases, then $\delta > 0$.

### 3 Conclusion

The interaction between an essential good and its complement is naturally lopsided. And yet in the Nash pricing game, the complementary product is able to capture over half of the surplus it creates.

The greatest challenge for the complementor firm arises if $A$ is able to enter the $B$ market. While the $A$ firm is not able to earn any profits in the $B$ market, it doesn’t have to. Firm $A$ can raise the price of $A$ in response to the reduced price of $B$. The surprising result is that giving away $B$ leads to joint profit maximization under a range of conditions. This is a problem for a $B$ firm hoping to be bought out by $A$. There is no incentive for $A$ to purchase $B$ in order to set its price optimally. In these cases, the most $A$ is willing to pay is its cost of entry.

If $A$ is not able to enter the $B$ market, there is only a small gain to moving first. There is a larger gain from creating two versions of $A$, one compatible and one incompatible with $B$, and charging more for the compatible version. The multiple avenues for $A$ to capture $B$’s surplus highlights the challenges facing a firm whose product depends on an essential good.

### 4 References

### 5 Appendix

**Proposition 3** Any pair of non-negative prices $(p_a, p_b)$ is a Nash Equilibrium if and only if $p_a + p_b = 1 + \lambda$ and $p_b \leq \lambda$.

**Proof.** If $p_a + p_b = 1 + \lambda$ and $p_b \leq \lambda$ then neither firm can unilaterally increase profits. Lowering price does not increase demand; raising price leads to zero demand and zero
profits. If \( p_a + p_b < 1 + \lambda \) then firm A can raise profits by increasing \( p_a \), as its demand will remain unchanged. If \( p_a + p_b > 1 + \lambda \), then either \( p_a > 1 \) or \( p_b > \lambda \). If \( p_B > 1 \), then firm A will increase its profits up from zero by setting \( p_a = 1 \). If \( p_a \leq 1 \) and \( p_B > \lambda \), then firm B will increase its profits up from zero by setting \( p_b = \lambda \). Finally, given that \( p_a + p_b = 1 + \lambda \) in any Nash Equilibrium, if \( p_b > \lambda \) then consumers will only be buying A. Thus firm B could increase its profits up from zero by setting \( p_b = \lambda \).

**Proposition 4** There is a unique Nash Equilibrium. Firm A charges \( 1/2 \) and firm B charges \( \lambda \).

**Proof.** First we show it is an equilibrium, then show it is unique.

If firm B charges \( \lambda \), then A faces a demand curve \( 1 - p_a \) and A’s optimal response is to charge \( 1/2 \). If A charges \( 1/2 \), then firm B faces a demand curve of \( 1 + \lambda - 1/2 - p_b = 1/2 + \lambda - p_b \) for \( p_b \leq \lambda \) and zero demand if \( p_b > \lambda \). (If firm B charges more than \( \lambda \), no customer will buy B as an add on.)

The optimal price for B is \( p_b = \min[(1/4 + \lambda/2), \lambda] \). The min is \( \lambda \) provided \( \lambda \leq 1/2 \). This demonstrates that \( (1/2, \lambda) \) is a Nash Equilibrium for \( \lambda \leq 1/2 \).

To see that this equilibrium is unique, observe that were A to charge some other prices than \( 1/2 \), firm B would then face a demand curve of \( 1 + \lambda - p_a - p_b \) which leads to an optimal price of \( p_b = \min[(1 + \lambda - p_a)/2, \lambda] \).

If \( p_a < 1/2 \), B will settle at \( \lambda \) which leads to \( p_a = 1/2 \). Thus the only possible alternative equilibria arise when \( p_a \geq 1/2 \) and \( p_b = (1 + \lambda - p_a)/2 \).

In that case, the residual demand for A is \( 1 + \lambda - p_b - p_a \) Thus the profit-maximizing price for A is

\[
p_a = (1 + \lambda - p_b)/2 = (1 + \lambda - (1 + \lambda - p_a)/2)/2.
\]

Simplifying leads to \( p_a = (1 + \lambda)/3 \leq 1/2 \) as \( \lambda \leq 1/2 \). Thus \( p_a \) can never be greater than \( 1/2 \) which leads \( p_b \) to \( \lambda \), demonstrating the uniqueness of the Nash equilibrium.

**Lemma 4** When \( \lambda = 1 \) there is a closed-form solution: \( p_a = 2 - \sqrt{2} \) and \( p_b = \sqrt{2} - 1 \)

**Proof.** For \( \lambda = 1 \), firm A’s first order conditions simplify to

\[
p_a^2 + (2 - 4p_a) - 2p_b = 0 \quad \text{when} \quad p_a + p_b \geq 1
\]

\[
p_a(3p_a + 4p_b) - 2 = 0 \quad \text{when} \quad p_a + p_b \leq 1
\]

(21)
Firm B’s first-order conditions simplify to:

\[ \begin{align*}
3p_b^2 + 4p_b(p_a - 2) + (3 - 2p_a) &= 0 \quad \text{when} \quad p_a + p_b \geq 1 \\
p_a^2 + 4p_b &= 2 \quad \text{when} \quad p_a + p_b \leq 1
\end{align*} \] (22)

Looking first at the case where \( p_a + p_b \geq 1 \), we have three solutions for \( p_a \) and \( p_b \). The solutions are \((p_a = 9/16, p_b = 3/2)\), \((p_a = 2 - \sqrt{2}, p_b = \sqrt{2} - 1)\), and \((p_a = 2 + \sqrt{2}, p_b = -1 - \sqrt{2})\). The first solution violates \( p_b \leq \lambda \), and the third condition has a negative value of \( p_b \). The middle solution exactly satisfies \( p_a + p_b = 1 \). This shows there is a unique solution for \( p_a + p_b \geq 1 \).

Looking next at the case where \( p_a + p_b \leq 1 \), we again have three solutions for \( p_a \) and \( p_b \). The solutions are \((p_a = -1, p_b = 1/4)\), \((p_a = 2 - \sqrt{2}, p_b = \sqrt{2} - 1)\), and \((p_a = 2 + \sqrt{2}, p_b = -1 - \sqrt{2})\). The first solution violates \( p_a \geq 0 \), and the third condition has a negative value of \( p_b \). The middle solution exactly satisfies \( p_a + p_b = 1 \). This shows there is a unique solution for \( p_a + p_b \leq 1 \). These two solutions coincide. ■

**Lemma 5** \( p_a + p_b \geq \lambda \) if and only if \( \lambda \leq 1 \).

**Proof.** When \( \lambda = 0 \), \( p_a = 1/2 \) and \( p_b = 0 \), so we know \( p_a + p_b \geq \lambda \). (The solution at \( \lambda = 0 \) follows directly from the second base case.) Graphing the solution for \( p_a, p_b, \) and \( p_a + p_b - \lambda \), we observe that \( p_a + p_b - \lambda \) is monotonicity decreasing. We know from the previous lemma that \( p_a + p_b - \lambda = 0 \) at \( \lambda = 1 \). ■

**Lemma 6** If the optimal \( p_a < 2/3 \), then the optimal \( p_b \) is 0.

**Proof.** Assume \( p_b > 0 \). Consider lowering \( p_b \) by \( \Delta \) and raising \( p_a \) by \( \Delta \), where \( \Delta \) is small. This has three first-order effects. First we charge more to those who continue to buy only \( A \). The gain is:

\[ \Delta * p_b * (1 - p_a) \] (23)

Second, we lose some customers who used to buy \( A \). The loss is

\[-p_a * p_b * \Delta \] (24)

Third, we sell \( B \) to more customers. The gain is

\[ p_b * \Delta * (1 - p_a) \] (25)
Combining these effects, our price change has a first-order effect on profits of:

\[ \Delta \ast p_b \ast (2(1 - p_a) - p_a) \]  

which is positive when \( 2 - 3p_a > 0 \).

This implies that so long as \( p_a < 2/3 \), the monopolist would increase its profits by lowering \( p_b \) and raising \( p_a \) by the same amount. But if for any positive \( p_b \) the firm would want to lower \( p_b \), then the optimal \( p_b \) must be 0. ■

**Theorem 1** 

The optimal monopoly prices are given by:

\[
p_a = \begin{cases} \frac{1}{4} \lambda + \frac{1}{2} & \text{when } \lambda \leq \frac{2}{3} \\ \frac{2}{3} & \text{when } \lambda \geq \frac{2}{3} \end{cases} \]  

\[
p_b = \begin{cases} 0 & \text{when } \lambda \leq \frac{2}{3} \\ \frac{1}{2} \lambda - \frac{1}{3} & \text{when } \lambda \geq \frac{2}{3} \end{cases} \]  

**Proof.** Lemma seven shows that for all \( p_a < 2/3 \), the optimal \( p_b = 0 \). As \( \lambda \to 0 \), the optimal \( p_a \to 1/2 \). (This follows because the \( B \) market becomes irrelevant and the monopoly prices to maximize profits on \( A \) alone.) Since \( p_a(0) = 1/2 \), for some range of \( \lambda \) around 0, the optimal \( p_a < 2/3 \), so by our lemma seven the optimal \( p_b \) will be 0 in that range. To demarche that range, observe that so long as \( p_b = 0 \), the first-order conditions that gives the optimal \( p_a \) are:

\[
1 - 2p_a + \frac{\lambda}{2} = 0 \quad \text{for } p_a > \lambda \\
\lambda - \frac{3}{2}p_a^2 = 0 \quad \text{for } p_a < \lambda
\]  

(29)

The solution to the second condition is \( p_a = \sqrt{\frac{2}{3}} \lambda \), which violates \( p_a < \lambda \) when \( 0 < \lambda < 2/3 \). The solution to the first condition is \( p_a = \frac{1}{4} \lambda + \frac{1}{2} \), which satisfies \( p_a < \lambda \) when \( \lambda < 2/3 \), and \( p_a = \frac{2}{3} \) at \( \lambda = 2/3 \). Also note that \( \frac{1}{4} \lambda + \frac{1}{2} < 2/3 \) for \( \lambda < 2/3 \), justifying the assumption that \( p_b = 0 \). Therefore \( p_a = \frac{1}{4} \lambda + \frac{1}{2} \) and \( p_b = 0 \) is the unique solution while \( \lambda \leq \frac{2}{3} \).

Turning to the case of \( \lambda > \frac{2}{3} \), it is possible that the optimal \( p_b > 0 \). If so, then we are at an interior solution for \( p_a \) and \( p_b \) and both first-order conditions are satisfied. As in the Nash case, the geometry of monopoly profits changes depending on whether \( p_a + p_b \leq \lambda \). The profits the joint monopoly will earn are:

\[
\frac{1}{\lambda} (p_a(\lambda - (p_a \ast p_b) - \frac{1}{2}p_b^2) + p_b(\lambda - p_b - \frac{1}{2}p_a^2)), \quad \text{and} \\
\frac{1}{\lambda} (p_a(\lambda - (1 - p_a) + \frac{1}{2}(\lambda - p_b)^2) + p_b((\lambda - p_b)(1 - p_a) + \frac{1}{2}(\lambda - p_b)^2))
\]  

(30) 

(31)
for \( p_a + p_b < \lambda \) and \( p_a + p_b > \lambda \), respectively. Differentiating these with respect to \( p_a \) and \( p_b \) leads to two first-order conditions per case. Firm A’s first-order conditions are:

\[
3p_a(p_a + 2p_b) = 2\lambda \quad \text{when} \quad p_a + p_b < \lambda
\]

\[
3p_b^2 + \lambda(2 - 4p_a + \lambda) = 4p_b\lambda \quad \text{when} \quad p_a + p_b > \lambda
\]

Firm B’s first-order conditions are:

\[
3p_a^2 + 4p_b = 2\lambda \quad \text{when} \quad p_a + p_b < \lambda
\]

\[
p_b(6p_a + 3p_b - 4) + \lambda(2 + \lambda) = 4(p_a + p_b)\lambda \quad \text{when} \quad p_a + p_b > \lambda
\]

When \( p_a + p_b \geq \lambda \), the first-order conditions are only satisfied if:

\[
p_a = \frac{2}{3} \quad \text{and} \quad p_b = \frac{1}{3}(2\lambda + \sqrt{2\lambda + \lambda^2}),
\]

\[
p_a = \frac{2}{3} \quad \text{and} \quad p_b = \frac{1}{3}(2\lambda - \sqrt{2\lambda + \lambda^2}),
\]

or if

\[
p_a = \frac{1}{4}\lambda + \frac{1}{2} \quad \text{and} \quad p_b = 0.
\]

Only the first solution does not violate \( p_a + p_b \geq \lambda \) when \( \lambda > 2/3 \).

When \( p_a + p_b \leq \lambda \), the first-order conditions are only satisfied if:

\[
p_a = \sqrt{\frac{2}{3}\lambda} \quad \text{and} \quad p_b = 0
\]

or if:

\[
p_a = \frac{2}{3} \quad \text{and} \quad p_b = \frac{1}{2}\lambda - \frac{1}{3}
\]

Both solutions satisfy \( p_a + p_b \leq \lambda \) when \( \lambda > 2/3 \). Therefore we are left with three candidate solutions. Calling these solutions 1, 2 and 3 respectively, we have corresponding profits \( \Pi_1, \Pi_2, \) and \( \Pi_3 \) of:

\[
\Pi_1 = \frac{1}{27}\lambda(6 - 2\sqrt{\lambda(2 + \lambda)} + \lambda(3 + \lambda - \sqrt{\lambda(2 + \lambda)}))
\]

\[
\Pi_2 = \left(\frac{2}{3}\lambda\right)^{3/2}
\]

\[
\Pi_3 = -\frac{1}{27} + \frac{1}{12}\lambda(4 + 3\lambda)
\]

To establish which solution maximizes profits, we compare their profits directly. Comparing solutions 1 and 2,

\[
\Pi_1 < \Pi_2 \iff \lambda(6(1 - \sqrt{6\lambda}) + \lambda^2 - (2 + \lambda)\sqrt{\lambda(2 + \lambda)} + 3\lambda) < 0
\]
which is true for all $\lambda > 2/3$ ($\Pi_1 < \Pi_2$ for all $\lambda$ greater than approximately 0.13). Therefore only the second and third solutions are possible optima. Looking at $\Pi_2$ and $\Pi_3$,

$$\Pi_2 < \Pi_3 \Leftrightarrow 4 < 3\lambda(12 - 8\sqrt{6\lambda} + 9\lambda) \Leftrightarrow \lambda > 2/3$$ (43)

This confirms that our third solution is unique for all $\lambda > 2/3$, and proves our theorem. □

**Theorem 2** A monopolist over both $A$ and $B$ always sets $p_b \geq c$ and $p_a \geq 1/2$.

**Proof.** As in the case without costs, consider lowering $p_b$ by $\Delta$ and raising $p_a$ by $\Delta$, where $\Delta$ is small. This has three first-order effects.

First we charge more to those who continue to buy only $A$. The gain is

$$\Delta * p_b * (1 - p_a).$$ (44)

Second, we lose some customers who used to buy $A$. The loss is

$$-p_a * p_b * \Delta.$$ (45)

Third, we sell $B$ to more customers. The gain is

$$(p_b - c) * \Delta * (1 - p_a).$$ (46)

Combining these effects, our price change has a first-order effect on profits of

$$\Delta * \{(p_b(2 - 3p_a) - c(1 - p_a))\}. \quad (47)$$

Since $\Delta$ can be either positive or negative, we know that

$$p_b = \frac{c(1 - p_a)}{2 - 3p_a} \quad (48)$$

Note that at $p_a = 1/2$ we have $p_b = c$, and $p_b > c$ if and only if $p_a > 1/2$.\footnote{Observe that $p_b$ is monotonic in $p_a$.} Therefore, either $p_a \geq 1/2$ and $p_b \geq c$ or $p_a < 1/2$ and $p_b < c$. To establish our theorem we only need show that the monopolist would never choose the second combination, namely $p_a < 1/2$ and $p_b < c$.

But the monopolist would never want to charge a price below 1/2 for $A$ when it is subsidizing $B$. It would do better by raising the price of $A$ up to 1/2. This raises profits on $A$. The gain from existing customers is more than 1/2 (as $A$'s market area exceeds 1/2 all
the way to the price of 1/2) and the loss from reduced demand is less \( p_a \), which is always less than 1/2. Thus the net effect on \( A \) is positive. A side effect of increasing the price of \( A \) is that sales of \( B \) will fall, but as the monopolist was losing money on each \( B \) sale, that, too, increases profits. ■

**Theorem 3** As \( \lambda \) increases from \( c \), \( p_b \) increases slowly: \( \frac{\partial p_b}{\partial \lambda} = 0 \) at \( \lambda = c \).

**Proof.** We prove this by inspection of the first-order conditions. The first-order condition for \( p_a \) leads to:

\[
p_a = \frac{1}{2} + \frac{(\lambda - p_b)^2}{4\lambda} - \frac{(p_b - c)}{2\lambda} (\lambda - p_b).
\]  
(49)

\[
\frac{dp_a}{d\lambda} = \frac{\lambda^2 + 2cp_b - 3p_b^2}{4\lambda^2} + \left( \frac{3p_b - c}{2\lambda} - 1 \right) \frac{dp_b}{d\lambda}
\]  
(50)

From equation 48, we know that

\[
\frac{dp_b}{d\lambda} = \frac{c}{(2 - 3p_a)^2} \frac{dp_a}{d\lambda}
\]  
(51)

Hence

\[
\frac{dp_a}{d\lambda} \left( 1 - \frac{3p_b - c - 2\lambda}{2\lambda} \frac{c}{(2 - 3p_a)^2} \right) = \frac{\lambda^2 + 2cp_b - 3p_b^2}{4\lambda^2}
\]  
(52)

From our previous theorem we know that \( p_b \to c \) as \( \lambda \to c \). Therefore as \( \lambda \to c \),

\[
\frac{dp_a}{d\lambda} \left( 1 - \frac{c - \lambda}{\lambda} \frac{c}{(2 - 3p_a)^2} \right) = \frac{\lambda^2 - c^2}{4\lambda^2}
\]  
(53)

Now observe that if \( c > 0 \), then as \( \lambda \to c \), \( \frac{dp_a}{d\lambda} = 0 \), while for \( c = 0 \), \( \frac{dp_a}{d\lambda} = 1/4 \). Hence, from equation 51, it follows that \( \frac{dp_b}{d\lambda} = 0 \) for both \( c > 0 \) and \( c = 0 \). ■

**Corollary 6** For positive \( c \), as \( \lambda \) increases from \( c \), \( p_a \) increases slowly: \( \frac{\partial p_a}{\partial \lambda} = 0 \) at \( \lambda = c > 0 \). At \( c = 0 \), \( \frac{\partial p_a}{\partial \lambda} = 1/4 \).

**Theorem 4** The \( A \) firm always charges strictly more for the compatible version.

**Proof.** Suppose we are in the coupon case so that we give a discount to people who buy both goods, i.e. \( p_{ac} < p_a \). Consider lowering \( p_a \) until it equals \( p_{ac} \). This has two effects. First, we charge less to those who used to buy the incompatible \( A \). This costs:

\[
(p_a - p_{ac}) * (1 - p_a) * (p_b + p_{ac} - p_a)
\]  
(54)
Second, we gain some customers who now buy the incompatible A. This increases profits by:

\[ p_{ac} \times (p_a - p_{ac}) \times \left( p_b + \frac{p_{ac} - p_a}{2} \right) \]  

(55)

Some customers switch from buying the compatible version to the incompatible version of A, but this is only a loss to firm B. Note that \((p_b + \frac{p_{ac} - p_a}{2}) > (p_b + p_{ac} - p_a)\), so this increases A’s profits as long as \(p_{ac} > (1 - p_a)\). But this must be true since \(p_{ac} > 1/2\). Thus discounts are never optimal.

Now, suppose \(p_a = p_{ac}\). Consider lowering \(p_a\) by \(\Delta\). This has two first-order effects.

First, firm A charges less to those who used to buy only A. This costs:

\[ \Delta \times p_b \times (1 - p_{ac}) \]  

(56)

Second, firm A gains customers who now buy A. This increases profits by:

\[ p_{ac} \times p_b \times \Delta \]  

(57)

Firm A loses \(\Delta\) on customers who switch from buying the compatible version to buying the incompatible version, but this is a second-order effect.

Combining these effects, the price change has a first-order effect on profits of:

\[ \Delta \times p_b \times (p_{ac} - (1 - p_{ac})) \]  

(58)

which is positive as long as \(p_{ac} > 1/2\).

This shows that firm A would never subsidize a version that is compatible with B, and in fact always wants to apply a surcharge. Prices solve:

\[ p_a = \begin{cases} 
\frac{1}{3}(p_{ac} + p_b - \lambda)(3p_{ac} + p_b - \lambda) + \frac{1}{7} & \text{when } \lambda \leq \frac{126}{121} \\
\frac{1}{3}(-3p_{ac} - 2p_b + \sqrt{(3p_{ac} + 2p_b)^2 + 6\lambda}) & \text{when } \lambda \geq \frac{126}{121}
\end{cases} \]  

(59)

\[ p_{ac} = \begin{cases} 
\frac{1}{3}(2 - 3p_a - 2p_b + 2\lambda - \sqrt{4 + 3p_a(3p_a - 4) + p_b^2 - 2p_b(1 + \lambda) + \lambda(2 + \lambda)}) & \text{when } \lambda \leq \frac{126}{121} \\
\frac{1}{3}(-3p_a^2 - 2p_b + 2\lambda) & \text{when } \lambda \geq \frac{126}{121}
\end{cases} \]  

(60)

\[ p_b = \begin{cases} 
\frac{1}{3}(2 - 2p_a - 2p_{ac} + 2\lambda - \sqrt{4 + 4p_a^2 - 2p_{ac} + 2p_{ac}(p_{ac} - 4 - \lambda) + (p_{ac} - \lambda)^2 + 2\lambda}) & \text{when } \lambda \leq \frac{126}{121} \\
\frac{1}{3}(-p_a^2 - 2p_{ac} + 2\lambda) & \text{when } \lambda \geq \frac{126}{121}
\end{cases} \]  

(61)
Theorem 5 If \( B \) is non-competitively supplied so that \( p_b > 0 \), for low enough \( \lambda \) firm \( A \) will profit from driving the price \( p_b \) to 0.

Proof. See Diagram (XX need to draw and insert), and note that consumers who value good \( B \) at value \( \frac{p_b - p_a}{1 - \theta} \) are indifferent between buying \( B \) and \( \overline{B} \). Therefore we can restrict attention to the case where \( \frac{p_b - p_a}{1 - \theta} > p_b \), since if this does not hold no consumers are buying good \( B \).

Consider lowering \( p_b \) all the way down to 0, lowering \( p_b \) by the same amount, and raising \( p_a \) by the same amount. This has two effects. First, we lose sales of \( A \) from the price increase:

\[-p_a \times \frac{1}{2} \times \frac{p_b}{\theta} \times p_b \]  \hspace{1cm} (62)

Second, we gain on those who were buying \( A \) and not buying \( B \):

\[ p_b \times \frac{p_b - p_a}{1 - \theta} \times (1 - p_a + \theta \frac{p_b - p_a}{1 - \theta}) \]  \hspace{1cm} (63)

Since \( \frac{p_b - p_a}{1 - \theta} > \frac{p_b}{\theta} \), if we rewrite the second force substituting for \( \frac{p_b - p_a}{1 - \theta} \), we only make the overall gain smaller. Therefore profits are less than:

\[ p_b \times \frac{p_b}{\theta} \times (1 - p_a + \theta \frac{p_b - p_a}{1 - \theta}) - p_a \times \frac{1}{2} \times \frac{p_b}{\theta} \times p_b \]  \hspace{1cm} (64)

simplifying, this is equal to:

\[ \frac{p_b^2}{\theta} (1 + p_b - \frac{3}{2} p_a) \]  \hspace{1cm} (65)

Examining this expression shows that for small \( p_a \), the firm earns positive profits from lowering \( p_b \). In particular, as long as \( p_a < \frac{2}{3} \) the firm strictly profits from forcing competition into the market for \( B \), driving \( p_b \) to 0. Since for small \( \lambda \), \( p_a \) is close to \( \frac{1}{2} \), the firm would choose to do so for small \( \lambda \). \( \blacksquare \)

Theorem 6 If \( B \) is competitively supplied so that \( p_b = 0 \), firm \( A \) will charge 0 for \( \overline{B} \) for low enough \( \lambda \).

Proof. Assume \( p_b = 0 \) and \( p_b > 0 \).

Consider lowering \( p_b \) by \( \Delta \) and raising \( p_a \) by \( \Delta \). This has three effects. First we charge more to those who continue to buy only
\[
p_a : \Delta \frac{p_\theta}{1 - \theta} * (1 - p_a + \frac{1}{2} p_\theta \frac{\theta}{1 - \theta}) \tag{66}
\]

Second, we lose some customers who used to buy A:

\[
-p_a * \Delta \frac{p_\theta}{1 - \theta} \tag{67}
\]

Third, we sell B to more customers:

\[
p_\theta * \Delta \frac{1}{1 - \theta} * (1 - p_a + p_\theta \frac{\theta}{1 - \theta}) \tag{68}
\]

Combining these effects, our price change has changed profits by:

\[
\Delta \frac{p_\theta}{1 - \theta} * (2(1 - p_a) - p_a + \frac{3}{2} p_\theta \frac{\theta}{1 - \theta}) \tag{69}
\]

which is positive as long as \( 2 - 3p_a + \frac{3}{2} p_\theta \frac{\theta}{1 - \theta} > 0 \).

In particular, if \( p_a < \frac{2}{3} \) then lowering \( p_\theta \) by \( \Delta \) and raising \( p_a \) by \( \Delta \) raises profits. Since for small \( \lambda \) \( p_a \) is close to \( \frac{1}{2} \), the firm would choose to set \( p_\theta = 0 \).

**Theorem 7** If one firm is a monopoly over both goods A and B, there is a \( U > 0 \) such that for all \( \lambda \in [0, U] \) the optimal price \( p_\theta(\lambda) \) for good B is 0.

**Proof.** Define \( p_\theta(0) \) as the \( \lim_{\lambda \to 0} p_\theta(\lambda) \) as \( \lambda \to 0 \) from above. First, note that \( p_\theta(0) = 0 \) follows from that fact that for all \( \lambda \), \( p_\theta(\lambda) \leq \lambda \). Next, \( 0 < p_\theta(0) < 1 \) follows trivially from the fact that \( f_a(\cdot) \) is continuous, since profits are 0 at both \( p = 0 \) and \( p = 1 \).

Given these boundary conditions, note that for every \( \varepsilon \) with \( 0 < \varepsilon < 1 - p_a(\lambda) \) we may choose a \( \lambda < \varepsilon \) such that:

\[
0 \leq p_\theta(\lambda) \leq \lambda < \varepsilon < (1 - p_a(\lambda)) \tag{70}
\]

Now, choose a \( \lambda \) such that the first inequality is strict (if we can’t do this, the theorem is automatically true.) We want to do this so that we can consider the following deviation: starting at the prices \( p_\theta(\lambda) \) and \( p_a(\lambda) \) with \( 0 < p_\theta(\lambda) \leq 1 - p_a(\lambda) \), we lower the price of B to 0, and raise the price of A by \( p_\theta(\lambda) \) to offset it. Note that since \( p_\theta(\lambda) \) and \( p_a(\lambda) \) are chosen optimally for \( \lambda \), this cannot increase profits.

Writing the effect on profits, there are a number of customers who used to purchase just good A that now stop purchasing A, they represent lost profits of:

\[
p_a(\lambda) \int_0^{p_\theta(\lambda)} (F_a(p_a(\lambda) + p_\theta(\lambda) - x) - F_a(p_a(\lambda))) * f_b(x) dx. \tag{71}
\]
As $\lambda \rightarrow 0$, this loss converges to

$$p_a(\lambda) * \frac{1}{2} p_b(\lambda) * f_a(p_a(\lambda)) * F_b(\lambda).$$  \hfill (72)

Offsetting this loss, there are a number of customers who used to buy only $A$ and now buy both $A$ and $B$, at the new higher price of $A$. This represents a gain in profits of:

$$p_b(\lambda) \int_0^{p_a(\lambda)} (1 - F_a(p_a(\lambda) + p_b(\lambda) - x)) * f_b(x) dx$$  \hfill (73)

As $\lambda \rightarrow 0$, this gain converges to

$$p_b(\lambda) * (1 - F_a(p_a(\lambda))) * F_b(\lambda)$$  \hfill (74)

Thus combining terms, the net effect on profits is

$$p_b(\lambda) * F_b(p_b(\lambda)) * [1 - F_a(p_a(\lambda)) - p_a(\lambda) * \frac{1}{2} * f_a(p_a(\lambda))]$$  \hfill (75)

Recall this is the change in profits from a deviation from the optimal prices for $A$ and $B$. It is weakly less than 0 if and only if

$$1 - F_a(p_a(\lambda)) - p_a(\lambda) * \frac{1}{2} * f_a(p_a(\lambda)) \leq 0$$  \hfill (76)

Now recall from our lemma that as $\lambda \rightarrow 0$, $p_a(\lambda) \rightarrow p_a(0)$, so as $\lambda$ gets small, this condition amounts to:

$$1 - F_a(p_a(0)) - p_a(0) * \frac{1}{2} * f_a(p_a(0)) \leq 0$$  \hfill (77)

But when $\lambda = 0$, the first-order condition for the optimally of $p_a(0)$ is:

$$1 - F_a(p_a(0)) - p_a(0) * f_a(p_a(0)) = 0$$  \hfill (78)

Note that this FOC implies that our previous expression for the change in profits must be positive for small enough $\lambda$. But this contradicts the optimally of $p_b(\lambda)$ and $p_a(\lambda)$; therefore the deviation we propose must be impossible. This implies that there must be a neighborhood $[0, U]$ around 0 such that for all $\lambda \in [0, U]$, $p_b(\lambda) = 0$, establishing our result.

\[\blacksquare\]

**Theorem 8** When the optimal $p_a > 1/2$, the optimal $\delta > 0$.

**Corollary 7** If $\frac{F_a}{1 - F_a(\cdot)}$ increases as $p_a$ increases, then $\delta > 0$.

**Proof.** To be added. \[\blacksquare\]