The Role of Spatial Demand on Outlet Location and Pricing

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Abstract

In this paper we consider the problem of outlet pricing and location in the context of unobserved spatial demand. Our analysis constitutes a scenario wherein capacity-constrained firms set prices conditioned on their location, demand and costs. This analysis enables us to develop maps of latent demand patterns across space. These maps are useful for engaging what-if policy simulations wherein firms can consider the effect of locating additional outlets on equilibrium profits and prices.

Using Bayesian spatial statistics, we apply our model to seven years of data regarding apartment location and prices in Roanoke, Virginia. We find that the 95% spatial decay in demand extends 7.5 miles in a region measuring slightly over 9.5 miles. We further find that capacity constraints are material in this market and can cost complexes upwards of $193 per apartment. As predicted, price elasticities and costs are biased downward when spatial covariance in demand is ignored. Our what-if analysis suggests that a proper accounting of spatial effects and capacity constraints leads to an entry recommendation that improves profitability by 66% over a model that ignores these effects.

Keywords: Outlet Location, Pricing, Spatial Statistics, Structural Models, Competition.
1 Introduction

Outlet location and pricing are of central concern to many firms. For example, the French retailer Carrefour SA added 5930 stores between 1999 and 2003 while Dollar General in the United States added 2,426 stores in the same period (Euromonitor 2005). Paramount to expansion efficacy is the effect of outlet location on sales, prices and profits, which are moderated by the underlying demand across regions in which a firm trades. Yet direct observation of demand across the areas in which firms trade is difficult: not only is this demand apportioned among existing outlets located at a small (relative to the trade space) number of fixed points in space, but the latent demand does not necessarily comport with the observed density of population. For example the presence of complementary stores or desirable traffic patterns may elevate demand in a specific locale.

Given the central role that the spatial distribution of demand plays in a firms’ location decision, we focus on the inference of latent spatial demand. Integrating Bayesian spatial statistics with a structural model of firm conduct allows us to capture a flexible distribution of latent spatial demand and use it to engage policy simulations regarding the sequential effect of locating an additional outlet on the demand, prices and profits for the new and existing outlets. This enables us to address questions such as the following:

- Can one infer the distribution of category demand across a given market or trade area from observations of retail sales at specific points in space? The answer to this question leads to a contour map of latent spatial demand that can be used to identify potential sites for new outlet entry or deletion.

- How do spatial demand and outlet location affect equilibrium prices and profits? We find that unobserved latent spatial demand leads to higher price variation over space, and that a proper accounting of spatial demand effects improves the profitability of an entry recommendation by 66% over a model that ignores spatial covariance in demand and costs.

There has been considerable prior work in marketing regarding outlet sales dating back to the gravity model (see, for example Bucklin 1972; Ghose and Craig 1983). Our work differs from this research in a number of respects. First, this research typically assumes prices to be exogenous such that prices of competing outlets do not change with the location of an outlet. To the extent
firms react to the new entry by changing prices, models that ignore this could misstate potential profits. Second, such models typically do not estimate latent demand across regions (i.e., the demand apportioned to an outlet arising from a particular point in space), but rather assume demand arises from observed differences in population across regions. In contrast, our work can accommodate unobserved sources of spatial demand. Third, we note that sales data necessary to estimate these models are often not observed as firms sometimes keep their outlet sales data private (e.g., Wal-Mart). Our approach need not require information on outlet sales to infer latent spatial demand.1

In this sense, our model of spatial demand is more related to analytical models of spatial location and demand in economics. Much of this work has been theoretical, and focuses upon the equilibrium location of outlet location and the corresponding prices (Hotelling 1929, d’Aspremont, Gabszewicz and Thisse 1979, Ansari, Economides, and Ghose 1994). Recently, empirical models in economics and marketing have begun to appear that focus on solving the sub-game of equilibrium prices conditioned on outlet location and capacity in order to infer latent spatial demand (Chan, Padmanabhan and Seetharaman 2005; Pinske, Slade and Brett 2002, Venkataraman and Kadiyali 2005, Thomadsen 2005a, b, Davis 2001).2 This subgame is a reasonable starting point for the outlet location problem for a couple of reasons. First, it must be solved before determining the optimal outlet location. Second, in a preponderance of markets, outlet locations are extant and fixed at the time a late entrant decides to enter, so the relevant managerial decision pertains to locating the next outlet and its capacity. Using the sub-game, one can explore the implications of adding an outlet on equilibrium prices and demand for the new and existing outlets. Third, competitive response latencies in constructing outlets can be large due to land acquisition, zoning, permitting and construction. Therefore, over a reasonable range of time it may be appropriate to consider the problem of adding a single outlet.

A central innovation in these empirical models of spatial demand is that they consider observed spatial demand effects arising from distance to some centroid such as a population center or an airport. Our work complements the foregoing stream of research in several key ways:

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1 By sales we refer to the observed number of units sold at a particular outlet. By demand, we refer to the distribution of product utility across various regions. It is the distribution of demand across space that drives sales at given locales.

2 Chan et al. (2005) explicitly consider the firm location decision from the perspective of a social planner.
• First, we supplement observed spatial demand factors via spatially correlated unobserved demand effects. Given there are a plethora of potential spatial influences, it is unlikely that a researcher can capture all or most of them. For example, the presence of a particular employer, hotel, traffic pattern, school, restaurants, family member, or friend could affect the choice of apartment and therefore equilibrium prices. Moreover, these influences tend to induce spatial covariance in demand shocks. In our data, these unobserved effects play a much greater role in demand than observed effects. In this regard, unobserved spatial heterogeneity is analogous to that of unobserved consumer heterogeneity in brand choice behavior where unobserved effects dominate observed effects such as demographics (Bucklin and Gupta 1992). Spatial covariance in demand and supply shocks has a number of important implications for the outlet location problem:

- Random spatial effects are biased toward zero when spatial covariance is ignored, leading to a downward bias in estimates of price elasticity. The bias arises because firms raise price in response to increased spatial demand. These positive supply-side price effects lead to a decrease in the apparent negative demand-side price effects. When spatial effects are underestimated, so too are the supply-side price effects. We demonstrate these effects via simulation and a data application. In sum, we capture spatial endogeneity in pricing via a structural link between random spatial demand effects and equilibrium prices.

- The downward bias in elasticities also leads to a downward bias in the estimate for marginal costs. Decreases in elasticities lead to higher apparent mark-ups. As price is the sum of costs plus markups, for a given price the higher estimated markup must be offset by a lower estimated cost. In the market we consider, survey data indicate marginal costs are 42% of revenue.

- In some categories, observed effects such as population centroids can not be used at all to capture latent spatial demand. In the cases of apartments, for example, population

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3 We differentiate between location-specific random effects (e.g., Bayern and Timmins 2003) commonly employed in sorting models and spatial random effects. Spatial models allow for spatial covariance in the location-specific effects. These covariances have implications for spatial prediction (such as relaxing the spatial IIA property) and outlet location and pricing. Our work further differs from this research stream inasmuch as we consider the supply side problem (price endogeneity) and capacity constraints in demand.
itself is endogenous. Similarly, in the case of hotels, demand comes from areas outside of a region. Via the inclusion of latent spatial random effects we can capture such phenomenon.

- The estimate of spatially correlated random effects in conjunction with spatial kriging yields a regional demand map which can serve as a valuable aid to i) find new locations and ii) afford insights regarding differences in demand across the map (e.g., a peak in a certain area might suggest an important omitted variable). In the absence of spatial covariance, predictions for demand are incumbent solely upon observed spatial differences and are likely to underestimate the true variation in demand. In our data the omission of spatial covariance attenuates the standard deviation of spatial demand by a factor of 25.

- Policy simulations pertaining to outlet location yield incorrect recommendations when spatial covariance is ignored. Omission of these effects implies substitution patterns that are independent of location (apart from observed spatial differences). For the logit demand system this results in a spatial IIA problem wherein share loss is apportioned not to proximal outlets but rather the largest ones. In our simulation, the policy recommendation from a model that incorporates spatial covariance yields a locale that would increase profits by 66% over a model that ignores spatial covariation.

- Second, we consider the issue of outlet capacity, as supply is not limitless. Extant models do not accommodate the possibility that capacity might be constrained (as would be the case with categories such as restaurants, hotels, or apartments). This can be problematic for several reasons.

  - In the context of policy simulations, as moving an outlet (or adding a new one) can not increase demand among extant outlets beyond their capacity.

  - An improper accounting of this constraint can also lead to biased estimates for price. If demand exceeds supply, firms can raise prices with little effect on demand. This leads to lower apparent price elasticities if capacity is ignored. The additional error arising from
poor predictions also leads to an increase in the estimated variance of the supply-side model.

- Our approach yields estimates of the costs of these capacity constraints (via estimates of the Kuhn-Tucker multiplier), which can be used to assess the merits of expansion.

• Third, we implement a fully Bayesian approach to estimating latent demand. By integrating Bayesian methods with structural models (Musalem, Bradlow and Raju 2005) we can capture highly flexible patterns of demand and estimate a wide array of models that might not otherwise be feasible. Moreover, the Bayesian approach has more desirable small sample properties and facilitates convergence (see Bajari 2003, Berry 2003, and Yang, Chen and Allenby 2003 for a more complete discussion of the benefits of a Bayesian approach). To our knowledge, this is the first application of Bayesian methods to structural models of location.

In sum, it is our goal to develop a flexible structural model of spatial demand in order to provide guidance to firms considering the potential location of new outlets (or changing the location of existing ones). In this sense, our work lies at the intersection of two developing research streams in marketing; spatial statistics (e.g., Bronnenberg and Mela 2004) and empirical economics (Chintagunta, Erdem, Rossi and Wedel 2006).

The paper proceeds as follows. First, we develop the demand-side model. We then generalize the model to consider the pricing problem faced by firms in the presence of capacity constraints, spatial demand effects, and the location of other firms. This is followed by a description of the simulator predicated upon these models and used to forecast demand and profits arising from new outlet locations. We then describe the apartment data used to calibrate the model, and follow this with both simulated and data-based results. The simulated data are used to show the biases that arise when one omits spatial covariance and capacity constraints and the real data show these spatial effects lead to considerable improvements in model performance. We conclude the results of our data application, suggested locations for additional outlets and future research directions.

2 Model

Our model presentation proceeds as follows. First, we present the consumer choice model. Specifically, we model discrete choice stochastic demand with spatially correlated random effects. These
spatial random effects enable us to capture unobserved differences in latent spatial demand across a firm’s trade area. Second, we outline the equilibrium conditions of supply and demand in the context of firms who own outlets that are capacity constrained. The equilibrium conditions are derived from the profit maximizing behavior of the firms and utility maximizing behaviors of the consumers. The market clearing condition establishes the economic equilibrium of demand and supply which leads to the structural estimation equations. These demand and supply-side equilibrium equations constitute a system of equations that we estimate using advances in Bayesian spatial statistics.

2.1 Demand Model: Consumer Outlet Choice

Suppose there exist $J$ outlets for a given type of goods (e.g., apartments, car dealers, hotel rooms, bank branches, etc.) in a given region. The location of the $j \in J$ outlet is denoted as $s_j$. Let the number and the set of the potential customers for a set of outlets in a region be $I$, $i = \{1,...,I\}$. Customer $i$’s random indirect utility for choosing outlet $j$ at location $s_j$ in period $t$ is

$$V_{tij} = X_{tj}\beta_1 + X_{tsj}\beta_2 - P_{tj}\gamma + \theta_{tsj} + \varepsilon_{tij}$$ (1)

In this equation, $X_{tj}$ represent the attributes of the $j$th outlet such as variety or amenities; $X_{tsj}$ represents the observed attributes specific to the location (such as the distance to a business center or a major highway; and $P_{tj}$ is a price index for an outlet. These are the fixed effects in the model. $\theta_{tsj}$, $j = 1,\ldots,J$ are the spatial random effects in the indirect utility that represent the unobserved location-specific desirability. $\theta_{tsj}$ can be structurally correlated with the prices $P_{tj}$ as a result of the endogeneity of price to random spatial demand effects; prices tend to increase when demand is high. The spatially correlated random effects $\theta_{tsj}$, $j = 1,\ldots,J$ can be modelled via a parametric isotropic model $N_J \left(0,\sigma^2_\theta R_J (\phi; d)\right)$ where $d$ is distance and $\phi$ is the vector of parameters (Banerjee, et. al. 2004) or a more general nonparametric Spatial Dirichlet Process model (Gelfand, et. al. 2004).
2005, Duan et al. 2005). When these correlations are positive, this distribution implies that the unobserved location-specific desirability is very similar for two locations very close to each other (i.e., outlets in this situation tend to draw share primarily from closer apartments). We note that these spatial structures are highly flexible and admit many potential latent demand surfaces.

The representative customer may choose the outside good, that is she may not buy in the region at all or she may consider expenditures on other types of goods sold in the region. Her indirect utility in this case is

\[
V_{t_i0} = M_t + \epsilon_{t_i0}. \tag{2}
\]

If the highest utility of choosing one outlet exceeds that of the outside-good, the customer will select the highest utility outlet.

The customer chooses the outlet with the highest utility. As is common in structural models of demand, \(\epsilon_{tij}\) is assumed independently from a Gumbell distribution (Berry, Levinshon and Pakes 1995, henceforth BLP). Though the value of \(\epsilon_{tij}\) is not known to the econometrician, the distribution of \(\epsilon_{tij}\) is common knowledge. The probability that customer \(i\) chooses outlet \(j\) as a function of the price index for that outlet is then

\[
W_{tj} (P_t, X_t, \Theta_t) = \frac{e^{X_{tj}\beta_1 + X_{tsj}\beta_2 - P_{tij}\gamma + \theta_{tsj}}}{\sum_{l=1}^{J} e^{X_{tl}\beta_1 + X_{tsl}\beta_2 - P_{tl}\gamma + \theta_{tsl}} + e^{M_t}}, \tag{3}
\]

where \(P_t, X_t, \Theta_t\) denote the vectors of the prices, properties and spatial desirability of all the existing stores in this region. For the market with consumers identical in utility, \(W_{tj}\) is also the market share for outlet \(j\) in the \(t\)-th period.

Therefore, by summing across the number of persons in the market, outlet \(j\)’s total demand is

\[
Q_{tj} (P_t, X_t, \Theta_t) = I \times W_{tj} (P_t, X_t, \Theta_t) \tag{4}
\]

As \(I, M_t\) and the intercept of the additive utility cannot be separately identified at the same time, we fix \(I\) and \(M_t\) as discussed in the Data section.

### 2.2 Supply Model: Bertrand Nash Game

In addition to consumers, the market comprises firms who compete on the basis of price, location and capacity. As noted previously, we focus upon the subgame of firm competition conditioned on location choice and capacity. Assume a market is comprised of \(F\) firms wherein each firm \(f\)
has \( F_f \) outlets. Each firm maximizes the total profit of its \( F_f \) outlets. Let us first consider firm \( f \)'s strategy. Conditional on the prices of the outlets not belonging to \( f \)'s chain, firm \( f \) faces the following profit maximization problem:

\[
\max \Pi_{tf} = \sum_{j \in F_f} (P_{tj} - c_{tj}) Q_{tj} (P_t, X, \Theta_t) \quad \text{s.t.} \quad Q_{tj} (P_t, X, \Theta_t) \leq K_j
\]

(5)

where \( c_{tj} \) represents the variable cost of goods sold and \( K_j \) is outlet \( j \)'s capacity constraint of \( K_j \), which is the total number of goods that the outlet can produce, sell or rent. The capacity is considered rigid over the short decision frame of a firm setting prices as the expansion or contraction of an outlet's physical plant involves construction/permitting time and considerable cost. This implies that the capacity consideration is relegated to an "upper stage" of the game wherein a firm sets location and capacity prior to choosing price in the subgame. When facing a different demand at time \( t \) as a result of setting the price \( P_{tj} \), \( K_j \) becomes an immutable short-term constraint to which the outlet is subject. The constrained maximization problem is therefore

\[
\max \Pi_{tj} = (P_{tj} - c_{tj}) Q_{tj} (P_t, X_t, \Theta_t) \quad \text{s.t.} \quad Q_{tj} (P_t, X_t, \Theta_t) \leq K_j
\]

(6)

As the firms' costs \( c_{tj} \) are not observed by the econometrician, we estimate them by assuming

\[
c_{tj} = Y_{tj} \beta_3 + \zeta_{tj}
\]

(7)

where \( \zeta_t \equiv (\zeta_{tj}; j = 1,\ldots,J) \) is distributed across space as \( N \left( 0, \sigma^2 \xi R_J (\psi) \right) \) and \( Y_{tj} \) are a set of cost shifters (for example, the attributes of the complex). The \( R_J (\psi) \) represent possible spatial dependencies in costs that may result from efficiencies in transport, access to local labor, or other unobserved spatial factors.

The Kuhn-Tucker conditions for this optimization problem with inequalities constraints are

\[
\frac{\partial \Pi_{tj}}{\partial P_{tk}} = \sum_{j \in F_f} (P_{tj} - Y_{tj} \beta_3 - \zeta_{tj}) \frac{\partial Q_{tj}}{\partial P_{tk}} + Q_{tk} - \sum_{m \in F_f} \lambda_{tm} \frac{\partial Q_{tm}}{\partial P_{tk}} = 0; k \in F_f
\]

(8)

with the Kuhn-Tucker multipliers, \( \lambda_{tm} \geq 0 \), and

\[
Q_{tm} (P_t, X_t, \Theta_t) = K_m, \text{ iff } \lambda_{tm} > 0 \quad \text{and} \quad Q_{tm} (P_t, X_t, \Theta_t) < K_m, \text{ iff } \lambda_{tm} = 0
\]

(9)

These multipliers reflect the marginal cost of the capacity constraint on apartment profitability and reflect the value of expanding a particular outlet's capacity by one unit.
As $Q_t = I \times W_t$, the representation of (8) with market shares is

$$\sum_{j \in F_f} (P_{tj} - Y_{tj} \beta_3 - \zeta_{tj}) \frac{\partial W_{tj}}{\partial P_{tk}} + W_{tk} - \sum_{m \in F_f} \lambda_{tm} \frac{\partial W_{tm}}{\partial P_{tk}} = 0; k \in F_f. \quad (10)$$

The Kuhn-Tucker conditions imply that firm $f$ chooses optimal prices $P^*_{tk}$ for the outlets with expected vacancy and the second best price $\tilde{P}_{tm}$ for outlets at capacity. $P^*_{tk}$ is the solution of the first order condition (10) above, whereas $\tilde{P}_{tm}$ satisfies the binding capacity constraint (9). $P^*_{tk}$, $\tilde{P}_{tm}$ and the non-zero multiplier $\lambda_m$ are solved simultaneously.

Let $\kappa_{tf}$ be the number of outlets that are full in firm's chain in period $t$. There are $F_f$ equations arising from the first order conditions for each firm and $\kappa_{tf}$ equations arising from the binding capacity constraints. To facilitate explication, we define a $F_f \times F_f$ matrix whose $(j, k)$th element is

$$\Omega^{(t,f)}_{j,k} = \frac{1}{\gamma} \times \frac{\partial W_{tj}}{\partial P_{tk}} = \begin{cases} W_{tj} (1 - W_{tj}), & \text{for } j = k \in F_f, \\ -W_{tj} (1 - W_{tj}), & \text{for } j \neq k \in F_f. \end{cases} \quad (11)$$

Let the vectors $\lambda_f = (\lambda_m, m \in F_f)$, $P_{tf} = (P_{tj}; j \in F_f)$, $W_{tf} = (W_{tj}; j \in F_f)$, $\zeta_{tf} = (\zeta_{tj}; j \in F_f)$ and the submatrix $Y_{tf} = (Y_{tj}; j \in F_f)$. Note for the outlets with spare capacity $\lambda_m = 0$ and for the outlets with full capacity $W_{tj} = \frac{K_j}{I}$. With these definitions, equation (10) can be rewritten more compactly as

$$\gamma \Omega^{(t,f)} (P_{tf} - Y_{tf} \beta_3 - \lambda_f) + W_{tf} = \gamma \Omega^{(t,f)} \zeta_{tf} \quad (12)$$

Likewise, for all the $F$ firms in the market, we define the following $J \times J$ matrix whose $(j, k)$th element is,

$$\Omega^{(t)}_{j,k} = \begin{cases} \frac{1}{\gamma} \times \frac{\partial W_{tj}}{\partial P_{tk}} (P_{tj} - P_{tj'; j \in F_f} - X \Theta_t), & \text{for } j, k \in \text{same } F_f, \\ 0, & \text{otherwise}. \end{cases} \quad (13)$$

The first order conditions for all the outlets can be written compactly as

$$\gamma \Omega^{(t)} (P_{t} - Y_{t} \beta_3 - \lambda_t) + W_{t} = \gamma \Omega^{(t)} \zeta_{t}. \quad (14)$$

If $\Omega^{(t)}$ is invertible, then we have

$$P_{t} - \lambda_t + \frac{1}{\gamma} \Omega^{(t)^{-1}} W_{t} = Y_{t} \beta_3 + \zeta_{t}. \quad (15)$$

The markup function is $B_t (\lambda_t, \gamma) = \lambda_t - \frac{1}{\gamma} \Omega^{(t)^{-1}} W_{t}$. Note that $\lambda_{tm} = 0$ for the outlets that have vacancy and $\lambda_{tm} > 0$ for the outlets with full occupancy. Let $J_{2t}$ be the number of outlets with
full occupancy, then there are $J_{2t}$ non-zero $\lambda_{tm}$’s and $J_{1t} \triangleq J - J_{2t}$ zero $\lambda_{tm}$’s. The economic interpretation of this equation is that the prices of the outlets with full capacity are higher than their optimal prices. Intuitively, this suggests firms who face demand in excess of supply will raise prices to the point wherein demand is equal to capacity; lower levels of price only serve to decrease revenue.

Of interest, equation (15) embeds all parameters in the supply and demand system. This suggests it is possible to estimate both the demand-side and cost side parameters even in the absence of any demand-side data (Thomadsen 2005).

3 Estimation

3.1 Demand Model

The demand-side estimation equations for all the outlets in the market are

$$W_{tj} = \frac{Q_{tj}}{T} = \frac{e^{X_{tj}\beta_1 + X_{tsj}\beta_2 - P_t\gamma + \theta_{tsj}}}{\sum_{l=1}^J e^{X_{tl}\beta_1 + X_{tsl}\beta_2 - P_t\gamma + \theta_{tsl} + e^{M_t}}};$$  \hspace{1cm} (16)

where $Q_{tj} = K_j$ if the outlet is at capacity (e.g. a full outlet) in that period.

Instead of using a maximum likelihood method to estimate model parameters, BLP propose an instrumental variables-based approach. One appealing aspect of this approach is that unobserved factors can affect both spatial random effects and prices and the IV approach allows one to control for this possibility by forming instruments for price. Another appealing aspect of this approach pertains to the supply-side, as the likelihood function for prices may suffer from the multiple equilibrium problem (Bajari 2003; Berry 2003). Let

$$\xi_t = X_t\beta_1 + X_{ts}\beta_2 - P_t\gamma + \Theta_t, \hspace{1cm} (17)$$

Then we have

$$W_{tj}(\xi_t) = \frac{e^{\xi_{tj}}}{\sum_{l=1}^J e^{\xi_{tl} + e^{M_t}}}; \hspace{1cm} (18)$$

Our goal is to attain estimates for $\xi_t$ so we can use the orthogonality conditions between the spatial errors in (17) and instruments to estimate $\beta_1$ and $\beta_2$ (that is, $E(Z^T\Theta_t) = 0$). BLP prove that estimates for $\hat{\xi}_{tj}$ are obtained by the using iterative procedure, $\hat{\xi}_{tj}^{(m+1)} = \hat{\xi}_{tj}^{(m)} + \ln \hat{W}_t - \ln \hat{W}_t \left(\hat{\xi}_{tj}^{(m)}\right)$ by showing this equation is a contraction mapping. However, we first condition $\xi_t$ on $\gamma$ in order
to estimate \( \gamma \) more efficiently by using information from both the supply and demand equations. Conditioning \( \xi_t \) on \( \gamma \), we redefine

\[
\xi_t + P_t \gamma \equiv \xi_t(\gamma) = X_t \beta_1 + X_{ts} \beta_2 + \Theta_t.
\]

We then adapt the BLP contraction mapping procedure as follows:

\[
\hat{\xi}_t^{(m+1)}(\gamma) = \hat{\xi}_t^{(m)}(\gamma) + \ln \hat{W}_t - \ln W_t \left( \hat{\xi}_t^{(m)}(\gamma) - \gamma P_t \right). 
\]

Conditioning the model parameters \( \beta_1, \beta_2 \) and \( \gamma \), we let

\[
\Theta_t(\beta_1, \beta_2, \gamma) = \hat{\xi}_t(\gamma) - X_t \beta_1 - X_{ts} \beta_2
\]

be mean independent of the instruments \( Z_t \) and have a spatial covariance structure \( N(0, \sigma_Z^R J(\phi)) \) conditional on the true \( \beta_1, \beta_2 \) and \( \gamma \). Following Romeo (2004), we assume \( Z_t^T \Theta_t \sim N(0, \sigma_Z^2 Z_t^T J(\phi) Z_t) \).

### 3.2 Supply Model

Note that the observed demand \( \hat{Q}_{tj}(P_t, X_t, \Theta_t) \) and observed market share \( \hat{W}_{tj}(P_t, X_t, \Theta_t) \) are not available to the firm when setting prices as these will be forecasts, not realizations. However, there is a relationship between these forecasts and the market share realizations. Should all consumers in the market abide by the decision rules set in this model, the central limit theorem for the multinomial model suggests that the observed market share \( \hat{W}_{tj} = \hat{Q}_{tj} \) converges to \( W_{tj} \) at rate \( \sqrt{I} \). When \( I \) is large, we can assume the observed market share \( \hat{W}_{tj} \) is a good estimate of the real market share. For constrained firms, \( \hat{W}_{tj} = \frac{K_j}{I} \) according to the Bertrand-Nash equilibrium. Thus, we will use the observed demand market share \( \hat{W}_{tj} \) to estimate the expected market share \( W_{tj} \) in Equation (15). Accordingly, the first order conditions from the arising from the profit function on the supply-side can be rewritten as:

\[
P_t - \lambda_t + \frac{1}{\gamma} \hat{\Omega}^{(t)-1} \hat{W}_t = Y_t \beta_3 + \zeta_t.
\]

where \( \hat{\Omega}^{(t)} \)'s elements are

\[
\hat{\Omega}^{(t)}_{j,k} = \begin{cases} 
\hat{W}_{tj} \hat{W}_{tk}, & \text{for } j \neq k \in \text{same } F_f; \\
-\hat{W}_{tj} \left( 1 - \hat{W}_{tj} \right), & \text{for } j = k; \\
0, & \text{otherwise.}
\end{cases}
\]
and the markup function for the outlet chain model is $B_t(\lambda_t, \gamma) = \lambda_t - \frac{1}{\gamma} \Omega(t)^{-1} W_t$.

As indicated in equation (22), the non-zero elements of the multipliers $\lambda_t$ (which we denote $\lambda_{2t}$) are functions of prices and other model parameters. This expression further indicates $\lambda_{2t}$ and $\zeta_{2t}$ can not be separately identified using the prices at capacity constrained outlets. Accordingly, i) only prices for outlets with spare capacity can be used for the supply-side estimation and ii) the supply-side instrumental variables include only the $J_{1t}$ submatrix of $Z_t$ (denoted $Z_{1t}$) representing those outlets with spare capacity. Once estimates of $\zeta_{1t}$ for spare capacity firms are obtained, the subvector of the spatial random effect $\zeta_{2t}$ for the outlets at capacity can be sampled using the Bayesian Kriging introduced in the section (4). Once the $\zeta_{2t}$ are thereby obtained, one can compute the distribution of Kuhn-Tucker multipliers by noting $\lambda_{2t} = P_{2t} + \frac{1}{\gamma} \hat{\Omega}^{(2t)-1} \hat{W}_{2t} - Y_{2t} \beta_3 - \zeta_{2t}$.

3.3 Joint Model

Conditioning on $\gamma$ and $\lambda_t$ and assuming the instruments are independent of the model errors, $Z^T_{1t} \zeta_t \sim N \left( 0, \sigma^2 \zeta Z^T_{1t} R_J (\psi) Z_{1t} \right)$ and $Z^T_t \Theta_t \sim N \left( 0, \sigma^2 \theta Z^T_t R_J (\phi) Z_t \right)$, we obtain the following joint distribution for the transformed data $P_{1t} - B_{1t}(\lambda_t, \gamma)$, the subvector of the prices and markup for the under-capacity outlets, and $\hat{\xi}_t(\gamma)$ from the demand-side:

$$
\left[ \begin{array}{c}
Z^T_{1t} [P_{1t} - B_{1t}(\lambda_t, \gamma)] \\
Z^T_t \hat{\xi}_t(\gamma)
\end{array} \right] \sim N \left( \left[ \begin{array}{c}
Z^T_{1t} Y_{1t} / \beta_3 \\
Z^T_t (X_{t1} \beta_1 + X_{ts} \beta_2)
\end{array} \right], \left[ \begin{array}{cc}
\sigma^2 \zeta Z^T_{1t} R_J (\psi) Z_{1t} & 0 \\
0 & \sigma^2 \theta Z^T_t R_J (\phi) Z_t
\end{array} \right] \right)
$$

We assume the spatial random effect $\theta_t$ in the indirect utility function is conditionally independent of the supply-side error term $\zeta_t$ in the cost structure. This does not mean these are independent; these are linked via the structural effect of unobserved spatial effects and other factors on equilibrium prices. Note both $\theta_t$ and $\zeta_t$ are posited to have spatial correlation.

Potential instruments $Z_t$ include observed spatial variables such as distance to known points such as population centroids (for both the outlet and the chain), attributes of distant outlets (from the same chain and from competitors), distant cost shifters (from the same chain and from competitors), and distant prices (from the same chain and from competitor). These are given
respectively by

\[
Z_{tj} = \left[ X_{ts_{t}}, \sum_{l \neq j, j \in F_f} X_{ts_{t}}, \sum_{l \neq j, j \in F_f} X_{t}, \sum_{\text{dist}(l) > r} X_{t}, \sum_{\text{dist}(l) > r} X_{t}, \sum_{\text{dist}(l) > r} Y_{t}, \sum_{\text{dist}(l) > r} Y_{t}, \sum_{\text{dist}(l) > r} P_{t}, \sum_{\text{dist}(l) > r} P_{t} \right].
\]

(25)

where \( l, j \notin F_f \) denotes that outlets \( l \) and \( j \) are not in the same outlet chain. We instrument using variables greater than a distance \( r \) from the location of \( j \) because distant outlets’ characteristics affect prices of a local outlet via competition but have little correlation with local spatial effects. Choosing the distance \( r \) requires some care. If \( r \) is too small, then proximal outlet attributes may be correlated with the spatial random error. On the other hand, if \( r \) is too large, the instrument correlation with prices will be small because competition is diminished; therefore parameter estimates will be inefficient. We recommend using the posterior estimates for the spatial autocorrelation to shed some insights regarding the choice of \( r \). If the spatial autocorrelation is high, the distance \( r \) should be higher. We rule out lagged own-prices as instruments because these are likely to be correlated with local spatial shocks.

The Bayesian model fitting is accomplished by a Gibbs sampler that is detailed in Appendix I.

4 Spatial Prediction

In this section, we consider the problem of a firm who wishes to select a new location, capacity and price. For each considered location this involves two steps. First, one must forecast spatial cost and demand random effects at the potential entry location. Second, one must compute the resulting equilibrium demand, prices and profits at that and other locations. We shall discuss each step in turn.

We consider competitive response in prices arising from the location of a new outlet but not competitive responses in the form of constructing new outlets or adding capacity. We do this for two reasons. First, the case of no response seems reasonable over an intermediate planning horizon as competitive response in outlet entry often takes many years owing to planning, property acquisition, permitting and zoning, and construction. Second, such an analysis is beyond the scope
of this research and is an important problem in its own right. Further, to address the problem of location response it is necessary to first solve the subgame of prices conditioned on entry and we do this below (see also Thomadsen (2005) and others who focus on the sub-game).

4.1 Bayesian Kriging of Demand and Cost Effects

The goal of spatial prediction is to make inferences about costs, $\zeta_{lj}$, and demand $\theta_{sj}$ over the entire region or space. This yields a map of latent demands and costs that can be used to obtain insights into the nature of the market in which the outlets compete. To conserve space, we illustrate the Kriging approach for $\theta_{sj}$; a similar procedure is used to sample $\zeta_{lj}$. Assume $n$ new firms were to consider entry in period $T$. To select the preferred locations $(\tilde{s}_1, \ldots, \tilde{s}_n)$, the firms would like to estimate the demand at each of the considered locations:

$$\tilde{Q}_{Tj} = I \times \frac{e^{X_1\beta_1 + X_2\beta_2 - P_T \gamma + \theta_{sj}^s}}{\sum_{l=1}^{J} e^{X_1\beta_1 + X_2\beta_2 - P_T \gamma + \theta_{sj}^s}} + e^{M_T}$$

In this demand system, $(\beta_1, \beta_2, \gamma)$ and $\theta_{Tsj}$, $l = 1, \ldots, J$ are sampled when we fit the model. However, $\theta_{T\tilde{s}_k}$, $k = 1, \ldots, n$ for the considered location are unknown as of yet. The estimation of the $\theta_{T\tilde{s}_k}$ is called spatial prediction or kriging.

We assume that the location preference error $\theta_s$ follows a Gaussian random process. Thus, for any two locations $s_1$ and $s_2$, $\theta_{s_1}$ and $\theta_{s_2}$ have a bivariate normal distribution with the covariance calculated from the covariance function of the Gaussian process. We further assume an exponential correlation function for the spatial random effects, $\exp(-\phi \cdot ||s - s'||)$, which defines the Orstein-Uhlenbeck process (Banerjee et. al. 2004). Therefore $\theta_{s_1}$ and $\theta_{s_2}$’s covariance is given by $\sigma^2 \exp(-\phi \cdot ||s_2 - s_1||)$. Likewise, if there are $J$ random variables $\theta_{s_1}, \ldots, \theta_{s_j}$, associated with location $s_1, \ldots, s_J$, the pairwise covariance of $\theta_{s_k}$ and $\theta_{s_j}$ is calculated as $\sigma^2 \exp(-\phi \cdot ||s_k - s_j||)$, which is also the $(k, j)$-th entry in covariance matrix of the multivariate normal distribution for $\theta_{s_1}, \ldots, \theta_{s_j}$. We denote this matrix as $\sigma^2 R_J(\phi)$.

For the spatial prediction problem in our model the observed firms occupy the locations $(s_1, \ldots, s_J)$ and the entering firms select $(\tilde{s}_1, \ldots, \tilde{s}_n)$. The corresponding location preferences are $(\theta_{s_1}, \ldots, \theta_{s_j})$ and $(\theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n})$, in which $(\theta_{s_1}, \ldots, \theta_{s_j})$ can be computed from equation (21) and the parameter draws from the Gibbs sampler. Therefore, $(\theta_{s_1}, \ldots, \theta_{s_j})$ are known. From the assumption of the

---

$^5$As indicated in the preceding section, spatial random cost effects can not be estimated directly for outlets at capacity, hence these must be predicted via Kriging in addition to the values at considered locations for new entry.
spatial Gaussian process, \((\theta_{s_1}, \ldots, \theta_{s_J})\) and \((\theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n})\) jointly have the following multivariate normal distribution:

\[
(\theta_{s_1}, \ldots, \theta_{s_J}, \theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n}) \sim N_{J+n} \left( 0, \sigma^2 R_{J+n} (\phi) \right)
\]  

(27)
in which \(\sigma^2 R_{J+n} (\phi)\) is a \((J+n) \times (J+n)\) covariance matrix. Each entry in \(R_{J+n} (\phi)\) is evaluated by the spatial covariance function, in this case \(\sigma^2 \exp (-\phi \cdot ||s-s'||)\). Note the covariance of the known \(\theta_{s_j}\) and the unobserved \(\theta_{\tilde{s}_k}\) is by definition \(\sigma^2 \exp (-\phi \cdot ||\tilde{s}_k - s_j||)\).

In order to sample \((\theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n})\) conditioning on the already sampled \((\theta_{s_1}, \ldots, \theta_{s_J})\), we rewrite \(\sigma^2 R_{J+n} (\phi)\) as follows:

\[
\begin{bmatrix}
\sigma^2 R_J (\phi) & \sigma^2 R_{J,n} (\phi) \\
\sigma^2 R_{J,n}^T (\phi) & \sigma^2 R_n (\phi)
\end{bmatrix}
\]  

(28)

where \(\sigma^2 R_{J,n} (\phi)\) includes the covariances for the observed \(\theta_{s_j}\) and the unobserved \(\theta_{\tilde{s}_k}\), and \(\sigma^2 R_n (\phi)\) is the covariance matrix for the unobserved \((\theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n})\).

Conditioning on the known \(\theta \triangleq (\theta_{s_1}, \ldots, \theta_{s_J})\), \(\tilde{\theta} \triangleq (\theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n})\) has the following distribution:

\[
\tilde{\theta} \sim N_n \left( R_{J,n}^T (\phi) R_J^{-1} (\phi) \theta, \sigma^2 \left[ R_n (\phi) - R_{J,n}^T (\phi) R_J^{-1} (\phi) R_{J,n} (\phi) \right] \right)
\]  

(29)

This is an \(n\)-variate conditional normal distribution with mean vector \(R_{J,n}^T (\phi) R_J^{-1} (\phi) \theta\) and covariance matrix \(\sigma^2 \left[ R_n (\phi) - R_{J,n}^T (\phi) R_J^{-1} (\phi) R_{J,n} (\phi) \right]\). Sampling \((\theta_{\tilde{s}_1}, \ldots, \theta_{\tilde{s}_n})\) from this distribution is called Bayesian spatial prediction or kriging.

Equation (29) also affords insights into the effect of spatial covariance on estimated price sensitivity. When spatial covariance is ignored \((R_{J+n} (\phi) = I_{J+n})\), the conditional spatial random effects are attenuated in expectation to zero. As prices rise with demand (because firms raise prices to capitalize on increased demand), an improper accounting of these effects countervails the fall in market share with price, leading to downward bias in price elasticity.

### 4.2 Predicting Demand and Prices

The entering firm’s decision problem consists of two steps: i) selecting the location and ii) setting price and capacity conditioned on location. We can solve this decision problem by backward induction: if the location has been selected, the firm will set the price and capacity to maximize its profit. Suppose one entering firm considers building a single outlet at \(\tilde{s}_k\). The second step of the problem is formulated as:

\[
\max \Pi_{T_k} = (P_{T_k} - c_k) \tilde{Q}_{T_k} (P_T, X, \Theta_T)
\]  

(30)
The expected demand at location \( \tilde{s}_k \) is given by

\[
\hat{Q}_{Tk} (P_T, X, \Theta_T) = I \times \frac{e^{X_k \beta_1 + X_k \beta_2 - P_{Tk} \gamma + \theta_{Tk}}}{\sum_{l=1}^{J} e^{X_l \beta_1 + X_l \beta_2 - P_{Tl} \gamma + \theta_{Tl}} + e^{X_k \beta_1 + X_k \beta_2 - P_{Tk} \gamma + \theta_{Tk}} + e^{M_T}}
\]  

(31)

There is no capacity constraint as the firm can build to demand. If the entering firm maximizes its profit with respect to \( P_{Tk} \) and selects the capacity \( \bar{K}_{Tk} = \hat{Q}_{Tk} (P_T, X, \Theta_T) \) accordingly, the first order condition is:

\[
\max \Pi_{Tk} = (P_{Tk} - c_k) \hat{Q}_{Tk} (P_T, X, \Theta_T) \Rightarrow \frac{\partial \Pi_{Tk}}{\partial P_{Tk}} = (P^*_{Tk} - c_k) \frac{\partial Q_j (P^*_{Tk}, P_T, X, \Theta_T)}{\partial P_{Tk}} + Q_j (P^*_{Tk}, P_T, X, \Theta_T) = 0
\]  

(32)

\[
(P_{Tk} - c_k) \frac{-\gamma \left[ \sum_{l=1}^{J} e^{X_l \beta_1 + X_l \beta_2 - P_{Tl} \gamma + \theta_{Tl}} + e^{M_T} \right]}{\sum_{l=1}^{J} e^{X_l \beta_1 + X_l \beta_2 - P_{Tl} \gamma + \theta_{Tl}} + e^{X_k \beta_1 + X_k \beta_2 - P_{Tk} \gamma + \theta_{Tk}} + e^{M_T}} + 1 = 0
\]  

(33)

The existing outlets will adjust their prices accordingly, knowing the location selection \( \tilde{s}_k \) of the entering firm. Their strategic pricing problem is:

\[
\max \Pi_{tf} = \sum_{j \in F_f} (P_{Tj} - c_{Tj}) \tilde{Q}_{Tj} (P_T, X_T, \Theta_T) \text{ s.t. } \tilde{Q}_{Tj} (P_T, X_T, \Theta_t) \leq K_j
\]  

(34)

where

\[
\tilde{Q}_{Tj} = I \times \tilde{W}_{Tj} = I \times \frac{e^{X_{Tj} \beta_1 + X_{Tj} \beta_2 - P_{Tj} \gamma + \theta_{Tj}}}{\sum_{l=1}^{J} e^{X_{Tl} \beta_1 + X_{Tl} \beta_2 - P_{Tl} \gamma + \theta_{Tl}} + e^{X_{Tj} \beta_1 + X_{Tj} \beta_2 - P_{Tj} \gamma + \theta_{Tj}} + e^{M_T}}
\]  

(35)

Ideally the optimal price of firm \( j \) is determined by the following Kuhn-Tucker condition:

\[
\frac{\partial \Pi_{tf}}{\partial P_{tk}} = \sum_{j \in F_f} (P_{Tj} - c_{Tj}) \frac{\partial \tilde{W}_{Tj}}{\partial P_{tk}} + \tilde{W}_{tk} - \sum_{m \in F_f} \lambda_m \frac{\partial \tilde{W}_{tm}}{\partial P_{tk}} = 0; k \in F_f
\]  

(36)

\[
I \cdot \tilde{W}_{Tm} (P^*_{Tj}, X_T, \Theta_T) = K_m, \text{ iff } \lambda_m > 0 \text{ and } I \cdot \tilde{W}_{Tm} (P^*_{Tj}, X_T, \Theta_T) < K_m, \text{ iff } \lambda_m = 0
\]  

(37)

The corresponding demand \( \hat{Q}_{Tj}^* = \hat{Q}_{Tj} \left( P^*_{Tj}, P^*_{Tk}, P^*_{T,-j}, X, \Theta_T \right) \) may exceed the capacity \( K_j \). As the market size \( I \) is unchanged, outlets with excess capacity previously will still have excess capacity because the addition of the new apartment complex reduces their demand. For the outlets that had full capacity, lowering their prices may not be profitable if the reduced demand \( \hat{Q}_{Tj}^* \)
continues to exceed capacity $K_j$. Therefore the prices of the firms with full occupancy either remain unchanged or adjust according to (36). The implementation of the optimization problem involves two steps: with a certain set of interim prices $P_{Tj}$, $P_{Tk}$, the demand $\tilde{Q}_{Tj}$ and $\tilde{Q}_{Tk}$ are calculated; if $\tilde{Q}_{Tj} \geq K_j$, the corresponding $P_{Tj}$ is solved from the binding constraint in (37) in the next step of the optimization; and if $\tilde{Q}_{Tj} < K_j$, the corresponding $P_{Tj}$ is solved from (36) in the next step.

An important consideration is the uniqueness of prices in this equilibrium as multiple price equilibria would imply different profit outcomes for each equilibrium. In Appendix II we identify conditions under which the expression for equilibrium prices in equation (36) constitutes a contraction mapping and how these conditions comport with uniqueness. In the range of our data it appears that these equilibria are unique.

Equation (33) for the entering firm and (36) for the existing firms constitute the strategic optimization problem that solves the new $P_{Tj}^*$, $P_{Tk}^*$, $\tilde{Q}_{Tk}^*$ and the profit $\Pi_{Tj}^*$, $\Pi_{Tk}^*$. $\Pi_{Tk}^*$ is a function of location $\tilde{s}_k$. Optimizing $\Pi_{Tk}^*$ with respect to the location $\tilde{s}_k$ solves the optimal market entry problem. In this problem we do not consider selecting the properties $X_k$ and $X_{\tilde{s}_k}$, which could greatly complicate the optimization problem. In reality the demand is considered a function of the properties, in order to provide differentiated goods for better profit. This is left for future research.

5 Data

To estimate our model we use panel data on apartment demand and prices. Apartments are a desirable category for illustrating the model because the number of outlets is sufficiently small to make estimation feasible but sufficiently large to obtain relatively reliable estimates of spatial effects. Moreover, unlike many previous applications of spatial demand models in economics, latent demand for apartments cannot be estimated as a function of the underlying observed spatial distribution of the population as the population itself is endogenous. We use data from Roanoke, Virginia for our analysis. This market shows good spatial coverage of apartments, little variability in supply, and a long time series of prices and vacancy rates are available for these markets. These characteristics make them ideal for our analysis.

The data for this study are provided by Real Data of Charlotte, North Carolina and are detailed
at www.aptindex.com. Real Data conducts an annual survey of apartments in a given market. The survey data consist of apartment attributes (such as whether the apartment has tennis courts, whether it has a pool, the age of the complex, etc.), addresses (which we convert to latitude and longitude), and prices. The Roanoke data cover 60 apartment complexes managed by 25 different firms, covering 7 years between 1999 and 2005. Figure 1 depicts the location and occupancy levels of the apartments in 2005; the shortest bar corresponds to 50 rented units and the largest apartment corresponds to 426 rented units. The average apartment capacity of these apartments is 152 units. Eight apartments are missing several years of data and are thus excluded from subsequent analysis.

[Insert Figure 1]

The apartment data is supplemented by two other data sources. First, we use the 2000 census data to attain the locations of schools and the market size, \( I \). The census data at 'www.census.gov/geo/www/tiger' contains the latitude and longitude of the Roanoke schools. From these data we computed the distance to the nearest school for use as an observed spatial covariate in our analysis. Other census data at 'www.fedstats.gov/qf/states/51' reports the number of non-home owning households. We use this to determine \( I \) (for the Roanoke area, this was 29,489 households in 2000).\(^6\)

Second, we determined the major employers in Roanoke from the Roanoke County Department of Economic Development (www.yesroanoke.com). For the largest employers with a single central location, we computed the mean distance from each complex to the major employers. Table 1 summarizes the variables we use in our analysis:

[Insert Table 1 About Here]

To construct a price index from these prices for various apartment sizes, we use the price of two bedroom apartments (which we report in Table 1). We choose this simple approach for a couple of reasons. First, to construct a measure that considers all apartments, we would need to create an index weight – and these weights may embed capacities, sales or other factors that 'contaminate' the pricing measure. Second, all of the apartments have two bedroom units, and the two bedroom size is always the modal size (many apartments have only two bedroom units). Hence, it is the most representative. When an apartment has multiple prices for the two bedroom units, we select

\(^6\)Our results are not particularly sensitive to reasonable changes in this value.
the median value (no information is available on the distribution of prices for a given apartment size at a complex). During the interval of the data, average apartment prices in Roanoke increased from $497 to $570 per unit though they remained constant the last three years. In the analysis, we adjust prices for inflation using CPI figures from the Bureau of Labor Statistics. Figure 2 presents a contour map of 2005 rents in nominal dollars.

[Insert Figure 2]

From Figure 2 we observe that the distribution of equilibrium prices is highly irregular, with the highest prices observed both near the downtown and on the periphery of town. A band of lower prices snakes from northwest to southeast. Given that prices change non-monotonically with distance from the city center, it is desirable to capture random spatial effects rather than relying upon the more commonly used approach wherein demand is a linear function of distance to a population centroid such as the center of town. The proposed spatial random effects approach can yield highly irregular demand and price surfaces. The spatial distribution of prices in Figure 2 also suggests there exists spatial covariance in the data.

Vacancy rates averaged roughly 6% with little change over time. In addition, total annual market capacity was fairly constant over time with a mean of 8985 units and a standard deviation of 337 units. No apartments were being constructed as of 2005. This suggests it is appropriate to model the sub-game of prices in our data conditioned on apartment locations and capacities. Figure 3 depicts a contour map of vacancy rates. Vacancy rates tend to be highest on the west side of town.

[Insert Figure 3]

6 Results

6.1 Simulation Results

To a) show how the omission of spatial covariance and capacity constraints biases model estimates and b) show that our model can recover the underlying parameters, we conduct a simulation using the approach described in Appendix III. We first simulate data for single apartment complex owners to illustrate the role of spatial covariance and capacity constraints and then proceed to simulate chain data to explore the effect of chain ownership.
6.1.1 Single Owner Simulation

Using these simulated data we estimate three models. In model 1, we include spatial covariance and capacity constraints. In model 2, we omit spatial covariance. In model 3, we omit capacity constraints. The Gibbs sampling chain proceed for 10000 iterations in each model though they appear to converge well before then (within 500 iterations). We discarded the first 2000 draws to ensure convergence. Priors are set to be as non-informative as possible. We use all available attributes and prices from preceding periods as instruments. Below, we report the parameter estimates and the ability of three models to recover the parameters.

[Insert Table 2 Here]

As indicated in Table 2a, the 95% posterior predictive interval for model 1 contains the true parameter values for all of the parameters in our properly specified model indicating that the model can recover the data generating mechanism. Contrasting the full model to one wherein we omit spatial correlation (model 2), we observe the predicted downward bias in median price effects. The magnitude of this effect in this parameter is 24%. We also observe a downward bias of about 23% in the estimate for marginal cost, $\beta_{30}$. The cause of this downward bias in estimated marginal costs can be seen by inspecting Equation (15). The downward bias in the estimate for price elasticity $\gamma$ lowers the expression on the left hand side of Equation (15). To maintain this equality, estimates for the right hand side must also be lower leading to a downward bias in the estimates for marginal cost. We also note that there is a decrease in the log marginal likelihood from $-1227.8$ to $-1258.7$. Contrasting the full model to one wherein we omit capacity constraints, we again observe the predicted downward bias in price elasticities though the effect is negligible, 2.4%. We conjecture this arises because only 5% of apartments in our simulation are at capacity (a number chosen to reflect our data). In addition, we note that the variance of the pricing error, $\sigma_\zeta$, is overestimated by a factor of 2, perhaps as a result of the additional error introduced by ignoring the capacity constraints. The large decrease in the log marginal likelihood, to $-1822.3$ is due in part to the increased number of observations in the likelihood (we include apartments at capacity on the demand-side).
6.1.2 Apartment Chain Simulation

We repeat the forgoing simulation in the context of an ownership structure wherein a firm owns multiple apartments and the simulation structure is detailed in Appendix III. We consider two such simulations. In the first one, we account properly for the chain ownership simulation and add instruments for chain-level attributes. The results of this simulation are presented in Table 2b. All the parameters lie in the 95% posterior predictive interval except $\phi_\zeta$. Of particular interest is the recovery of the estimate for the highest Kuhn-Tucker multiplier. This parameter captures the marginal cost of the constraint to the firm; in this case chain 39 in year 8 could gain $170.05 in profit if it could add one apartment. In the second simulation we ignore the ownership structure and assume all apartments are independent. When the chain structure is ignored, we observe a 10% downward bias in price elasticities, and the 95% posterior predictive interval for the price coefficient no longer includes the true value. This downward bias arises because competition is softer in the context of chain ownership, leading to higher prices for the same level of demand. Higher prices coupled with no change in demand implies a lower price elasticity. We note that the model fit for the chain model is greater when the ownership structure is properly specified.

In sum, the simulation data evidence a) that our model recovers parameters well and b) biases in parameter estimates that arise from ignoring spatial covariance and capacity constraints are in the predicted direction.

6.2 Roanoke Results

We next apply the data detailed in Section 5 to estimate our model of outlet demand and pricing. We ran the sampling chain for 10000 iterations. Inspection of the sequence of draws indicates good convergence after about 500 iterations though we discarded the first 2000 iterations. Moreover, little autocorrelation in the draws was evident. In addition to estimating the full model, we provide two benchmarks; no spatial correlation and no capacity constraints. Comparison of these models to the full model yields insights into i) the magnitude of parameter biases that can arise when capacity and spatial covariance in demand and prices are ignored and ii) the degree to which spatial covariance improve model performance.
6.2.1 Parameter Estimates

Table 3 presents the results of our estimation. The full model is the best fitting model and its improvement over the model with no spatial covariance is sizable (the improvement in the log marginal likelihood is 15.8). This affords evidence that spatial covariance matters in practice.

[Insert Table 3 About Here]

Demand-side Estimates. Most attributes play a significant role in apartment choice. Of these, gym and tennis play the greatest role in apartment choice. Inclusion of a gym in a complex adds nearly twice the utility of a pool or free heat. Increasing distances from schools yields lower utility though the 95% posterior predictive interval lies just outside of zero.7 The effective range of the median spatial decay, wherein 95% of the spatial effect has decayed is given by $3/\phi$ (Banerjee et al. 2004), or 7.3 miles. This suggests that the demand-side spatial covariance is sizable, as the maximum distance between apartments is 9.6 miles. The price parameter is positive, indicating that an increase in price lowers the likelihood of apartment choice (as this enters our likelihood function with a negative sign). Consistent with our previous findings, price elasticity is biased toward zero when one ignores spatial covariance and capacity constraints.

Supply-side Estimates. According to a 1998 survey of apartment managers conducted by the Institute of Real Estate Management (IREM), annual operating expenses for apartments average 42% of revenue and these costs include administrative expenses, operating expenses, maintenance expenses, tax/insurance, and payroll and amenities. For the average rent of $491 in our data, this implies our estimates for variable costs should be in the neighborhood of $206 per apartment. Given a mean rent of $491 and an average apartment age of 25 years, the estimates in Table 3 imply an average marginal cost of an apartment is $210 (153+14.3*100/Age) close to the $491 implied by the IREM survey. Viewed another way, estimated variable costs are about 43% of revenue compared to the 42% reported by the IREM. Table 3 further indicates that heat increases the variable costs of the apartment and that newer apartments have higher operating costs, perhaps due to greater amenities.

The highest median Kuhn-Tucker multiplier is $193 for apartment 45 in year 1. This apartment happened to be one of the two highest rent apartments at capacity, which is perhaps why the capac-

7When estimating the model we noted that the correlation between distance to schools and distance to employers was high so we omitted the latter as it had less explanatory power.
ity constraint is so costly. As expected, cost estimates are biased downward when spatial covariance is ignored and the cost error variance, \( \sigma_\zeta \), increases when capacity constraints are ignored. The increased cost variance might play a role in the finding that the 95% posterior prediction interval for costs, \( \beta_{30} \), does not exclude zero.

The spatial effects on the supply-side (prices), with an effective range of 1.3 miles, are smaller than those on the demand-side. We conjecture that costs are a more "global" variable in the sense that all apartments likely face a similar cost of capital, utilities, labor and supplies. On the demand-side, however, certain regions are more desirable than others leading to higher spatial correlation.

### 6.2.2 Latent Spatial Effects

Using the median draw for the spatial random effects, we create a contour plots of spatial random demand \( \theta_{s_j} \) and cost effects \( \zeta_{tj} \) and offer a discussion of these results. Figure 4 plots the \( \theta_{s_j} \) and then interpolates latent demand between these observations via triangle based cubic interpolation.8

```
[Insert Figure 4 Here]
```

The Figure indicates two modes of high latent demand; north of the downtown and southwest of downtown. It is interesting to compare Figure 4 which computes the latent spatial demand effects with Figures (2) and (3) which depict pricing and vacancy. The highest demand region southwest of town not only has high latent demand but also higher vacancy rates because it has more apartments. In contrast, the region north of town not only evidences high latent demand, but also low vacancy rates. The region southwest of town supports the highest rents while the western portion of the high latent demand region north of town also has high (albeit smaller) rents. Given the western portion of the region north of town has higher rents, low vacancies and high latent demand, this suggests it might be an especially desirable location for new apartments and we explore this line of inquiry next.

---

8 A more precise approach is to krig demand in more locations to obtain greater coverage of latent spatial effects prior to interpolating demand between these points (we pursue kriging in Section 7). However, the Figure provides a good approximation of latent demand without the need for kriging.
7 Managerial Implications

One can use our approach to engage policy experiments pertaining to the selection of a new outlet location as it structurally links prices to outlet entry. For example, a firm could compare across available properties the effect of locating an additional outlet on demand, prices and profits at a) the additional outlet and b) other outlets in the chain. In our analysis we compare the desirability of these entry options subject to the caveat that no other outlets enter. However, even in the context of competitive response, one could simulate the effect of various competitive location responses to the firm’s choices of next outlet location. This suggests that many scenarios could be played to simulate entry effects.

One might conjecture that firms already occupy the optimal locations, so that the policy simulation is of little value. We think this is unlikely primarily because the ebb and flow of persons into the market and changes in the attributes of various locales (e.g., new roads) over the years likely renders the extant distribution of existing outlets sub-optimal. It is therefore likely to be useful to conjecture what the next best location may be (e.g., can we verify our conjecture in Section 6.2.2 that the best region is north of town).

7.1 Locating Apartments in Roanoke

Using the procedure discussed in Section 4.2, we assess the effect of an additional apartment on equilibrium demand, prices and profits. We begin by creating a 10 by 10 grid of potential apartment entry locations within the convex hull defined by extant apartment locales in Roanoke. For each location on the 100 point grid we compute equilibrium profits associated with an entry at that location. This computation assumes a modal apartment design for the new complex, that is, using the modal feature set in the data. We execute this procedure twice; once for our full model and once for a model that omits spatial covariance and capacity constraints. This enables us to contrast the recommendation of each model. The pricing contraction mapping algorithm described in the Appendix II appears to converge to a unique equilibrium over a range of starting values (this was also the case for simulated data). Figure 5 depicts the predicted equilibrium profits at these locations.

[Insert Figure 5 Here]
The top half of Figure 5 portrays the equilibrium profits for the various entry options using the full model. The areas of the solid circles correspond to predicted profits, with a maximum of $35,010, a median of $12,921, and a minimum of $3,194. The bottom half of Figure 5 depicts predicted profits for the various entry options using the model with no spatial effects and no capacity constraints. The predicted profits from this model range from $14,719 down to $13,655. The median predicted profit is $14,188 which is higher than the median predicted in the full model. Most strikingly, the constrained model evidences a lack of variation in predicted profits as it ignores unobserved information regarding latent demand and costs from extant apartments. Instead, it uses only observed spatial effects such as distance to the nearest school and these observed effects appear to be quite small relative to the unobserved spatial variation. The standard deviation in predicted profits is over 25 times larger for the spatial covariance model. Using this model, one would inadvertently conclude some of the lower profit locations would yield a good profit. Further, the Figure indicates several "false modes," wherein the full model predicts small profits.

The open circles in the top and bottom half of Figure 5 enclose the locales with the highest predicted profit in each of the respective models. For both the full model and the model without spatial random effects, the highest predicted profit is north of town; an area with high latent demand, low vacancies and moderate to high rents. The prediction points are in fact contiguous. However, were one to adopt the recommendation of the model without capacity constraints or spatial covariance, expected profits would be $21,153 ($21,153 represents the predicted profits from the full model using the recommended locale from the constrained model). Using the recommendation from the spatial model increases expected profits to $35,010 or 66%. When one considers that these estimates are cash flows and that firms expand into many different cities, the profit implications of our model could be considerable. It is also worth noting that the region southwest of downtown also has a smaller mode of higher profits and that this is not observable with the constrained model data.

8 Conclusions

In this manuscript we address the problem of the firm’s outlet location problem. A necessary step in this process is to solve the sub-game problem of firm pricing conditioned on firm locations. Firms can then use this model to simulate the effect of locating an incremental outlet on equilibrium
prices, demand and profits. Our model applies these concepts in the context of apartment data and affords recommendations about the next best location for entry.

Our work extends prior research in a number of respects. Unlike gravity models, we explicitly and structurally consider competitive response in prices. Unlike analytical models, our approach is data-focused to enable insights into data-driven decisions in a number of different marketing contexts. We extend structural model of the subgame of outlet location and prices to include, among other things, spatial covariation and capacity constraints. It is desirable to consider spatial correlation for unobserved random demand and costs effects for several reasons. First we show that when ignored, it can lead to price endogeneity, biasing the estimates of price effects and marginal costs downward. Second, these effects are interesting in their own right, leading to a managerially informative map of latent demand. Neglecting these effects obscures considerable spatial demand variation inherent in our data. It is also desirable to consider capacity constraints. A proper accounting of these effects mitigates a downward bias in price elasticity estimates, and the Kuhn-Tucker multipliers provide an estimate of the cost of the capacity constraint. Presumably, firms with higher costs would be more inclined to consider additional capacity. In addition, failure to consider capacity constraints and spatial effects leads to suboptimal recommendations for firm location.

To achieve these aims we integrate Bayesian spatial statistics with structural models of competition in a model of outlet demand and pricing. Simulations indicate that the resulting model recovers parameters well and that ignoring spatial effects and capacity constraints leads to biased parameter estimates. We then apply this model to a novel apartment data set that includes prices, demand and capacities over a 6 year period. We find that the inclusion of spatial covariance in demand improves model fit and that omitting spatial effects and capacity constraints lead to biased parameter estimates as predicted.

Given we seek to assess the policy implications of outlet location, we use a spatial kriging approach coupled with our supply-side model to explore the impact of locating an additional outlet at potential sites of interest to a firm (or more systematically upon a grid). These simulations reveal a) the best entry locations relate to high latent spatial demand and a dearth of nearby outlets, b) accommodating capacity and unobserved spatial effects in policy simulations improves the profitability of recommended entry locales by 66% in our data, and c) ignoring spatial covariance
leads to little variation in predicted profits, understating the actual standard deviation in predicted profits across space by a factor of 25.

A number of limitations exist that also represent opportunities for future research.

- We consider the pricing sub-game only and do not model the outlet location game. We do this because the sub-game is an important problem in its own right and is useful for policy simulations over the intermediate term. An important extension would be to consider the entry problem as well. We think this would be a difficult extension as it involves a dynamic program to solve for sequential entry and prices and it is likely that solutions to this problem would not be unique. Moreover, the distribution of latent demand can render such equilibria obsolete after a few years.

- Many outlets sell multiple goods or have products that appeal to different markets (e.g. supermarkets or medical centers). It would be desirable to ascertain whether competition across goods leads to different outcomes for location and pricing than in the indexed single good case modelled herein.

- We do not consider unobserved consumer heterogeneity as, in practice, such effects are hard to identify with aggregate data available (Bodapati and Gupta 2004). However, our inclusion of spatial random effects leads to a demand system that allows for complex spatial substitution patterns and avoids the IIA assumption in spatial choice, which is sufficient for our objectives. Though this is not our immediate focus, an analysis which adds consumer heterogeneity could enrich insights into the nature of competition across outlets in some contexts.

- We presume attributes such as pool and tennis are exogenous. It would be desirable to model not only the location of an outlet, but also its optimal design. We believe that this exercise in combinatorial optimization would prove challenging, especially with respect to documenting the uniqueness of these equilibria.

- Our model is of limited applicability when alternative channels such as the Internet comprise a significant portion of demand. Multi-channel models (Ansari, Mela and Neslin 2005) could be overlaid with the spatial models to develop unique insights into this setting.
In sum, the approach we develop incorporates spatial effects into a model of outlet competition and affords insights into the role of unobserved spatial effects on outlet pricing and demand. We hope that our research will spark some subsequent research related to these and other remaining issues.
Appendix I: Sampler in Section 3.3

Step 1. Sample \((\beta_1, \beta_2)\). Let \(X_{tA} = [X_t, X_{ts}]\) and \(\beta_A = (\beta_1, \beta_2)\). Suppose \(\beta_A\) is distributed \(N(\hat{\beta}_A, \hat{\Sigma}_A)\) a priori. The likelihood is

\[
\exp\left(-\frac{1}{2\sigma_\theta} \sum_{t=1}^{T} \left(\xi_t(\gamma) - X_{tA}\beta_A\right)^T Z_t (Z_t^T R_J (\phi) Z_t)^{-1} Z_t^T \left(\xi_t(\gamma) - X_{tA}\beta_A\right)\right).
\]

The prior is conjugate to the likelihood. The full conditional for \((\beta_1, \beta_2)\) is \(N(\hat{\beta}_A, \hat{\Sigma}_A)\), where

\[
\hat{\Sigma}_A = \left(\frac{1}{\sigma_\theta^2} \sum_{t=1}^{T} X_{tA}^T Z_t (Z_t^T R_J (\phi) Z_t)^{-1} Z_t^T X_{tA} + \hat{\Sigma}_A^{-1}\right)^{-1},
\]

\[
\hat{\beta}_A = \hat{\Sigma}_A \left(\frac{1}{\sigma_\theta^2} \sum_{t=1}^{T} X_{tA}^T Z_t (Z_t^T R_J (\phi) Z_t)^{-1} Z_t^T \xi_t(\gamma) + \hat{\Sigma}_A^{-1}\hat{\beta}_A\right).
\]

Step 2. Sample \(\beta_3\). Suppose \(\beta_3\) is distributed \(N(\hat{\beta}_3, \hat{\Sigma}_3)\) a priori. The likelihood is

\[
\exp\left(-\frac{1}{2\sigma_\zeta^2} \sum_{t=1}^{T} (P_{1t} - B_{1t}(\lambda_t, \gamma) - Y_{1t}\beta_3)^T Z_{1t} (Z_{1t}^T R_{J_{1t}} (\psi) Z_{1t})^{-1} Z_{1t}^T (P_{1t} - B_{1t}(\lambda_t, \gamma) - Y_{1t}\beta_3)\right).
\]

The prior is conjugate to the likelihood. The full conditional for \(\beta_3\) is \(N(\hat{\beta}_3, \hat{\Sigma}_3)\), where

\[
\hat{\Sigma}_3 = \left(\frac{1}{\sigma_\zeta^2} \sum_{t=1}^{T} Y_{1t}^T Z_{1t} (Z_{1t}^T R_{J_{1t}} (\psi) Z_{1t})^{-1} Z_{1t}^T Y_{1t} + \hat{\Sigma}_3^{-1}\right)^{-1},
\]

\[
\hat{\beta}_A = \hat{\Sigma}_3 \left(\frac{1}{\sigma_\zeta^2} \sum_{t=1}^{T} Y_{1t}^T Z_{1t} (Z_{1t}^T R_{J_{1t}} (\psi) Z_{1t})^{-1} Z_{1t}^T (P_{1t} - B_{1t}(\lambda_t, \gamma)) + \hat{\Sigma}_3^{-1}\hat{\beta}_3\right).
\]

Step 3. Sample \(\sigma_\theta^2\). Let the prior be \(\sigma_\theta^2 \sim IG(\alpha_\theta, \beta_\theta)\). Let \(L\) be the number of variables in \(Z_t\). Then,

\[
[\sigma_\theta^2|\beta_1, \beta_2, \phi, X_t, X_{ts}, \xi_t(\gamma), Z_t, t = 1, \ldots, T] \sim IGamma(\alpha_\theta, \beta_\theta),
\]

29
where $\tilde{\alpha}_\theta = \alpha_\theta + \frac{T L}{2}$, and
\[
\tilde{\beta}_\theta = \beta_\theta + \frac{1}{2} \sum_{t=1}^{T} \left( \tilde{\xi}_t (\gamma) - X_t \alpha_\beta \right)^T Z_t \left( Z_t^T R_J (\phi) Z_t \right)^{-1} Z_t^T \left( \tilde{\xi}_t (\gamma) - X_t \alpha_\beta \right).
\]

**Step 4. Sample $\phi$.** Use a discrete sampler. Assume $\phi$ can only take $n$ values: $(\phi_1, \ldots, \phi_n)$. For each $\phi_t$, calculate the posterior probability
\[
|Z_t^T R_J (\phi_t) Z_t|^{-\frac{1}{2}} \exp \left( -\frac{1}{2\sigma_\theta^2} \sum_{t=1}^{T} \left( \tilde{\xi}_t - X_t \alpha_\beta \right)^T Z_t \left( Z_t^T R_J (\phi_t) Z_t \right)^{-1} Z_t^T \left( \tilde{\xi}_t - X_t \alpha_\beta \right) \right).
\]
Sample $\phi$ with replacement from $(\phi_1, \ldots, \phi_n)$.

**Step 5. Sample $\sigma_\zeta^2$.** Let the prior be $\sigma_\zeta^2 \sim IG(\alpha_\zeta, \beta_\zeta)$. Then,
\[
[\sigma_\zeta^2, \beta_\zeta, \psi, Y_t, P_t, B_t (\gamma), Z_t, t = 1, \ldots, T] \sim IGamma(\tilde{\alpha}_\zeta, \tilde{\beta}_\zeta),
\]
where $\tilde{\alpha}_\zeta = \alpha_\zeta + \frac{T L}{2}$, and
\[
\tilde{\beta}_\zeta = \beta_\zeta + \frac{1}{2} \sum_{t=1}^{T} \left( P_{it} - B_{it} (\lambda_t, \gamma) - Y_{it} \beta_3 \right)^T Z_{it} \left( Z_{it}^T R_{Jit} (\psi) Z_{it} \right)^{-1} Z_{it}^T \left( P_{it} - B_{it} (\lambda_t, \gamma) - Y_{it} \beta_3 \right).
\]

**Step 6. Sample $\psi$.** Use a discrete sampler. Assume $\psi$ can only take $n$ values: $(\psi_1, \ldots, \psi_n)$. For each $\psi_t$, calculate the posterior probability
\[
|Z_{it}^T R_{Jit} (\psi_t) Z_{it}|^{-\frac{1}{2}} \exp \left( -\frac{1}{2\sigma_\zeta^2} \sum_{t=1}^{T} \left( P_{it} - B_{it} (\lambda_t, \gamma) - Y_{it} \beta_3 \right)^T Z_{it} \left( Z_{it}^T R_{Jit} (\psi_t) Z_{it} \right)^{-1} Z_{it}^T \left( P_{it} - B_{it} (\lambda_t, \gamma) - Y_{it} \beta_3 \right) \right).
\]
Sample $\psi$ with replacement from $(\psi_1, \ldots, \psi_n)$.

**Step 7. Sample $\gamma$: Metropolis-Hastings Algorithm.** Suppose $\gamma$ has a log-normal prior (as we expect demand to fall with price our prior for the sign of $\gamma$ is positive). Use a random walk chain to generate the $n$-th draw of $\gamma^{(n)}$. After recalculating $\tilde{\xi}_t (\gamma^{(n)})$ and $B_t (\gamma^{(n)})$, the acceptance probability is given as:
\[
\min \left\{ \prod_{t=1}^{T} N(Z_{it}^T [P_{it} - B_{it}(\lambda_t,\gamma^{(n)})])Z_{it}^2 Y_{it} \beta_3 \sigma_\zeta^2 Z_{it}^T R_{Jit}(\psi) Z_{it}) N(Z_{it}^T \tilde{\xi}_t (\gamma^{(n)}) | Z_{it}^T (X_t \beta_1 + X_{it} \beta_2) \sigma_\zeta^2 Z_{it}^T R_J(\phi) Z_{it}) \pi(\gamma^{(n)}) \right. \\
\left. \prod_{t=1}^{T} N(Z_{it}^T [P_{it} - B_{it}(\lambda_t,\gamma^{(n-1)})])Z_{it}^2 Y_{it} \beta_3 \sigma_\zeta^2 Z_{it}^T R_{Jit}(\psi) Z_{it}) N(Z_{it}^T \tilde{\xi}_t (\gamma^{(n-1)}) | Z_{it}^T (X_t \beta_1 + X_{it} \beta_2) \sigma_\zeta^2 Z_{it}^T R_J(\phi) Z_{it}) \pi(\gamma^{(n-1)}) , 1 \right\}.
\]
Appendix II: Contraction Mapping

To establish that the equilibrium prices are unique for the simulation we would like to show that prices,

\[ P_{tj}^{(m+1)} (\gamma) = P_{tj}^{(m)} (\gamma) + \ln c_t - \ln \left( P_{tj}^{(m)} - \frac{1}{1 - W_{tj} (P_{tj}^{(m)})} \right) \]  
(A1)

\[ P_{tj}^{(m+1)} (\gamma) = P_{tj}^{(m)} (\gamma) + \ln \frac{K_j}{T} - \ln W_{tj} \left( P_{tj}^{(m)} \right) \]  
(A2)

are contraction mappings within a closed subset of the Euclidian space and under certain conditions of prices and market shares, this closed subset is reasonable in reality. Then by the contraction mapping theorem, there is a unique solution of prices in this closed subset (please refer to Berry et al. (1995) for the proof).

Let \( f_j (p_t) = P_{tj} + \ln c_{tj} - \ln \left( P_{tj} - \frac{1}{1 - W_{tj}(P_t)} \right) \), then the derivative of \( f_j (p_t) \) with respect to \( P_{tk} \) is

\[
1 - \frac{1}{P_{tj} - \frac{1}{T_{tj}(P_t)}} \frac{1}{1 - W_{tj}(P_t)}; \quad \text{if } k = j
\]

\[
\frac{1}{P_{tj} - \frac{1}{T_{tj}(P_t)}} \frac{W_{tj}(P_t) W_{tk}(P_t)}{(1 - W_{tj}(P_t))^2}; \quad \text{if } k \neq j
\]

It is clear that both terms are greater than zero. In order that we have the contraction property for the right side of (A1), we need

\[
\sum_{k=1}^{J} \frac{\partial f_j (p_t)}{\partial p_{tk}} = 1 - \frac{1}{P_{tj} - \frac{1}{T_{tj}(P_t)}} \left[ \frac{1}{1 - W_{tj}(P_t)} - \sum_{k \neq j} W_{tj}(P_t) W_{tk}(P_t) \frac{1}{(1 - W_{tj}(P_t))^2} \right]
\]

\[
= 1 - \frac{1}{P_{tj} - \frac{1}{T_{tj}(P_t)}} < 1
\]

Note \( P_{tj} - \frac{1}{T_{tj}(P_t)} = c_{tj} \) if there is a solution for \( P_{tj} \). As long as \( P_{tj} - \frac{1}{T_{tj}(P_t)} > 0 \), the conditions for contraction mapping is satisfied. We have \( J \) such conditions for all \( P_{tj} \). These conditions define a subset of \( R^J \) space. Within the closure of this subset, we have a contraction mapping and there is a unique solution for \( P_t = (P_{tj}, j = 1, \ldots, J) \). In most business situations
where the market share $W_{tj}$ is not near one and $P_{tj}$ is large, the contraction mapping conditions are satisfied.

Equation (A2) is also a contraction mapping as shown in the BLP paper (Berry et. al. 2005). Thus, both the (A1) and (A2) are contraction mappings. However, on each iteration of price, prices might switch from the former contraction mapping to the latter or vice-versa. Though (A1) and (A2) each represents a contraction mapping, the alternating system of equations may not and we are unable to prove that the equilibrium pricing solution is unique. However, in our simulations and in our data application, prices converge to an equilibrium and the equilibrium appears to be robust across starting values.

**Appendix III: Simulation Design**

To approximate the size and structure of the data we randomly distribute 40 apartment locations in a rectangular area shown in Figure 6 for the single owner case and Figure 7 for the chain owner case. For purposes of parsimony, the single complex-specific attribute considered is the presence of a garage. The single simulated location attribute is the distance to the town center, marked by a diamond in Figure 6. The utility function is then

$$V_{tij} = X_{tj} \beta_1 + X_{tsj} \beta_2 - P_{tj} \gamma + \theta_{tsj} + \epsilon_{tij}$$  \hspace{1cm} (A3)

where $X_{tj}$ is a 0-1 indicator variable that represents the presence of the garage and $X_{tsj}$ is the distance to the town center. $X_{tj}$ and $X_{tsj}$ do not vary over time. The diagonal distance across this rectangular region is 30 units. We set $\beta_1 = 1.0$, $\beta_2 = -0.3$, and $\gamma = 0.01$. The spatial random effect $\theta_{tsj}$ is distributed $N(0, \sigma^2 \theta R_J(\phi))$. Using the exponential covariance function for the Gaussian process, we set $\sigma_\theta = 0.4$ and $\phi_\theta = 0.4$. We simulate data for 10 independent periods with 10 independently sampled $\theta_t$’s.

[Insert Figures 6 and 7 Here]

The prices for the 40 apartments and 10 periods are determined by the full model. The cost for each apartment is simulated to be

$$c_{tj} = \beta_{30} + Y_{tj} \beta_{31} + \zeta_{tj}$$  \hspace{1cm} (A4)
where \( Y_{tj} \) is a scalar representing the size of the apartment complex’s rooms (we presume larger rooms cost more to maintain). We specify \( Y_{tj} \) to assume three values: \((1, 2, 3)\). We set \( \beta_{30} = 150 \) and \( \beta_{31} = 30 \). \( \zeta_{tj} \) is distributed \( N \left( 0, \sigma^2_{\zeta} R_j (\varphi) \right) \), where \( \sigma_{\zeta} = 10, \varphi = 0.4 \). The capacity constraint \( K_j \) for all the apartment is set to 200.

The price \( P_{tj} \) and demand \( Q_{tj} \) are generated by our demand-supply model. An iterative procedure

\[
P_t^{(m+1)} (\gamma) = P_t^{(m)} (\gamma) + \ln c_t - \ln \left( P_t^{(m)} - \frac{1/\gamma}{1 - W_t \left( P_t^{(m)} \right)} \right)
\]

(A5)

generates all the \( P_{tj} \)'s in each period. We then impute demand, \( Q_{tj} (P_t) \). If \( Q_{tj} \) exceeds \( K_j \), the price for the apartment, (A5), becomes

\[
P_t^{(m+1)} (\gamma) = P_t^{(m)} (\gamma) + \ln \frac{K_j}{I} - \ln W_t \left( P_t^{(m)} \right)
\]

(A6)

All the prices \( P_{tj} \)'s are generated again and \( Q_{tj} \)'s are computed. If any \( Q_{tj} \) exceed \( K_j \), we replace the mapping in (A5) by (A6), and iterate demand and prices again. We continue in this fashion until all the equilibrium prices and quantities satisfy the capacity constraints. Using these contraction mappings, prices in our simulation converge to a equilibrium that is unique over a wide range of starting values. Note that Equations (A5) and (A6) are generalized to accommodate the chain ownership structure in the chain simulation.
References


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<tr>
<th>Attribute</th>
<th>Observations</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Max</th>
<th>Min</th>
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<td>491</td>
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<td>46%</td>
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<td>Year Built</td>
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<td>1976</td>
<td>11</td>
<td>1952</td>
<td>2002</td>
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<tr>
<td>Average Distance to Major Employers (Miles)</td>
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<td>Distance to Nearest School (Miles)</td>
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<td>Maximum Distance Between Apartments (Miles)</td>
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Table 2a - Single Ownership Simulation Result

<table>
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<tr>
<th>Parameter Variable</th>
<th>Demand-side Simulation Value</th>
<th>Full Model Posterior Median (95% Posterior Prediction Interval)</th>
<th>No Spatial Correlation Value</th>
<th>No Capacity Constraints Value</th>
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<tbody>
<tr>
<td>β₁</td>
<td>Garage</td>
<td>1.00</td>
<td>0.99 (0.93,1.05)</td>
<td>0.94 (0.85,1.03)</td>
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<tr>
<td>β₂</td>
<td>Distance to City</td>
<td>-0.30</td>
<td>-0.30 (-0.33,-0.27)</td>
<td>-0.30 (-0.32,-0.27)</td>
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<td>γ</td>
<td>Price</td>
<td>0.01</td>
<td>0.0097 (0.0085,0.0113)</td>
<td>0.0074 (0.0068,0.0082)</td>
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<td>σθ</td>
<td>Std. Dev. θ</td>
<td>0.40</td>
<td>0.39 (0.35,0.45)</td>
<td>0.45 (0.40,0.52)</td>
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<tr>
<td>φθ</td>
<td>Spatial Decay</td>
<td>0.40</td>
<td>0.58 (0.32,0.90)</td>
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Supply-side

<table>
<thead>
<tr>
<th>Parameter Variable</th>
<th>Demand-side Simulation Value</th>
<th>Full Model Posterior Median (95% Posterior Prediction Interval)</th>
<th>No Ownership Structure</th>
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<tbody>
<tr>
<td>β₃₀</td>
<td>Variable Cost</td>
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<td>145 (130,160)</td>
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<tr>
<td>β₃₁</td>
<td>Apartment Size</td>
<td>30.3 (29.1,31.4)</td>
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<tr>
<td>σₜ</td>
<td>Std. Dev. ζ</td>
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<td>φₜ</td>
<td>Spatial Decay</td>
<td>0.40</td>
<td>0.53 (0.32,0.79)</td>
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Log Marginal Likelihood

-1227.8

= -1258.7

-1822.3

Table 2b - Multiple Complex Ownership Simulation Results

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<th>No Ownership Structure</th>
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<td>Garage</td>
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<td>0.98 (0.91,1.04)</td>
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<td>Distance to City</td>
<td>-0.30</td>
<td>-0.30 (-0.33,-0.26)</td>
</tr>
<tr>
<td>γ</td>
<td>Price</td>
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<td>0.0097 (0.0089,0.0105)</td>
</tr>
<tr>
<td>σθ</td>
<td>Std. Dev. θ</td>
<td>0.40</td>
<td>0.45 (0.39,0.54)</td>
</tr>
<tr>
<td>φθ</td>
<td>Spatial Decay</td>
<td>0.40</td>
<td>0.34 (0.21,0.54)</td>
</tr>
</tbody>
</table>

Supply-side

<table>
<thead>
<tr>
<th>Parameter Variable</th>
<th>Supply-side Simulation Value</th>
<th>Full Model Posterior Median (95% Prediction Interval)</th>
<th>No Ownership Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>β₃₀</td>
<td>Variable Cost</td>
<td>150</td>
<td>144 (134,154)</td>
</tr>
<tr>
<td>β₃₁</td>
<td>Apartment Size</td>
<td>30.0 (28.8,31.3)</td>
<td>30.5 (28.8,30.4)</td>
</tr>
<tr>
<td>σₜ</td>
<td>Std. Dev. ζ</td>
<td>9.28 (8.29,10.50)</td>
<td>10.93 (9.23,16.67)</td>
</tr>
<tr>
<td>φₜ</td>
<td>Spatial Decay</td>
<td>0.40</td>
<td>0.67 (0.47,0.93)</td>
</tr>
<tr>
<td>λ⁺</td>
<td>Kuhn-Tucker</td>
<td>170</td>
<td>154 (136,171)</td>
</tr>
</tbody>
</table>

Log Marginal Likelihood

-1424.4

-1272.4

¹The full and no ownership log marginal likelihood are not comparable as the full model uses some chain-level instruments. Using only single-ownership instruments in the Full Model yields similar estimates as using the chain instruments in the full model and a log-marginal likelihood of −1236.2, suggesting that capturing the chain structure improves model fit.
Table 3 - Roanoke Data Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Full Model</th>
<th>No Spatial Correlation</th>
<th>No Capacity Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand-side</td>
<td></td>
<td>Posterior Median (95% Posterior Prediction Interval)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>Clubhouse</td>
<td>0.66 (0.27, 1.02)</td>
<td>0.13 (-0.42,0.65)</td>
<td>0.63 (0.21,0.98)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>Tennis</td>
<td>0.82 (0.67,0.98)</td>
<td>0.68 (0.48,0.88)</td>
<td>0.80 (0.64,0.95)</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>Pool</td>
<td>0.56 (0.36,0.80)</td>
<td>0.83 (0.54,1.11)</td>
<td>0.54 (0.33,0.77)</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>Gym</td>
<td>1.05 (0.79,1.32)</td>
<td>1.00 (0.61,1.41)</td>
<td>1.04 (0.78,1.31)</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>Heat</td>
<td>0.50 (0.40,0.60)</td>
<td>0.41 (0.24,0.57)</td>
<td>0.50 (0.39,0.60)</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>Distance to School</td>
<td>-0.06 (-0.14,0.03)</td>
<td>-0.01 (-0.09,0.07)</td>
<td>-0.05 (-0.14,0.03)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Price</td>
<td>0.0038 (0.0033,0.0052)</td>
<td>0.0031 (0.0027,0.0036)</td>
<td>0.0037 (0.0028,0.0046)</td>
</tr>
<tr>
<td>$\sigma_\theta$</td>
<td>Std. Dev. $\theta$</td>
<td>0.32 (0.26,0.42)</td>
<td>0.36 (0.32,0.42)</td>
<td>0.31 (0.26,0.42)</td>
</tr>
<tr>
<td>$\phi_\theta$</td>
<td>Spatial Decay</td>
<td>0.41 (0.21,1.21)</td>
<td>—</td>
<td>0.61 (0.21,1.21)</td>
</tr>
<tr>
<td>Supply-side</td>
<td>Varies</td>
<td>Posterior Median (95% Posterior Prediction Interval)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{30}$</td>
<td>Variable Cost</td>
<td>153 (96,236)</td>
<td>86 (29,148)</td>
<td>92 (-9,175)</td>
</tr>
<tr>
<td>$\beta_{31}$</td>
<td>Heat</td>
<td>58.5 (27.0,90.1)</td>
<td>50.8 (14.9,86.3)</td>
<td>29.4 (-7.3,67.8)</td>
</tr>
<tr>
<td>$\beta_{31}$</td>
<td>100/Age</td>
<td>14.3 (7.3,21.1)</td>
<td>15.2 (7.7,22.3)</td>
<td>24.8 (15.4,33.9)</td>
</tr>
<tr>
<td>$\sigma_\zeta$</td>
<td>Std. Dev. $\zeta$</td>
<td>80.9 (70.8,93.7)</td>
<td>82.9 (72.1,95.9)</td>
<td>97.0 (84.4,112.7)</td>
</tr>
<tr>
<td>$\phi_\zeta$</td>
<td>Spatial Decay</td>
<td>2.4 (1.41,3.00)</td>
<td>—</td>
<td>2.2 (1.21,3.00)</td>
</tr>
<tr>
<td>$\lambda_{\text{max}}$</td>
<td>Kuhn-Tucker</td>
<td>193 (34,353)</td>
<td>198 (35,368)</td>
<td>—</td>
</tr>
<tr>
<td>Log Marginal Likelihood</td>
<td></td>
<td>-895.5</td>
<td>-911.3</td>
<td>-962.2</td>
</tr>
</tbody>
</table>
Figure 1: 2005 Location and Occupancy of Apartments in Roanoke, Virginia
Figure 2: Distribution of 2005 Rental Prices ($) in Roanoke, Virginia
Figure 3: Distribution of 2005 Vacancy Rates (%) in Roanoke, Virginia
Figure 4: Latent Spatial Demand for Apartments in Roanoke, Virginia
Figure 5: Kriged Profits for Potential Apartment Locations in Roanoke Virginia
Figure 6: Apartment Locations In Single Owner Simulation
Figure 7: Chain locations (apartments from the same chain are indicated by the same symbol).