Advertising and price targeting on the Hotelling line

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Substantially, but not fully, complete.
Abstract

We study two types of targeting in this paper, exogenous targeting and endogenous targeting. By exogenous targeting we mean targeting by product preferences: advertising levels and prices geared to specific consumer preferences. Exogenous targeting presumes observability of consumer preferences (or of factors correlated with them). This is the sort of targeting that is much in the news in contemporary media, and is also the subject of a recent academic literature, e.g., Chen and Iyer (2002), Iyer et al. (2005). By endogenous targeting we mean the subtler, covert targeting that happens naturally, when firms choose different levels of awareness advertising, and optionally follow that up with randomized prices (McAfee 1994, Fershtman 1993, Boyer and Moreaux 1999). This kind of targeting can happen even when firms are not exogenously targeting anyone.

In this paper we examine how these two types of targeting play out along the Hotelling line. We show that exogenous targeting of consumers with a stronger preference for your brand generally entails advertising more and pricing higher; endogenously, this enables the stronger brand to target its “loyalists” while the weaker brand targets “switchers.” In general, endogenous targeting creates differentiation, even among homogeneous consumers, while exogenous targeting vitiates it, even among heterogeneous consumers. The targeting strategy game has the characteristics of a Prisoners’ Dilemma when exogenous targeting costs are low: while each firm would prefer that neither did any exogenous targeting, the only equilibrium involves both firms doing full targeting, and being worse off as a result (both individually and collectively).
1 Introduction

That advertising can serve a variety of functions is well understood by now (Bagwell 2007). Any advertisement, by definition, promotes awareness of the advertised brand. A subset of advertisements also provides information about product attributes. A second subset may persuade consumers through sheer volume of advertising (Nelson 1974, Sutton 1991). Finally, a third subset may complement product consumption (Becker and Murphy 1993). In this paper we examine advertising that operates in (at most) the first two ways.\footnote{The reason for the qualification “at most” is that while the simplest interpretation of our model is that advertising makes consumers aware of the product and educates them about the product’s attributes, it is not the only interpretation. For example, we might imagine a situation where advertising only makes consumers aware of the product, and, when they go to the store on their weekly shopping trip, they get the necessary product and price information at the shelf. Alternatively, consumers could even be already aware of all products in the market. The function of advertising, then, is simply to activate the consumer’s memory, to bring the product from the recesses of the mind to “top-of-mind.”}

The goal is to study advertising and price targeting, increasingly common, especially in online environments (Hartmann et al. 2011, Nair et al. 2013, Sahni et al. 2014). Two kinds of targeting are distinguished: exogenous targeting and endogenous targeting. By exogenous targeting we mean targeting by product preferences: advertising levels and prices geared to specific consumer preferences. For example, advertising (pricing) more or less to consumers with strong versus weak preferences for your brand. Exogenous targeting presumes observability of consumer preferences, or of factors correlated with them, such as demographics, geography, and purchase history. This is the sort of targeting that is commonly discussed in the media, especially under the rubric of online targeting (in its many avatars: “cookie-based targeting,” “behavioral targeting,” “retargeting,” etc), although offline targeting (such as by direct mail) is also possible. It is also the subject of a recent academic literature: Chen and Iyer (2002), Iyer et al. (2005). By endogenous targeting we mean the subtler, covert targeting that happens automatically when firms choose different levels of awareness advertising, and optionally follow that up with randomized prices. This sort of targeting can happen even when firms are not (exogenously) targeting anyone (McAfee 1994, Fershtman 1993, Boyer and Moreaux 1999).
In this paper we examine how these two types of targeting play out along the Hotelling line (Figure 1). Arguably, the Hotelling model provides the simplest, most natural, and perhaps even the most general setting in which to examine targeting issues. Consumer preferences for the two firms vary along the line. Yet their behavior is not pre-determined: any consumer can choose any firm depending on the prices charged.\footnote{By contrast, in Iyer et al.’s (2005) model, consumers come pre-classified as loyalists and switchers, and each consumer has the same reservation price for each firm. These assumptions have significant effects on targeting strategies, as we discuss later.} By investigating how targeting strategy varies along the line, we can improve our understanding of how firms should alter their advertising and pricing strategies when targeting consumers with strong versus weak preferences vis-a-vis competitors. In addition, recognizing that exogenous targeting is potentially costly, our model allows us to ask whether targeting itself is advisable, and if so, what should be the extent of targeting—both in consumer space—how many to target and which to target—and in variable space: advertising targeting alone, price targeting alone, or both.

Firms in our model target consumers at different locations exogenously, and endogenously within each targeted location, by creating “loyalists” and “switchers” out of homogeneous consumers. The endogenous targeting is shaped by the exogenously targeted advertising expenditures, which, in turn, determines the exogenously targeted prices. We first characterize what the exogenously targeted advertising and price equilibrium looks like at each location. For $x = 1/2$, adapting results from McAfee (1994), this equilibrium is asymmetric—even though consumers at this location view the two firms symmetrically. Unfortunately, as a prediction of outcomes, this result is not very useful: there are two pure-strategy equilibria, mirror-images of each other, and the two firms have different preferences between them. Therefore, in Section 3, we extend McAfee’s result by finding the unique mixed-strategy advertising equilibrium, which is symmetric, and treat that as the outcome of the targeting game at $x = 1/2$. In Section 4, we characterize all the equilibria for locations $x \neq 1/2$. For such locations, when the asymmetry between the firms is large, there is only one pure-strategy advertising equilibrium. In this equilibrium, the stronger brand—the one closer to the targeted $x$—advertises more than the weaker brand, and makes more profit. However, for less
extreme locations, in addition to this equilibrium, there is another pure-strategy equilibrium, and, once again, the firms’ preferences between these equilibria conflict. Fortunately, there is a mixed-strategy equilibrium that settles the dispute, and we use this equilibrium as the outcome of the targeting game for $0 < |x - 1/2| < \delta$.\footnote{The mixed-strategy equilibria are continuous in the sense that the mixed-strategy equilibrium at $x = 1/2$ is the limit of the mixed-strategy equilibria at $x \neq 1/2$ as $x \to 1/2$.}

Having characterized the (targeted) advertising and price equilibria at each location of the Hotelling line, we turn in Section 5 to the question of what targeting strategy should the firms follow. At one extreme is no targeting, i.e., all consumers on the Hotelling line are treated the same, and everyone faces the same advertising level and prices. This is the model studied by Grossman and Shapiro (1984). At the other extreme is full targeting, i.e., advertising and price targeting at each location, as per our solutions in Sections 3 and 4. A third targeting option is the one considered by Chen and Iyer (2002): price targeting at each location, but no advertising targeting. Obviously, targeting of any kind is more costly than no targeting, but, given economies of scope in targeting, the cost to implement advertising or price targeting might be the same as the cost to implement advertising and price targeting. But if that is the case, the following proposition is immediate: given Chen-Iyer targeting by both firms, each will have an incentive to deviate unilaterally to full targeting. However, the choice between full targeting and no targeting is not straightforward. If targeting cost is a fixed cost, independent of how many locations are targeted, and if this cost is large, then no targeting is the only equilibrium targeting strategy. On the other hand, if this cost is small, then the only equilibrium is full targeting—even though in this equilibrium both firms are worse off than if they both didn’t target. On the other hand, if targeting is a variable cost, proportional to the number of locations being targeted, then it is possible to envisage an equilibrium where some locations are targeted but not others. We conjecture that in this case, both firms will target consumers near the middle of the market while leaving consumers closer to them untargeted.

In what follows, any proofs not in the text are in an Appendix.
2 Model

Consider a standard Hotelling (1929) duopoly on a unit line (Figure 1). A unit mass of consumers is distributed uniformly along the line. Two firms, 1 and 2, each produce one product at constant marginal cost; we normalize this cost to zero without loss of generality.

Consumers are unaware of a product unless exposed to its advertising. When a firm spends $\alpha \theta^2$ dollars in advertising ($\alpha > 0$), it makes a fraction $\theta \in [0, 1]$ of consumers aware of its existence and product attributes. Once aware of product 1 (resp. product 2), consumers develop a reservation price for it. When they do, consumers at location $x$ have a reservation price $V - tx$ for a unit of product 0 and $V - t(1 - x)$ for a unit of product 1 (we will assume $V - t > 0$ so that even a consumer located farthest from a firm is willing to consider it).

Thus, our conception of advertising is that it is informative, as in Butters (1977), Grossman and Shapiro (1984), and Iyer et al. (2005): information is conveyed through advertising content, not through advertising volume (as in Nelson’s 1974 signaling model or Sutton’s 1991 persuasive model).

Advertising and price can (but need not) be exogenously targeted at each location. In other words, consumers are potentially identifiable by their location $x \in [0, 1]$, and should the firms choose to do so, they can offer consumers customized advertising and price levels based on location, $\theta_i(x)$, $p_i(x)$. Non-targeted locations receive a common advertising level $\theta_i$ and a common price $p_i$. Within each location, advertising and pricing is untargeted, i.e., all consumers at $x$ have the same probability of being exposed to a firm’s advertising—this probability could, of course, be different depending on whether $x$ is a targeted location or not—and when do, they receive the same price. Firms compete in two stages. First, each firm $i$ chooses its targeting strategy, $X_i$, the set of consumer locations (possibly empty) to target, and the advertising levels, $\theta_i(x)$ and $\theta_i$, for targeted and untargeted locations, respectively. Second, having observed each other’s advertising levels, the firms choose their

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4 As mentioned in footnote 1, other interpretations of our model are possible.

5 Our modeling of advertising differs from Iyer et al. (2005) in one way. Advertising expenditure is a continuous variable in our model, whereas in their’s it is discrete: either zero or one of three positive amounts. The big difference this makes is in the awareness levels: in Iyer et al. (2005), all consumers in a targeted segment are either aware or not aware, whereas in our model, $\theta_i$ can take any value between 0 and 1.
prices, $p_i(x)$ and $p_i$, for targeted and untargeted locations, respectively. The prices chosen are then incorporated in the advertising content, if needed (see footnote 3), and each firm executes its advertising plan.

In what follows, we will first analyze the advertising and price equilibrium at each location $x \in [0,1]$—starting with $x = 1/2$ in Section 3, and proceeding to $x \neq 1/2$ in Section 4—assuming $x$ is being (exogenously) targeted by both firms. This analysis itself proceeds in two stages: examining the price equilibrium first, for given advertising levels $(\theta_1(x), \theta_2(x))$, and using that equilibrium to then examine the advertising equilibrium. Finally, after completing the advertising and price equilibrium analysis at each location we turn to the question of targeting strategy: what set of locations, if any, to target. Only then will we consider targeting costs. Until then, all profits are to be interpreted as gross of targeting costs.

3 Targeting $x = 1/2$

Consumers at $x = 1/2$ value both firms equally. Denote that value by $v = V - t(1/2)$.

Let $\theta_1$ and $\theta_2$ be the fractions of consumers at $x = 1/2$ aware of firms 1 and 2, respectively, as a result of their ad spending. This sub-market, then, divides itself naturally into four segments: (1) aware of firm 1, but not 2: size $\theta_1(1 - \theta_2)$; (2) aware of firm 2, but not 1: size $\theta_2(1 - \theta_1)$; (3) aware of both firms: size $\theta_1 \theta_2$; and (4) aware of neither firm: size $(1 - \theta_1)(1 - \theta_2)$. Consumers in segment 1 (resp. 2) behave as if they are “loyal” to firm 1 (resp. 2); consumers in segment 3 behave as if they are “switchers” (Narasimhan 1988). Consumers in segment 1 will buy firm 1’s product as long as its price, $p_1$, does not exceed $v$; similarly, consumers in segment 2 will buy firm 2’s product as long as its price $p_2$ doesn’t exceed $v$. Finally, consumers in segment 3 will buy whichever product is cheaper.

The pricing game, following well-known arguments (Narasimhan 1988), only has a mixed-strategy equilibrium. Each firm has a lower bound on prices, $\underline{p}_i$, given by

$$\theta_1 ((1 - \theta_3 - i) + \theta_3 - i) \underline{p}_i = \theta_i (1 - \theta_3 - i)v$$

Alternatively, these segments can be given a consideration set interpretation: a segment of size $\theta_1(1 - \theta_2)$ (resp. $\theta_2(1 - \theta_1)$) only has firm 1 (resp. firm 2) in its consideration set, whereas a segment of size $\theta_1 \theta_2$ considers both firms’ products.
i.e.,
\[ p_i = v(1 - \theta_{3-i}) \quad \text{for } i = 1, 2. \]

Assume \( \theta_1 \geq \theta_2 \). Then \( p_1 \geq p_2 \), so effectively the former serves as the lower bound of the support for both firms’ mixed strategies (prices below \( p_1 \) are dominated for both firms). However, since firm 2 has the lower lower bound, it “controls” the switching segment (in addition to its loyalist segment). Its expected revenue (gross of advertising costs) is therefore \( v(1 - \theta_2)\theta_2 \). Firm 1, controlling only its loyalists, has an expected revenue of \( v\theta_1(1 - \theta_2) \).

Proceed now to the first stage, the choice of advertising levels, assuming still that those choices fall within the gamut of \( \theta_1 \geq \theta_2 \). The firms are maximizing the following profit functions:

\[
\begin{align*}
\pi_1 &= v\theta_1(1 - \theta_2) - \alpha \theta_1^2, \\
\pi_2 &= v(1 - \theta_2)\theta_2 - \alpha \theta_2^2.
\end{align*}
\]

These result in the following first-order conditions:

\[
\begin{align*}
v(1 - \theta_2) - 2\alpha \theta_1 &= 0, \\
v(1 - 2\theta_2) - 2\alpha \theta_2 &= 0.
\end{align*}
\]

Solving them we get a candidate advertising equilibrium,

\[
\begin{align*}
\theta_2^* &= \frac{v}{2(\alpha + v)}, \\
\theta_1^* &= \min \left\{ \frac{v(1 - \theta_2^*)}{2\alpha}, 1 \right\} \\
&= \min \left\{ \frac{v}{2(\alpha + v)} \left( \frac{2\alpha + v}{2\alpha} \right), 1 \right\}.
\end{align*}
\]

Note that \( \theta_1^* \leq 1 \) if and only if \( v \leq \alpha(1 + \sqrt{5}) \). Therefore, as \( \alpha \downarrow 0 \), \( \theta_1^* \to 1 \) and \( \theta_2^* \to 1/2 \), and we replicate Fershtman’s (1993) and Boyer and Moreaux’s (1999) observation that even

\footnote{In Iyer et al. (2005), since loyalists and switchers are pre-determined, when both are targeted, there is no interaction between them. So, for instance, the advertising and pricing strategy to switchers is entirely independent of the advertising and pricing strategy to loyalists. By contrast, here, since loyalists and switchers arise endogenously, they obviously interact in determining the exogenously targeted advertising and pricing strategies of the two firms.}
when advertising costs are negligible, only one of the firms would make the entire market aware of its product.

The candidate advertising equilibrium will be an equilibrium if no firm will deviate from the assumed order, $\theta_1 \geq \theta_2$, to the opposite order, $\theta_1 < \theta_2$, i.e., move from region $R_1 \triangleq \{(\theta_1, \theta_2) \in [0, 1]^2 : \theta_1/\theta_2 \geq 1\}$ to region $R_2 \triangleq \{(\theta_1, \theta_2) \in [0, 1]^2 : \theta_1/\theta_2 \leq 1\}$. Such a move would represent a change in each firm’s (endogenous) targeting strategy: firm 1 would be changing its target from loyalists to switchers, while firm 2 would be doing the opposite.

We show in the Appendix that neither firm would deviate from $(\theta_1^*, \theta_2^*)$ in this way. Therefore, $(\theta_1^*, \theta_2^*)$ is an equilibrium. The key feature of this equilibrium is, of course, that $\theta_1^* > \theta_2^*$: symmetric firms, catering to a homogeneous market, choosing asymmetric advertising levels! The key insight it enables is that firms can (endogenously) target different segments of a homogeneous market via advertising levels alone. Doing so enables them to mute the price competition between them. Effectively, the firm with the higher advertising level creates a larger “loyal segment,” and its pricing strategy “targets” this segment. Its higher advertising level reduces its rival’s loyal segment, while making the switcher segment larger. This induces its rival to “target” the switcher segment.

The firms’ profits in the equilibrium are:

$$\pi_2^* = \frac{v^2}{4(\alpha + v)}. \quad (2)$$

$$\pi_1^* = \begin{cases} 
\frac{v^2(2\alpha + v)^2}{16\alpha(\alpha + v)^2} & \text{if } v \leq (1 + \sqrt{5})\alpha \\
\frac{v^2 - 2\alpha^2}{2(\alpha + v)} & \text{otherwise}
\end{cases}$$

Note that $\pi_1^* > \pi_2^*$. This is not surprising: not only does firm 1 get a higher expected margin, $v$ (versus $v(1 - \theta_2^*)$), it also gets a higher expected market share, $\theta_1^*(1 - \theta_2^*)$ (versus $\theta_2^*$).

Of course, in a symmetric model, it is too much to expect that one firm can get away with having everything its own way. Sure enough, $(\theta_2^*, \theta_1^*)$ is also an equilibrium: in this equilibrium firm 1 chooses $\theta_2^*$, firm 2 chooses $\theta_1^*$, and it is firm 2 that makes the greater profit. In other words, while the asymmetric equilibrium is interesting in shedding light on how endogenous targeting works, as a prediction of targeting outcomes at $x = 1/2$, it is not satisfactory. Fortunately, another solution exists, a mixed strategy equilibrium, and this is
symmetric. In this equilibrium each firm randomizes over the same two advertising levels, with the same probabilities, and makes the same expected profit: $\pi^*_2$ above.

**Proposition 1** When both firms target consumers at $x = 1/2$,

1. There are two asymmetric pure-strategy equilibria, mirror-images of each other, with advertising levels given by (1). In these equilibria, the firm that advertises more makes the higher profit.

2. In addition, there is a symmetric mixed strategy equilibrium where each firm randomizes between $\theta^*_1$ and $\theta^*_2$ with probabilities $\lambda^*_1 < 1/2$ and $\lambda^*_2 > 1/2$, respectively, and has the same expected profit, $\pi^*_2$ in (2) above.

4 Targeting $x \neq 1/2$

Now suppose both firms are targeting an $x \neq 1/2$ with their advertising and pricing strategies. Denote $V-tx$ by $v_1(x)$, $V-t(1-x)$ by $v_2(x)$ and $v_1(x) - v_2(x) = t(1-2x)$ by $\Delta(x)$. Consumers at $x < 1/2$ (resp. $x > 1/2$) perceive firm 1 (resp. firm 2) to be the stronger brand by an amount $|\Delta(x)|$. Note that $v_1(x), v_2(x) > 0$ for all $x$ given our assumption $V-t > 0$. Without loss of generality, fix an $x < 1/2$. Then, $v_1(x) > v_2(x)$. In what follows, we will suppress the $x$-dependence of all quantities in order to focus on a given $x$.

In the pricing game, because consumers at $x$ value the two products differently, firm 1’s lower bound on prices, $v_1(1-\theta_2)$, is different from firm 2’s lower bound, $v_2(1-\theta_1)$. Also, the switchers are not indifferent between the firms; they prefer firm 1 by $\Delta$. As a result, who controls the switcher segment depends on whether $v_1(1-\theta_2) - \Delta$ exceeds $v_2(1-\theta_1)$ or not, i.e., $\theta_1/\theta_2 > v_1/v_2$. Define regions $R_1 \triangleq \{ (\theta_1, \theta_2) \in [0,1]^2 \mid \theta_1/\theta_2 \geq v_1/v_2 \}$ and $R_2 \triangleq \{ (\theta_1, \theta_2) \in [0,1]^2 \mid \theta_1/\theta_2 \leq v_1/v_2 \}$. In $R_1$, firm 2 controls the switcher segment (as well as its loyalists) while firm 1 controls only its loyalists; in $R_2$, firm 1 controls the switchers (and its loyalists) while firm 2 controls only its loyalists. In $R_1$, prices below $v_1(1-\theta_2) - (v_1 - v_2)$ are dominated for firm 2, so the lower bound of its support is $v_2-v_1\theta_2$, while the lower bound of firm 1’s support continues to be $v_1(1-\theta_2)$; in $R_2$, prices below $v_2(1-\theta_1) + (v_1 - v_2)$ are dominated for firm 1, so the lower bounds of their
supports are $v_1 - v_2 \theta_1$ and $v_2(1 - \theta_1)$, respectively. This yields the following profit functions for $i = 1, 2$:

$$
\pi_i^{R_i}(\theta_i, \theta_{3-i}) = v_i(1 - \theta_{3-i}) \theta_i - \alpha \theta_i^2
$$

$$
\pi_i^{R_{3-i}}(\theta_i, \theta_{3-i}) = (v_i - v_{3-i} \theta_i) \theta_i - \alpha \theta_i^2
$$

Now we turn to the firms’ advertising choices. Let $(\theta_1, \theta_2) \in R_1$. Maximizing the $R_1$-profit functions with respect to $\theta_1$ and $\theta_2$ we get the following first-order conditions,

$$
v_1(1 - \theta_2) - 2\alpha \theta_1 = 0,
$$

$$
v_2 - 2v_1 \theta_2 - 2\alpha \theta_2 = 0,
$$

and from them, the solution

$$
\theta_2^* = \frac{v_2}{2(\alpha + v_1)},
$$

$$
\theta_1^* = \min \left\{ \frac{v_1(1 - \theta_2^*)}{2\alpha}, 1 \right\},
$$

$$
\theta_1^* = \min \left\{ \left( \frac{v_1}{2\alpha} \right) \left( \frac{2(\alpha + v_1) - v_2}{2(\alpha + v_1)} \right), 1 \right\},
$$

as a candidate advertising equilibrium. It is easy to check that $\theta_1^*/\theta_2^* > v_1/v_2$, i.e., that $(\theta_1^*, \theta_2^*) \in R_1$, as assumed initially. To complete the proof that $(\theta_1^*, \theta_2^*)$ is indeed an equilibrium, we need to verify that neither firm would want to deviate unilaterally from these advertising levels to an advertising level that puts them in region $R_2$. We show in the Appendix that this is indeed the case. Therefore, $(\theta_1^*, \theta_2^*)$ is an equilibrium. In this equilibrium, the firms’ profits are,

$$
\pi_2^* = \frac{v_2^2}{4(\alpha + v_1)},
$$

$$
\pi_1^* = \begin{cases} 
\frac{v_1^2}{4\alpha} \left( \frac{2(\alpha + v_1) - v_2}{2(\alpha + v_1)} \right) & \text{if } \left( \frac{v_1}{2\alpha} \right) \left( \frac{2(\alpha + v_1) - v_2}{2(\alpha + v_1)} \right) \leq 1 \\
\frac{v_1^2 - 2\alpha^2 + v_1(v_1 - v_2)}{2(\alpha + v_1)} & \text{otherwise}
\end{cases}
$$

The stronger brand advertises more ($\theta_1^* > \theta_2^*$) and makes more profit ($\pi_1^* > \pi_2^*$). Therefore we have,
**Proposition 2** For $x < 1/2$, the advertising levels $(\theta_1^*, \theta_2^*)$ in (4) constitute an equilibrium. In this equilibrium, the stronger brand advertises more than the weaker brand and makes more money. As $x$ decreases, firm 1’s profits increase, but firm 2’s profits decrease.

We shall refer to this equilibrium as the putative equilibrium; in this equilibrium the stronger brand (endogenously) targets its loyalists while the weaker brand (endogenously) targets the switchers. However, it might be asked, Is it possible for there to be a role reversal—for the stronger brand to target the switchers and for the weaker brand to target its loyalists? In other words, is an equilibrium possible in $R_2$? More generally, are there other equilibria besides the putative equilibrium? We examine these questions below.

**Equilibrium in $R_2$ and mixed strategy equilibrium**

First let us examine the possibility of a pure-strategy advertising equilibrium in $R_2$. Maximizing the firms’ $R_2$-profit functions with respect to $\theta_1$ and $\theta_2$ yields the following first-order conditions:

\[
\begin{align*}
 v_1 - 2v_2 \theta_1 - 2\alpha \theta_1 &= 0, \\
 v_2 (1 - \theta_1) - 2\alpha \theta_2 &= 0.
\end{align*}
\]

If $v_1 \geq 2(\alpha + v_2)$, then $\theta_1 \equiv 1$, which implies $\pi_2(\theta_2) \leq 0$ for all $(1, \theta_2) \in R_2$. However, firm 2 can guarantee itself positive profits by deviating to $(1, \theta_2^*) \in R_1$. Hence, for $v_1 \geq 2(\alpha + v_2)$, an equilibrium in $R_2$ cannot exist.

So suppose $v_1 < 2(\alpha + v_2)$. Now the first-order conditions yield the following candidate

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8 Compare this to the advertising equilibria in *Chen and Iyer’s (2002)* model where there is no exogenous targeting of advertising, only of price. There, two kinds of pure-strategy equilibria appear, one symmetric and a pair of (mirror-image) asymmetric equilibria. In the asymmetric equilibria, one firm targets its loyalists and the other the switchers, throughout the line, whereas, in the symmetric equilibrium, the stronger firm targets the switchers and the weaker firm targets the loyalists (the opposite of our targeting strategy).
equilibrium,

\[
\theta^*_{1^*} = \frac{v_1}{2(\alpha + v_2)}, \\
\theta^*_{2^*} = \min \left\{ \left( \frac{v_2}{2\alpha} \right) (1 - \theta^*_{1^*}), 1 \right\}, \\
\theta^*_{2^*} = \min \left\{ \left( \frac{v_2}{2\alpha} \right) \left( \frac{2(\alpha + v_2) - v_1}{2(\alpha + v_2)} \right), 1 \right\}
\]

(6)

In this equilibrium, the firms’ profits are:

\[
\pi^*_{1^*} = \frac{v_1^2}{4(\alpha + v_2)},
\]

(7)

\[
\pi^*_{2^*} = \begin{cases} 
\frac{v_2^2}{4\alpha} \left( \frac{2(\alpha + v_2) - v_1}{2(\alpha + v_2)} \right)^2 & \text{if } \left( \frac{v_1}{2\alpha} \right) \left( \frac{2(\alpha + v_2) - v_1}{2(\alpha + v_2)} \right) \leq 1 \\
\frac{v_2^2 - 2\alpha^2 - v_2(v_1 - v_2)}{2(\alpha + v_2)} & \text{otherwise}
\end{cases}
\]

For \( v_1 \geq 2v_2 \), \( \theta^*_{2^*} \) is the first term in the minimum,\(^9\) and \( \theta^*_{2^*} \leq \theta^*_{1^*}(v_2/v_1). \) Hence, firm 2’s best response to \( \theta^*_{1^*} \) in \( R_2 \) is \( \theta^*_{2^*}(v_2/v_1) = v_2/2(\alpha + v_2). \) However, firm 2 can do better by deviating to \( \theta^*_{2^*} \), placing \( (\theta^*_{1^*}, \theta^*_{2^*}) \) in \( R_1 \), where it can guarantee itself \( v_2^2/4(\alpha + v_1). \)\(^10\) Hence for \( 2v_2 \leq v_1 < 2(\alpha + v_2) \) as well, there is no equilibrium in \( R_2 \).

So an equilibrium in \( R_2 \) can only exist for \( v_1 < 2v_2 \). Proposition 3 below asserts that when \( v_1 - v_2 \) is small enough, \( (\theta^*_{1^*}, \theta^*_{2^*}) \) is indeed an equilibrium in \( R_2 \). The following proposition characterizes all pure-strategy Nash equilibria in advertising strategies.

**Proposition 3** For \( x < 1/2 \),

(a) There exists a threshold \( \delta \in (0, v_2) \) such that for \( v_1 - v_2 \geq \delta \), the putative equilibrium at (4) is the only advertising equilibrium.

(b) However, for \( v_1 - v_2 < \delta \), there are two advertising equilibria in pure strategies, the putative equilibrium at (4) and the equilibrium at (6).

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\(^9\)\( \theta^*_{2^*} \) is the first term in the minimum if and only if \( v_1 > 2v_2 - 2\alpha(v_2 + 2\alpha)/v_2 \).

\(^10\) \( \theta^*_{1^*}(v_2/v_1) \) yields profit \( v_2\theta^*_{1^*}(v_2/v_1) (1 - \theta^*_{1^*}) - \alpha\theta^*_{2^*}(v_2/v_1)^2 = (v_2^2/4(\alpha + v_2)^2)(\alpha - (v_1 - 2v_2)) < v_1^2/4(\alpha + v_1). \)
When the stronger brand is significantly stronger, Proposition 3 says that the only equilibrium involves the stronger brand targeting its loyalists while the weaker brand targets switchers. Effectively, the stronger brand is saying that its reservation price advantage can only be brought home by targeting its loyalists—people who are willing to pay the reservation price, no questions asked. This calls for higher advertising levels. From the perspective of the weaker brand, if \( v_2 \) is far away from \( v_1 \), then it is also “small” on an absolute basis. The weaker brand’s loyalists are simply less valuable; therefore it has a greater incentive to focus on the switcher segment rather than on its loyalists.

The second part of Proposition 3 says that when the stronger brand is insufficiently differentiated from the weaker brand, then it is possible for a role reversal, with the stronger brand targeting the switchers and the weaker brand targeting its loyalists. In this scenario, equilibrium is being driven purely by market segmentation considerations—the desire to mute price competition by making opposite targeting choices than your competitor. This incentive is symmetric; both brands have this incentive. That is why the putative equilibrium continues to exist. When the weaker (stronger) brand chooses to target its loyalists, the stronger (weaker) brand is happy to target the switchers because (a) the switcher segment has expanded with the weaker (stronger) brand’s advertising expenditure, and (b) the switcher segment has become more lucrative because, with \( v_1 \approx v_2 \), the weaker (stronger) brand’s lower bound on prices is close to its own. As might be expected, as \( v_2 \to v_1 \), the two equilibria morph into each other—become mirror-images, as in the case of \( x = 1/2 \).

While the first part of Proposition 2 seems to resolve the question of how firms with different levels of appeal will behave when targeting consumers, the second part muddies the waters. Given two equilibria, which will be played? This leads to the question, Do the two firms have the same preference between the two equilibria? Unfortunately, no. Proposition 4 says that the stronger firm prefers the (I) equilibrium while the weaker firm prefers the (II) equilibrium, even though total profits are higher in the former.

**Proposition 4** Between the two advertising equilibria, \((\theta_1^*, \theta_2^*)\) and \((\theta_1^{**}, \theta_2^{**})\), total profits are higher in the former. However, the stronger brand prefers the former while the weaker firm prefers the latter.
Given the conflicting preferences of the two firms between the two pure-strategy equilibria, it is natural to look to a mixed strategy equilibrium to settle the dispute. Indeed, this is what Proposition 5 does.

**Proposition 5** When there are two pure-strategy advertising equilibria, there also exists a mixed strategy advertising equilibrium in which the stronger firm randomizes between $\theta_1^*$ and $\theta_1^{**}$ and the weaker brand randomizes between $\theta_2^*$ and $\theta_2^{**}$. In this equilibrium, firm 1’s expected profit is a weighted average of its profits in the two pure-strategy equilibria whereas firm 2’s expected profit is equal to its profit in the putative equilibrium $(\theta_1^*, \theta_2^*)$.

In the mixed strategy equilibrium, firm 2 gets its expected profit in the $(\theta_1^*, \theta_2^*)$ pure-strategy equilibrium, whereas firm 1 gets a convex combination of its expected profit in the two pure-strategy equilibria. While firm 2 is indifferent between the mixed strategy equilibrium and its worst-case outcome—the putative equilibrium—firm 1 prefers the mixed strategy equilibrium to its worst-case outcome—the $(\theta_1^{**}, \theta_2^{**})$ equilibrium. In this way, the mixed strategy equilibrium does settle the dispute between the firms—albeit in firm 1’s favor, but this is only to be expected since it is the stronger brand.

## 5 Targeting strategy

We know now what exogenous targeting of advertising and pricing decisions by both firms looks like along the Hotelling line (Figure 1). For $x = 1/2$ and locations nearby, $x \in (x_\delta, 1 - x_\delta)$, these decisions are described by the mixed strategy equilibrium of Propositions 1 and 5. For locations $x \in [0, x_\delta]$ and for locations $x \in [1 - x_\delta, 1]$, these decisions are described by the putative equilibrium of Proposition 2 and its transpose, respectively. The boundary points $x_\delta$ and $1 - x_\delta$ are, of course, related to the $\delta$ defined in Proposition 3.

![Figure 1: Targeting equilibria along the Hotelling line](image-url)
What we don’t know yet is whether any targeting should be done at all, and if so, what locations should be targeted, and with what variable(s)—advertising, price, or both? These are the questions of targeting strategy.

At the most general level, answers to these questions require specifying a set of targeted locations for each firm, $X_i$, $i = 1, 2$, and the variable(s) that are being targeted at those locations. If $X_i$ is empty, then firm $i$ is doing no targeting: all $x \in [0,1]$ receive the same advertising and price levels, $\theta_i$ and $p_i$, respectively. At the other extreme, if $X_i = [0,1]$, and both advertising and price are being targeted at each location, then each location $x \in [0,1]$ receives a customized advertising level and a customized price from firm $i$, $\theta_i(x)$ and $p_i(x)$, respectively. An intermediate option would be if $X_i$ is a nonempty proper subset of $[0,1]$ and, say, only price is being targeted to all $x \in X_i$. In this case, firm $i$ offers: (i) each $x \in X_i$, a customized price $p_i(x)$, (ii) each $x \in [0,1]$, a uniform advertising level $\theta_i$, and (iii) each $x \in [0,1] \setminus X_i$, a uniform price $p_i$. We envisage firms making their targeting strategy decisions together with the associated advertising decisions, first, before they make their price decisions.

As noted earlier, targeting strategy has not received much attention in the literature. Typically, the game begins with a targeting or no targeting decision already made. For example, Grossman and Shapiro (1984) assume no targeting: in their model, $X_1 = X_2 = \phi$ and each firm offers a uniform $(\theta_i, p_i)$ to all $x \in [0,1]$. On the other hand, Chen and Iyer (2002) assume that each firm targets all consumers on price, but not on advertising. In their model, $X_1 = X_2 = [0,1]$, and each firm chooses a $\theta_i$ for the entire market and a customized $p_i(x)$ for each $x \in [0,1]$. Below we examine whether the Grossman-Shapiro no-targeting assumption or the Chen-Iyer price-targeting-only assumption survive the endogenizing of targeting strategy.

The cost of targeting is obviously important to targeting strategy. As Propositions 1–5 document, targeting a location requires information about the reservation prices of consumers at that location. In addition, firms must have the ability to identify observable markers for consumers at the targeted locations, which they can then feed to their targeting technology. The targeting technology in turn must have the ability to direct ads and prices to particular locations.

11The exception is, of course, Iyer et al. (2005).
groups of consumers, so that, consumers at $x$ receive the advertising and price, $\theta_i(x)$ and $p_i(x)$, meant for them, and not, say, the the advertising level $\theta_i(x')$ or price $p_i(x')$ meant for $x'$. The advent of the Internet has increased firms’ abilities to do all these things at a low cost (Nair et al. 2013).

**Full targeting versus partial targeting**

It is reasonable to assume that full targeting—on both advertising and price—and partial targeting—on advertising or price—cost the same. In other words, if it costs $c$/location to target on advertising or price, then it costs the same $c$/location to target on advertising and price. One reason is, both advertising targeting and price targeting require the same data—the reservation prices of customers at the targeted locations. Another reason is, the technology for directing ads and prices to particular consumers is essentially the same. For instance, an e-mail bearing a promotional price directed to a select group of consumers,\footnote{See, for example, Sahni et al. (2014), where they document that many e-mail promotions work like ads.} could also be an ad for the product.

But this immediately implies that if there is a targeting strategy equilibrium in which both firms target a common set of locations, then they must target those locations fully.

**Proposition 6** In any targeting strategy equilibrium in which $X_1 \cap X_2 \neq \phi$, both firms must target locations in $X_1 \cap X_2$ fully, i.e., on both advertising and price.

This immediately implies the following Corollary.

**Corollary 1** The targeting strategy examined in Chen and Iyer (2002) cannot be an equilibrium targeting strategy.

What drives the proposition is that a non-targeted advertising level or price is always beaten by a targeted advertising level or price: customization beats non-customization. To see this, consider the advertising equilibria in Chen and Iyer (2002). As noted, these authors envisage a targeting strategy in which both firms target the whole Hotelling line—but on price only; advertising is not targeted by any firm: all locations receive the same advertising levels. Now note that just because advertising levels are not targeted by location doesn’t mean
that no endogenous targeting occurs at each location. In fact, any pair of \((\theta_1, \theta_2)\), whether targeted to a specific location or not, as long as it is received at that location, implies specific endogenous targeting choices at that location (depending on whether \(\theta_1/\theta_2 > v_1/v_2\) or \(\theta_1/\theta_2 \leq v_1/v_2\)), and these choices drive the price equilibrium that follows. The same is true for Chen and Iyer’s (2002) advertising equilibria, which are of two types: symmetric and asymmetric. The symmetric equilibrium implies that at \(x = 1/2\) neither firm endogenously targets any consumers and at \(x \neq 1/2\) both firms endogenously target switchers amongst those closest to them and loyalists amongst those farthest to them; their asymmetric equilibrium implies that one firm endogenously targets loyalists throughout the line and the other firm endogenously targets switchers throughout the line. Per our Proposition 1, symmetric advertising levels is never an equilibrium for \(x = 1/2\) when both firms target their advertising to that location. Furthermore, \(V > 2t\) (assumed in Chen and Iyer 2002) implies \(v_1(x) < 2v_2(x)\) for all \(x \in [0, 1]\), so our Proposition 3 says that the putative equilibrium (4) is the only equilibrium when both firms target their advertising to locations \(x \neq 1/2\). In this equilibrium each firm endogenously targets the loyalists amongst those closest to it and the switchers amongst those farthest from it—the opposite of the endogenous targeting choices embedded in Chen and Iyer’s (2002) symmetric equilibrium, and quite different from the uniform targeting embedded in the asymmetric equilibrium.

Figure 2: Firm 1’s advertising strategy along the Hotelling line with full targeting (solid line) and Chen-Iyer targeting (dotted line)
symmetric equilibrium) at each point on the Hotelling line. Notice that at the ends of the line, the two equilibria depart considerably. These differences in advertising strategy, in turn, have a significant impact on the pricing strategy (which is targeted in both cases). Figure 3 shows this. Firm 1’s Chen-Iyer prices to consumers at \( x < 1/2 \) are relatively high even though this firm is endogenously targeting switchers in those locations; this is because those prices are being supported by firm 2’s targeting of its loyalists at those locations—how high those prices can go is limited only by the relatively dim view those loyalists have of firm 2. Under full targeting, firm 1 targets loyalists at \( x < 1/2 \). Since these consumers like firm 1, it can charge a higher price to them. Figure 4 shows the minimum price charged at each location—the price the less preferred firm charges at that location. Under full targeting, the less preferred firm is targeting switchers at each location; under Chen-Iyer targeting, this firm is targeting its loyalists. It is clear that under full targeting these consumers receive a significantly better offer from the less preferred firm. The lack of advertising competition at each point in Chen-Iyer targeting increases prices to switchers.

Figure 3: Firm 1’s expected price along the Hotelling line with full targeting (solid line) and Chen-Iyer targeting (dotted line)

Note, however, that in going from partial targeting to full targeting, the firms are confronted with a Prisoners’ Dilemma. Even though both firms will deviate from partial targeting to full targeting unilaterally, their profits when they do so will be lower than their profits when they did partial targeting only. For instance, when we compare the firms’ profits in the Chen-Iyer symmetric equilibrium to their profits in the full targeting equilibrium (for the
Figure 4: The less preferred firm’s price at each location with full targeting (solid line) and Chen-Iyer targeting (dotted line) parameter settings $V/\alpha = 1.5$ and $V/t = 2.5$) we get Figure 4.

Figure 5: Comparing firm 1’s profits in the full targeting equilibrium (solid line) versus Chen and Iyer’s (2002) partial targeting equilibrium (dotted line)

Targeting versus no targeting

Now let us compare targeting with no targeting (as in Grossman and Shapiro 1984). By no targeting we mean, of course, “no location-specific targeting.” Iyer et al (2005) call this “uniform targeting.”

What are the equilibrium advertising and price levels with no targeting? Assuming that each firm is competing for the segment that is aware of both firms (which will be the case if
\( \alpha \) is not too large), each firm maximizes
\[
\theta_i \left[ (1 - \theta_{3-i}) + (\theta_{3-i}) \left( \frac{p_{3-i} - p_i + t}{2t} \right) \right] (p_i) - \alpha \theta_i^2 \text{ for } i = 1, 2.
\]

For given \((\theta_1, \theta_2)\), this yields the price equilibrium:
\[
p_i = \left( \frac{4t}{3} \right) \left( \frac{1}{2\theta_i} + \frac{1}{\theta_{3-i}} - \frac{3}{4} \right) \text{ for } i = 1, 2,
\]
and, ultimately, the symmetric advertising equilibrium,
\[
\theta_i = \left( \frac{4t}{7t - 12\alpha} \right) \left( 2 - \sqrt{\frac{9t + 12\alpha}{4t}} \right) \text{ for } i = 1, 2,
\]
as long as \( V - t \geq p_i \) and \( 7t/12 > \alpha > t/4 \).

The no-targeting equilibrium, even though it doesn’t target any consumer location specifically, implicitly targets the consumers marginal between the firms. This has the effect of inducing the firms to hold back on their advertising expenditures in the quest for margins (see the price equation above). As a result consumers at the ends get a price windfall but are relatively starved of advertising. Profits are positive because the products are differentiated
\( (t > 0) \).

Suppose targeting costs are fixed costs and it costs \( F > 0 \) to target a location or the the entire market. Then, obviously, if \( F \) is large enough, neither firm will deviate from the no targeting strategy, and no targeting will be an equilibrium targeting strategy. On the other hand, if \( F \) is small enough, then either firm will deviate from no targeting, and full targeting will be the only equilibrium strategy. To see this, start with the no targeting advertising and price levels above. Consider the consumers at \( x = 1/2 \). It is easy for either firm to secure a higher profit by targeting these consumers with a price \( p'_i = p_{3-i} - \epsilon \), holding everything else constant. This captures the entire switcher segment of size \( \theta_i \theta_{3-i} \), instead of half of it, while retaining the loyalist segment \( \theta_i (1 - \theta_{3-i}) \) at a price arbitrarily close to \( p_i \). Obviously, even more profitable deviations exist if both advertising and price are targeted, and not just at \( x = 1/2 \), but throughout the line.

However, note the Prisoners’ Dilemma the firms find themselves in. By fully targeting consumers at each location, they substantially vitiate the differentiation built into the

\[ \text{13By contrast, in Iyer et al. (2005), profits are zero because products are undifferentiated.} \]
Hotelling model: the $t$ parameter does not moderate the price competition at each location (as it did under no targeting when only the marginal location was being fought over)—it only alters relative reservation prices, not price elasticities. While endogenous targeting via advertising can restore some measure of differentiation, it cannot go all the way. This logic leads to the following proposition.

**Proposition 7** Assume targeting costs are fixed costs and equal to $F$. Then, if $F$ is large enough, the only equilibrium targeting strategy is no targeting. On the other hand, if $F$ is small enough, then full targeting of all locations is the only equilibrium targeting strategy. In this equilibrium, each firm is worse off compared to when neither was targeting anyone.

Now consider the possibility that targeting is a variable cost, depending on the number of locations targeted. In particular, suppose it costs $F$ to target the entire Hotelling line, but only $fF < F$ to target a fraction $f < 1$. Suppose $F$ exceeds the “large enough” lower bound defined in Proposition 7. Now, given that targeting fractions of the market costs less than $F$, there exists a set of locations where deviations from the no-targeting equilibrium are profitable. We conjecture that in this case a positive interval of locations straddling the middle of the market will be targeted by both firms, leaving each firm a set of uncontested consumer locations where it is strong, which it will not target individually.

6 Conclusion

We have examined targeted advertising and price strategies along the Hotelling line, and addressed the question of equilibrium targeting strategy. Targeted advertising and price strategies differ, of course, from their untargeted counterparts. In the latter, even though no one is being targeted explicitly, a “marginal consumer location” is being targeted implicitly—the firms’ equilibrium advertising and price strategies are shaped by the competition for consumers at this location. Even though these marginal consumers perceive the two firms as differentiated (as consumers in a Hotelling model are wont to do given $t > 0$), they have relatively weak preferences for both firms, and price competition further reduces prices, Consumers at inframarginal locations get a price windfall, but at the cost of depressed advertising levels. With targeting, each targeted location is marginal, and—as long as the
targeting technology works as it should—there is no spill-over from one targeted location to
another targeted location, nor from targeted locations to untargeted locations. Furthermore,
consumers within each location are homogeneous. The effect is to make every contested
targeted location a hot battleground. But not necessarily a zero-profit battleground. This
is because advertising, when it generates awareness and influences consideration, has the
magical property of creating heterogeneous segments out of homogeneous consumers. This
allows each firm to use its targeted advertising strategy to sub-target consumers within each
location, leading to differentiated targeted pricing strategies. Even consumers in the middle,
with the weakest preferences for both firms, yield positive profits to both firms. Of course,
those profits are even higher near the ends of the line: now competition is muted not only
by the endogenous targeting, but also by the exogenous differentiation between the firms.
However, these consumers pay higher prices than they did under no targeting because they
don’t get the benefit of free-riding on the lower prices offered consumers in the middle.

Equilibrium targeting strategy in variable space is driven by the simple idea that at every
location, full targeting—advertising and price targeting—always beats partial targeting—
advertising or price targeting. The reason is economies of scope in targeting costs, and
one firm’s full customization always beats another firm’s partial customization. A targeted
strategy also beats an untargeted strategy at every location for customization reasons, but
targeting costs intervene. When targeting costs are fixed, either no targeting by both firms or
full targeting by both firms at every location is the equilibrium—the latter leading to worse
outcomes than the former, a Prisoners’ Dilemma effect. When targeting costs are variable, we
conjecture that the equilibrium targeting strategy is to fully target consumers in the middle,
leaving consumers with more extreme preferences untargeted.
Appendix

Proof of Proposition 1

If firm 1 deviates to an advertising level \( \theta'_1 \) such that \( \theta'_1/\theta^*_2 \leq 1 \), then the lower bound of both firms’ pricing distributions will be \( v(1 - \theta'_1) \) and firm 1’s expected profit will be \( v(1 - \theta'_1)\theta'_1 - \alpha \theta^*_2^2 \). Maximizing this subject to the constraint \( \theta'_1 \leq \theta^*_2 \) yields \( \theta'_1 = \theta^*_2 = v/2(\alpha + v) \) with expected profit \( v(1 - \theta^*_2)\theta^*_2 - \alpha \theta^*_2^2 \). For \( v/\alpha < 1 + \sqrt{5} \), the difference in firm 1’s profits between the candidate equilibrium and its best \( R_2 \) deviation is

\[
\frac{v^2(1 - \theta^*_2)^2}{4\alpha} - \left( v(1 - \theta^*_2)\theta^*_2 - \alpha \theta^*_2^2 \right)
= \frac{v^2}{4\alpha} \left( 1 - \theta^*_2 \left( 1 + \frac{2\alpha}{v} \right) \right)^2
> 0.
\]

For \( v/\alpha \geq 1 + \sqrt{5} \), firm 1’s profit in the candidate equilibrium is \( v(1 - \theta^*_2) - \alpha \), so the difference in firm 1’s profits between the candidate equilibrium and its best \( R_2 \) deviation is

\[
v(1 - \theta^*_2) - \alpha - \left( v(1 - \theta^*_2)\theta^*_2 - \alpha \theta^*_2^2 \right)
= (1 - \theta^*_2)(v(1 - \theta^*_2) - \alpha (1 + \theta^*_2))
= (1 - \theta^*_2)(v - \alpha - \theta^*_2(v + \alpha))
= (1 - \theta^*_2)(v/2 - \alpha) > 0.
\]

Hence firm 1 will not deviate from \((\theta^*_1, \theta^*_2)\) to \( R_2 \). How about firm 2? If \( \theta^*_1 = 1 \) and firm 2 deviates to a \( \theta'_2 \) such that \( \theta^*_1/\theta'_2 \leq 1 \), then it will not have any loyalists, and its expected profit will be zero—obviously not a profitable deviation. If \( \theta^*_1 < 1 \), then its expected profit from the deviation will be \( v\theta'_2(1 - \theta^*_1) - \alpha \theta^*_2^2 \). Maximizing this subject to the constraint \( \theta'_2 \geq \theta^*_1 \) yields \( \theta'_2 = \theta^*_1 \). So the maximum expected profits of firm 2 from such a deviation will be \( v\theta^*_1(1 - \theta^*_1) - \alpha \theta^*_1^2 \). The difference in firm 2’s profits between the candidate equilibrium and \( 14 \) It is easy to verify that the unconstrained maximizer of \( v\theta'_2(1 - \theta^*_1) - \alpha \theta^*_2^2 \), \( v(1 - \theta^*_1)/2\alpha \), is less than \( \theta^*_1 \) when \( \theta^*_1 < 1 \).
its best $R_2$ deviation is
\[
\frac{v^2}{4(\alpha + v)} - \left( v\theta_1^*(1 - \theta_1^*) - \alpha \theta_1^{*2} \right)
\]
\[
= v^2 \left( \frac{1}{4(\alpha + v)} - \frac{\theta_1^*(1 - \theta_1^*)}{v} + \frac{\alpha \theta_1^{*2}}{v^2} \right).
\]
The minimum value of this expression is zero, achieved for $\theta_1^* = v/2(\alpha + v)$ ($\theta_1^*$ is the $\text{argmin}$ of the expression inside the parentheses seen as a function of $\theta_1$). Since $\theta_1^*$ is larger than $\theta^0$, we have shown that firm 2 will not deviate from $(\theta_1^*, \theta_2^*)$ either.

Now consider the mixed strategy equilibrium. For a given $\theta_{3-i}$, firm $i$’s profit function can be written as
\[
\pi_i(\theta_i; \theta_{3-i}) = \begin{cases} 
 v(1 - \theta_{3-i}) \theta_i - \alpha \theta_i^2 & \text{for } (\theta_i, \theta_{3-i}) \in R_i \\
 v(1 - \theta_i) \theta_i - \alpha \theta_i^2 & \text{for } (\theta_i, \theta_{3-i}) \notin R_i
\end{cases}
\]
To determine the mixed strategy equilibrium, we will first calculate the best response function for each firm $i$, $B_i(\theta_{3-i}) = \arg\max_{\theta_i} \pi_i(\theta_i; \theta_{3-i}) = \arg\max\{\pi_i(B_i^{R_i}(\theta_{3-i}); \theta_{3-i}), \pi_i(B_i^{R_{3-i}}(\theta_{3-i}); \theta_{3-i})\}$, where $B_i^{R_i}(\theta_{3-i})$ is the best response in region $R_i$, and $B_i^{R_{3-i}}(\theta_{3-i})$ is the best response in region $R_{3-i}$. Now,
\[
B_i^{R_i}(\theta_{3-i}) = \begin{cases} 
 \frac{v(1-\theta_{3-i})}{2\alpha} & 0 \leq \theta_{3-i} \leq \frac{v}{v+2\alpha} \\
 \theta_{3-i} & \frac{v}{v+2\alpha} < \theta_{3-i} \leq 1
\end{cases}
\]
\[
B_i^{R_{3-i}}(\theta_{3-i}) = \begin{cases} 
 \theta_{3-i} & 0 \leq \theta_{3-i} \leq \frac{v}{2(v+\alpha)} \\
 \frac{v}{2(v+\alpha)} & \frac{v}{2(v+\alpha)} < \theta_{3-i} \leq 1
\end{cases}
\]
The pure-strategy equilibria in the text are simply the intersections of $B_i^{R_i}(\theta_2)$ and $B_i^{R_2}(\theta_1)$.

Firm $i$’s profit at its two best responses to $\theta_{3-i}$ are:
\[
\pi_i(B_i^{R_i}(\theta_{3-i}); \theta_{3-i}) = \begin{cases} 
 \frac{v^2(1-\theta_{3-i})^2}{4\alpha} & 0 \leq \theta_{3-i} \leq \frac{v}{v+2\alpha} \\
 v\theta_{3-i} - (v + \alpha) \theta_{3-i}^2 & \frac{v}{v+2\alpha} < \theta_{3-i} \leq 1
\end{cases}
\]
\[
\pi_i(B_i^{R_{3-i}}(\theta_{3-i}); \theta_{3-i}) = \begin{cases} 
 v\theta_{3-i} - (v + \alpha) \theta_{3-i}^2 & 0 \leq \theta_{3-i} \leq \frac{v}{2(v+\alpha)} \\
 \frac{v^2}{4(v+\alpha)} & \frac{v}{2(v+\alpha)} < \theta_{3-i} \leq 1
\end{cases}
\]
\[
\pi_i(B_i^{R_i}(\theta_{3-i}); \theta_{3-i}) \text{ decreases in } \theta_{3-i} \text{ and } \pi_i(B_i^{R_{3-i}}(\theta_{3-i}); \theta_{3-i}) \text{ is increasing or constant in } \theta_{3-i}.
\]
former. So there exists a $\theta^i$ such that for $\theta_{3-i} < \theta^i$, firm $i$ prefers its best response in region $R_i$, but for $\theta_{3-i} > \theta^i$, it prefers its best response in region $R_{3-i}$; for $\theta_{3-i} = \theta^i$, firm $i$ is indifferent between its two best responses. This $\theta^i$ is the solution to
\[ \frac{v^2 (1 - \theta_{3-i})^2}{4\alpha} = \frac{v^2}{4(v + \alpha)}, \]i.e., $1 - \sqrt{\frac{v}{v+\alpha}}$. At this $\theta_{3-i}$, each firm is indifferent between its two best responses $\theta^i = \frac{v}{2\alpha} \sqrt{\frac{v}{v+\alpha}}$ and $\theta^2 = \frac{v}{2(v+\alpha)}$. For all $\theta_{3-i} > \theta^i$, firm $i$’s best response is $\theta^*_i$. Hence all $\theta < \theta^i$ are strictly dominated by $\theta^*_2$ for both firms. Since $\theta^*_1$ is the best-response to $\theta^*_2$, by iterated dominance, all $\theta > \theta^i$ other than $\theta^*_i$ can also be eliminated. In other words, only $\theta^*_1$ and $\theta^*_2$ are the only points in the support of each firm’s mixed strategy (see Table 1).

Note that, by symmetry, we have $\pi_1(\theta^*_i, \theta^*_j) = \pi_2(\theta^*_j, \theta^*_i) \quad \forall i, j = 1, 2$. First, we have
\[
\begin{align*}
\pi_1(\theta^*_i, \theta^*_i) &= v(1 - \theta^*_i)\theta^*_i - \alpha(\theta^*_i)^2 \\
&= v^2 \left(\frac{2(\alpha - v)(\alpha - v)}{16\alpha^2(v + \alpha)}\right) \\
\pi_1(\theta^*_i, \theta^*_j) &= v(1 - \theta^*_i)\theta^*_j - \alpha(\theta^*_j)^2 \\
&= v^2 \left(\frac{2\alpha + v}{16\alpha(v + \alpha)^2}\right) \\
\pi_1(\theta^*_j, \theta^*_i) &= \pi_1(\theta^*_j, \theta^*_j) \\
&= v(1 - \theta^*_j)\theta^*_j - \alpha(\theta^*_j)^2 \\
&= \frac{v^2}{4(v + \alpha)} 
\end{align*}
\]

Since $(\theta^*_i, \theta^*_j)$ and $(\theta^*_2, \theta^*_i)$ are pure Nash equilibria, we have
\[ \pi_1(\theta^*_1, \theta^*_1) < \pi_1(\theta^*_2, \theta^*_1) < \pi_1(\theta^*_1, \theta^*_2) \] (12)

Now the weights $\lambda^*_1$ and $\lambda^*_2$ should be determined such that the expected payoffs by choosing $\theta^*_1$ and $\theta^*_2$ be equal, i.e.,
\[
\lambda^*_1 \pi_1(\theta^*_1, \theta^*_1) + \lambda^*_2 \pi_1(\theta^*_1, \theta^*_2) = \lambda^*_1 \pi_1(\theta^*_2, \theta^*_1) + \lambda^*_2 \pi_1(\theta^*_2, \theta^*_2), \] (13)
\[
\lambda^*_1 + \lambda^*_2 = 1. \] (14)

Since $\pi_1(\theta^*_2, \theta^*_1) = \pi_1(\theta^*_2, \theta^*_2) = v^2/4(\alpha + v)$, we have $\lambda^*_1 \pi_1(\theta^*_1, \theta^*_1) + \lambda^*_2 \pi_1(\theta^*_1, \theta^*_2) = \frac{v^2}{4(\alpha + v)}$. Setting $\lambda^*_2 = 1 - \lambda^*_1$ and solving this equation, we have $\lambda^*_1 = \frac{\alpha}{2\alpha + v} < 1/2 < \frac{\alpha + v}{2\alpha + v} = \lambda^*_2$. 

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Proof of Proposition 2

If firm 1 deviates to an advertising effort $\theta'_1$ such that $\theta'_1 / \theta'_2 < v_1 / v_2$, then it will control the switcher segment, and the lower bound of its price distributions will be $v_1 - v_2 \theta'_1$. Hence its expected profit will be $(v_1 - v_2 \theta'_1) \theta'_1 - \alpha \theta'_1^2$. Maximizing this subject to the constraint $\theta'_1 \leq \theta_2^*(v_1 / v_2)$ yields $\theta_1^* = \theta_2^*(v_1 / v_2) = v_1 / 2(\alpha + v_1)$ with expected profit $(v_1 / v_2)^2 (v_2(1 - \theta_2^*)\theta_2^* - \alpha \theta_2^*^2)$.

The difference in firm 1’s profit in the putative equilibrium versus its best deviation in $R_2$ can be written as

$$v_1^2 (1 - \theta_2^*)^2 / 4 \alpha - (v_1 / v_2)^2 (v_2(1 - \theta_2^*)\theta_2^* - \alpha \theta_2^*^2)$$

$$= \frac{v_1^2}{4 \alpha} (1 - \theta_2^*)^2 (1 + \frac{2 \alpha}{v_2})^2$$

$$> 0.$$  

Hence firm 1 will not deviate from the putative equilibrium.

What about firm 2? If it deviates to $\theta'_2$ in region $R_2$, then the stronger brand will control the switchers segment and the weaker brand’s expected profit will be $v_2 \theta'_2 (1 - \theta_1^*) - \alpha \theta_2^*^2$. Maximizing this subject to the constraint $\theta'_2 \geq \theta_1^*(v_2 / v_1)$ yields $\theta_2^* = \theta_1^*(v_2 / v_1)$, so the maximum expected profits with such a deviation will be $(v_2^2 / v_1)\theta_1^*(1 - \theta_1^*) - \alpha \theta_1^*^2(v_2 / v_1)^2$.

The difference between firm 2’s profit in the putative equilibrium versus its best deviation in $R_2$ is therefore

$$\frac{v_2^2}{4(\alpha + v_1)} - \left( \frac{v_2^2}{v_1} \left( \theta_1^*(1 - \theta_1^*) \right) - \alpha \theta_1^*^2 \left( \frac{v_2}{v_1} \right)^2 \right)$$

$$= v_2^2 \left( \frac{1}{4(\alpha + v_1)} - \frac{\theta_1^*(1 - \theta_1^*)}{v_1} + \frac{\alpha \theta_1^*^2}{v_1^2} \right).$$

The minimum value of the expression in parentheses is zero, achieved for $\theta_1^* = v_1 / 2(\alpha + v_1)$ ($\theta_1^*$ being the argmin of the expression inside the parentheses seen as a function of $\theta_1^*$). Since the value of $\theta_1^*$ in the putative equilibrium is larger than $\theta_0$, we have shown that firm 2 will not deviate from the putative equilibrium either.

Proof of Proposition 3

We have already shown that the putative equilibrium at (3) always exists. So we only need to identify whether, and when, $(\theta_1^{**}, \theta_2^{**})$ can also be a Nash equilibrium. As already argued,
a necessary condition for this is \( v_1 - v_2 < v_2 \). Assume this is the case.

The stronger brand, should it deviate to region \( R_1 \), its best deviation will be \( \theta_1 = v_1(1 - \theta_2^{**})/2\alpha \) yielding \( v_1^2 (1 - \theta_2^{**})^2/4\alpha \). (It is easy to check that \( \theta_1 = v_1(1 - \theta_2^{**})/2\alpha \) satisfies the region \( R_1 \) requirement \( \theta_1/\theta_2^{**} > v_1/v_2 \).) The difference between the stronger brand’s profits in region \( R_1 \) and its profits in (6) is

\[
\frac{v_1^2(1 - \theta_2^{**})^2}{4\alpha} - \left((v_1 - v_2\theta_1^{**})\theta_1^{**} - \alpha\theta_1^{**2}\right) = \frac{v_1^2(1 - \theta_2^{**})^2}{4\alpha} - v_1\theta_1^{**} + \theta_1^{**2}(v_2 + \alpha) = \frac{v_1^2(1 - \theta_2^{**})^2}{4\alpha} - v_1\left(1 - \frac{2\alpha\theta_2^{**}}{v_2}\right) + \left(1 - \frac{2\alpha\theta_2^{**}}{v_2}\right)^2(v_2 + \alpha) = \frac{v_1^2(1 - \theta_2^{**})^2}{4\alpha} - \frac{v_1^2}{4(\alpha + v_2)}.
\]

Therefore, the stronger brand will not deviate to region \( R_1 \) if and only if

\[
(1 - \theta_2^{**})^2 \leq \frac{\alpha}{(\alpha + v_2)} \iff \theta_2^{**} \geq \frac{\sqrt{\alpha + v_2 - \sqrt{\alpha}}}{\sqrt{\alpha + v_2}}. \tag{15}
\]

Since \( \theta_2^{**} = \frac{v_2}{2(\alpha + v_2)} + \frac{v_2(v_2 - (v_1 - v_2))}{4\alpha(\alpha + v_2)} \), it is immediate that (15) is satisfied as \( v_1 - v_2 \to 0 \). By continuity, there is a \( \delta_S > 0 \) such that for \( v_1 - v_2 \leq \delta_S \), the stronger firm will not deviate from \( (\theta_1^{**}, \theta_2^{**}) \).

Now consider the weaker brand’s deviations. Its best deviation in \( R_1 \) is \( \theta_2^{*} \) which yields \( v_2^2/4(\alpha + v_1) \). The weaker brand will not deviate to region \( R_1 \) if and only if

\[
\frac{v_2^2(1 - \theta_2^{**})^2}{4\alpha} \div \frac{v_2^2}{4(\alpha + v_1)} \geq 1 \iff (1 - \theta_2^{**})^2 \geq \frac{\alpha}{\alpha + v_1} \iff \theta_2^{**} \leq \frac{\sqrt{\alpha + v_1 - \sqrt{\alpha}}}{\sqrt{\alpha + v_1}} \tag{16}
\]

Since \( \theta_2^{**} = \frac{v_2}{2(\alpha + v_2)} \), it is immediate that (16) is satisfied as \( v_1 - v_2 \to 0 \). By continuity, there is a \( \delta_W > 0 \) such that if \( v_1 - v_2 \leq \delta_W \), then the weaker firm does not deviate from \( (\theta_1^{**}, \theta_2^{**}) \).

Let \( \delta = \min\{\delta_S, \delta_W\} \). Therefore, \( (\theta_1^{**}, \theta_2^{**}) \) is a Nash equilibrium if and only if \( \Delta \leq \delta \).
Proof of Proposition 4

Let $\pi^R_{ij}$ denote the profit of firm $i$ in the region $R_j$ equilibrium $(i, j \in \{1, 2\})$. It is straightforward to show that $(\pi^R_{11} + \pi^R_{21}) > (\pi^R_{12} + \pi^R_{22})$. We also have

$$\frac{\pi^R_{11}}{\pi^R_{12}} = \frac{(\alpha + v_2) [2(\alpha + v_1) - v_2]}{4\alpha(\alpha + v_1)^2} = 1 + \frac{4v_2 (\alpha + v_1) (v_1 - v_2) + v_2^2 (\alpha + v_2)}{4\alpha(\alpha + v_1)^2} > 1,$$

since $v_1 > v_2$. Therefore, $\pi^R_{11} < \pi^R_{12}$. Similarly,

$$\frac{\pi^R_{22}}{\pi^R_{21}} = \frac{v_2^2 (1 - \theta_1^{**})^2}{4\alpha} \div \frac{v_2^2}{4(\alpha + v_1)} > 1,$$

for $\Delta < \delta$. Therefore, $\pi^R_{22} > \pi^R_{21}$. So the stronger and weaker brands have opposite preferences for which Nash equilibrium should be played.

Proof of Proposition 5

The first task in computing the mixed-strategy equilibrium is to determine the advertising supports of the two firms. For each firm, for every advertising level of its opponent, there is a best response in region $R_1$ and there is a best response in region $R_2$. Denote these best response functions by $B^R_{ij}(\cdot)$, for $i = 1, 2$, $j = 1, 2$. These regional best response functions are defined by

$$B^R_{1j}(\theta_{3-i}) = \arg \max_{(\theta_i; \theta_i \geq \theta_2(v_1/v_2))} \pi_i(\theta_i; \theta_{3-i})$$

$$B^R_{1j}(\theta_{3-i}) = \arg \max_{(\theta_i; \theta_i \leq \theta_2(v_1/v_2))} \pi_i(\theta_i; \theta_{3-i})$$

for $i = 1, 2$. Since the firms’ profit functions are given by

$$\pi_i(\theta_i; \theta_{3-i}) = \begin{cases} v_i (1 - \theta_{3-i}) \theta_i - \alpha \theta_i^2 & (\theta_i, \theta_{3-i}) \in R_i \\ v_i (1 - \theta_i) \theta_i - \alpha \theta_i^2 & (\theta_i, \theta_{3-i}) \notin R_i \end{cases} \tag{17}$$

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it is easily verified that

\[ B_1^{R_1}(\theta_2) = \begin{cases} \frac{v_1(1-\theta_2)}{2\alpha} & 0 \leq \theta_2 \leq \frac{v_2}{2(\alpha+v_2)} \\ \min\{\left(\frac{v_1}{v_2}\right)\theta_2, 1\} & \frac{v_2}{2(\alpha+v_2)} < \theta_2 \leq 1 \end{cases} \]

\[ B_1^{R_2}(\theta_2) = \begin{cases} \frac{v_1(1-\theta_2)}{2\alpha} & 0 \leq \theta_2 \leq \frac{v_2}{2(\alpha+v_2)} \\ \frac{\frac{v_1}{v_2}\theta_2}{2(\alpha+v_2)} & \frac{v_2}{2(\alpha+v_2)} < \theta_2 \leq 1 \end{cases} \]

\[ B_2^{R_1}(\theta_1) = \begin{cases} \frac{v_2(1-\theta_1)}{2\alpha} & 0 \leq \theta_1 \leq \frac{v_1}{2(\alpha+v_1)} \\ \frac{\frac{v_1}{v_2}\theta_1}{2(\alpha+v_1)} & \frac{v_1}{2(\alpha+v_1)} < \theta_1 \leq 1 \end{cases} \]

\[ B_2^{R_2}(\theta_1) = \begin{cases} \frac{v_2(1-\theta_1)}{2\alpha} & 0 \leq \theta_1 \leq \frac{v_1}{2(\alpha+v_1)} \\ \frac{\frac{v_1}{v_2}\theta_1}{2(\alpha+v_1)} & \frac{v_1}{2(\alpha+v_1)} < \theta_1 \leq 1 \end{cases} \]

The corresponding profits are:

\[ \pi_1(B_1^{R_1}(\theta_2); \theta_2) = \begin{cases} \frac{v_1^2(1-\theta_2)^2}{4\alpha} & 0 \leq \theta_2 \leq \frac{v_2}{2(\alpha+v_2)} \\ \max\left\{\frac{v_1^2}{v_2^2}, \frac{v_1^2}{v_2^2}, v_1(1-\theta_2) - \alpha\right\} & \frac{v_2}{2(\alpha+v_2)} < \theta_2 \leq 1 \end{cases} \]

\[ \pi_1(B_1^{R_2}(\theta_2); \theta_2) = \begin{cases} \frac{v_1^2(1-\theta_2)^2}{4\alpha} & 0 \leq \theta_2 \leq \frac{v_2}{2(\alpha+v_2)} \\ \frac{v_1^2}{v_2^2} & \frac{v_2}{2(\alpha+v_2)} < \theta_2 \leq 1 \end{cases} \]

\[ \pi_2(B_2^{R_1}(\theta_1); \theta_1) = \begin{cases} \frac{v_2^2(1-\theta_1)^2}{4\alpha} & 0 \leq \theta_1 \leq \frac{v_1}{2(\alpha+v_1)} \\ \frac{v_2^2}{v_1^2} & \frac{v_1}{2(\alpha+v_1)} < \theta_1 \leq 1 \end{cases} \]

\[ \pi_2(B_2^{R_2}(\theta_1); \theta_1) = \begin{cases} \frac{v_2^2(1-\theta_1)^2}{4\alpha} & 0 \leq \theta_1 \leq \frac{v_1}{2(\alpha+v_1)} \\ \frac{v_2^2}{v_1^2} & \frac{v_1}{2(\alpha+v_1)} < \theta_1 \leq 1 \end{cases} \]

Note that as \( \theta_2 \) becomes large, firm 1 would prefer \( R_2 \) to \( R_1 \) and as \( \theta_1 \) becomes large firm 2 would prefer \( R_2 \) to \( R_1 \).\[^{15}\] Therefore, as shown in Figure 6, there exist \( 0 < \theta_2^I < \theta_1^I < 1 \) such that firm 1 prefers its region-\( R_1 \) best-response, \( B_1^{R_1}(\theta_2^I) \), to its region-\( R_2 \) best-response, \( B_1^{R_2}(\theta_2^I) \), if and only if \( \theta_2 \leq \theta_2^I \), and firm 2 prefers its region-\( R_1 \) best-response, \( B_2^{R_1}(\theta_1^I) \), to its region-\( R_2 \) best-response, \( B_2^{R_2}(\theta_1^I) \) if and only if \( \theta_1 \geq \theta_1^I \). These quantities are given by

\[ 1 - \sqrt{\frac{\alpha}{v_2 + \alpha}} = \theta_2^I < \theta_1^I = 1 - \sqrt{\frac{\alpha}{v_1 + \alpha}} \]

\[^{15}\]For instance, for \( \theta_2 > \max\{v_2/(v_2+\alpha), (v_1-\alpha)/v_1\} \), firm 1’s maximum profits in \( R_1 \) are negative whereas they are positive in \( R_2 \).
Figure 6: The two firms’ best-response profits in the two regions

Now note that not only $B_{R_2}^2(\theta_2^*) = v_1/2(v_2 + \alpha) = \theta_1^*$, but also for $\theta_2 \geq \theta_2^I$, firm 1’s global best response is $\theta_1^*$; similarly, not only $B_{R_1}^1(\theta_1^*) = v_2/2(v_1 + \alpha) = \theta_2^*$, but also for $\theta_1 \geq \theta_1^I$, firm 2’s global best response is $\theta_2^*$. In other words, all $\theta_1 \in (B_{R_2}^2(\theta_2^I), B_{R_1}^1(\theta_1^I))$ and all $\theta_1 < B_{R_2}^1(\theta_1^I)$ are strictly dominated for firm 1 and similarly all $\theta_2 \in (B_{R_1}^1(\theta_1^I), B_{R_2}^2(\theta_2^I))$ and all $\theta_2 < B_{R_2}^2(\theta_2^I)$ are strictly dominated for firm 2. Then, by iterative dominance, the only $\theta_1 \geq B_{R_1}^1(\theta_1^I)$ to survive is $\theta_1^*$ (firm 1’s best response to $\theta_2^*$) and the only $\theta_2 \geq B_{R_2}^2(\theta_1^I)$ to survive is $\theta_2^{**}$ (firm 2’s best response to $\theta_1^{**}$). Therefore, in the mixed strategy equilibrium, each firm randomizes over the strategies in Table 2. It is straightforward to see that all cells except the $(\theta_1^{**}, \theta_2^{**})$ cell belong in $R_1$. Therefore, along column 1, and hence in the mixed-strategy equilibrium, firm 2’s profit is $v_2^2/4(\alpha + v_1)$, its profit in the putative equilibrium. Therefore, the probabilities
\( \lambda_i^* \) and \( \lambda_i^{**} \) are given by

\[
\lambda_i^* \pi_2(\theta_i^*, \theta_{2^*}^*) + \lambda_i^{**} \pi_2(\theta_i^{**}, \theta_{2^{**}}^*) = \frac{v_2}{4(\alpha + v_1)}, \quad (18)
\]

\[
\lambda_i^* + \lambda_i^{**} = 1. \quad (19)
\]

Calculating the probabilities \( \lambda_2^* \) and \( \lambda_2^{**} \) is somewhat more complicated:

\[
\lambda_2^* \pi_1(\theta_1^*, \theta_2^*) + \lambda_2^{**} \pi_1(\theta_1^{**}, \theta_2^{**}) = \lambda_2^* \pi_1(\theta_1^{**}, \theta_2^*) + \lambda_2^{**} \pi_1(\theta_1^{**}, \theta_2^{**}), \quad (20)
\]

\[
\lambda_2^* + \lambda_2^{**} = 1, \quad (21)
\]

where:

\[
\pi_1(\theta_1^*, \theta_2^*) = v_1(1 - \theta_2^*)\theta_1^* - \alpha(\theta_1^*)^2
\]

\[
\pi_1(\theta_1^{**}, \theta_2^{**}) = v_1(1 - \theta_2^{**})\theta_1^{**} - \alpha(\theta_1^{**})^2
\]

\[
\pi_1(\theta_1^{**}, \theta_2^{**}) = v_1(1 - \theta_2^{**})\theta_1^{**} - \alpha(\theta_1^{**})^2
\]

\[
\pi_1(\theta_1^{**}, \theta_2^{**}) = (v_1 - v_2\theta_1^{**})\theta_1^{**} - \alpha\theta_1^{**2} = \frac{v_1^2}{4(\alpha + v_2)}. \quad (22)
\]

It is easy to verify that \( \pi_1(\theta_1^{**}, \theta_2^*) > \pi_1(\theta_1^{**}, \theta_2^{**}) \). Therefore, firm 1’s expected profit in the mixed strategy equilibrium is greater than its profit in the equilibrium \( (\theta_1^{**}, \theta_2^{**}) \), but less than its profit in the putative equilibrium.
References


Randomization | $\lambda_2^*$ | $\lambda_1^*$  
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**Table 1:** The advertising mixed strategy equilibrium at $x = 1/2$

Randomization | $\lambda_2^{**}$ | $\lambda_1^{**}$  
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**Table 2:** The Mixed Strategy Under Asymmetry