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THE PRICING OF SUPERSHARES

by

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The Pricing of Supershares

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Abstract. The new "supershare" securities proposed by Hakansson [1977, 1976] are subject to the same sort of riskless-hedge combinations as are other forms of secondary securities such as stock options. In consequence, the prices of supershares must, even in the absence of distributional assumptions, obey certain rational pricing relationships with each other and with the underlying primary security. When the primary security is assumed to follow a geometric Brownian motion process, exact supershare valuation formulae of the Black-Scholes [1973] type are obtained. The "hedge portfolio algebra" of Garman [1976] is employed to make the analysis concise.

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1. INTRODUCTION

Hakansson [1977, 1976] has proposed the concept of a superfund (and its inflation-indexed counterpart, the purchasing power fund) as a vehicle whereby the securities market may be made considerably more complete by the issuance of a relatively few new securities. Basically, the idea is this: the superfund purchases a large, diversified portfolio of ordinary securities (usually the market portfolio) and issues "vertical" claims against the value of the portfolio at some liquidation date, i.e., claims which pay off only if that final portfolio value lies within pre-specified narrow boundaries. Hakansson shows that most of the complex securities that an investor might wish to hold may be constructed via combinations of the superfund's new securities, which he terms supershares.

At first blush, supershares would seem to be exotic securities and their valuation might therefore be dismissed as a mere mathematical recreation. Such an impression would be quite mistaken, however. For example, if we are interested in the welfare efficiency of securities markets, Hakansson [1977] shows that the superfund can be expected to realize sizable efficiency gains. In particular, when investors' otherwise heterogeneous utility functions depend only on wealth and their beliefs are homogenous only within sets of states leading to identical wealth levels, full efficiency obtains. Thus in this case a large number of supershares written against the market portfolio will behave very much like Arrow-Debreu [1964, 1959] certificates (over superstates) and their values will approximate the value of such certificates. In such a world where a superfund exists, a large body of finance literature that is sometimes known as the "state-contingent" approach is suddenly transformed from highly abstract theory to almost direct practicality.

In the present note we shall set forth valuation conditions for supershares. These will take the form of (a) necessary and sufficient rational pricing conditions for the avoidance of riskless arbitrage; and (b) under stronger assumptions, explicit valuation formulae for supershares. The methods employed are those introduced by Garman [1976] and Cox and Ross [1975], respectively.

Our assumptions are as follows:

(A1) At date 0, the superfund purchases a portfolio of ordinary assets. Without loss of generality, we shall assume that the market dominates this portfolio in units such that $\frac{3}{k_0}$, $k_0 > 0$, is the date 1 payoff of each portfolio unit, where $\delta$ is the date 1 total value of the superfund which holds $k_0$ such units. Also at date 0, the superfund issues $k_i > 0$ supershares which pay off $\frac{3}{k_i}$ at date 1 if and only if $b_i < \delta < b_{i+1}$, $i = 1, 2, ..., N$. (By convention, let $b_0 = b_1 < b_2 < \cdots < b_N < b_{N+1} = \infty$.)

(A2) No distributions (dividends etc.) are made by the superfund prior to date 1.

(A3) At date 0, an investor may risklessly borrow or lend any amount at the interest rate $\alpha$, so that $d \equiv (1 + \alpha)$ is the amount he must repay at date 1 for a loan of 1, or the amount he will receive back if he is the lender of 1.

1. For an alternative method of increasing market completeness, see Ross [1975]. Instead of supershares, Ross proposals a set of options written against a specialized portfolio, with some effects similar to the superfund. Ross shows that whenever investor tastes and beliefs distinguish only amongst states having differential payoffs over securities, there exists a portfolio against which options may be written to create a fully efficient market.

2. Not necessarily included in this class are the "path dependent" securities, such as one which pays $1 if the Dow Jones average goes over 1000 within the next year. This possibility has been put forward in private communication from S. Ross and J. Cox. An alternative conjecture is that the maturities of the superfunds become finer on the time line, path-dependent securities will offer no substantial increase in market completeness; however, this question is not addressed here.

3. In the limit as payoff boundaries converge and as the number of supershare types is increased. (Of course in theory, an infinite number of states cannot be spanned by a finite set of securities.) But when is a market "nearly" complete? At some point, the introduction of a new security becomes more expensive than the social value of the increased completeness it provides. The proper formulation and solution of this question is of some importance.
(A4) The superfund is not too large; in particular, investors believe they are free to directly hold any number of units at the same portfolio of assets as held by the superfund, again denominated such that each unit pays off \( \bar{s}/k_0 \) at date 1.

(A5) There are no transactions costs and no restrictions on the short selling of any security, or use of proceeds thereof.

2. THE RATIONAL PRICING OF SUPERSHARES

Under assumptions (A1) - (A5), supershares constitute a set of PL-options as defined in Garman [1976], and so may be described by the "hedge portfolio algebra" of that reference via their corresponding option vectors.\(^4\)

\[
\begin{align*}
T_1 &= (-d_p, -b_j/k_1, 0, \ldots, 0|1/k_1, -1/k_1, 0, \ldots, 0) \\
T_2 &= (-d_p, b_j/k_2, -b_j/k_2, 0, \ldots, 0|0, 1/k_2, -1/k_2, 0, \ldots, 0) \\
T_3 &= (-d_p, 0, b_j/k_3, -b_j/k_3, 0, \ldots, 0|0, 0, 1/k_3, -1/k_3, 0, \ldots, 0) \\
& \quad \vdots \\
T_N &= (-d_p, 0, 0, \ldots, 0, b_j/k_N|0, 0, \ldots, 0, 1/k_N),
\end{align*}
\]

where \( p_i \) is the date 0 price of a single supershare which pays off only if \( b_i \leq \bar{s} < b_{i+1} \). In addition, the portfolio units available to investors may be represented by the PL-option vector

\[
T = \begin{bmatrix}
T_0 \\
T_1 \\
\vdots \\
T_N
\end{bmatrix}
\]

where \( p_o \) is the unit price of each. Let

\[
\sum_{i=1}^{N} k_i p_i = k_0 p_0 \quad \text{(1a)}
\]

and

\[
\sum_{i=1}^{N} \frac{k_i p_i}{b_i} \leq 1 \quad \text{(1b)}
\]

summarize the necessary and sufficient conditions for rational pricing of supershares.

\(^4\) An option vector is basically a short-hand method for describing piece-wise linear (PL) functions of an underlying random variable. In the case at hand, the PL functions represent the net payoff of a supershare, in date 1 terms, as a function of \( \bar{s} \). The first \( N \) components in the option vector give the "jump" in the PL payoff function at \( b_i \), \( i = 1, 2, \ldots, N \), while the second \( N \) components provide the change in function slope at those points.

\(^5\) The basis function matrix transforms a set of PL-options into the values of their corresponding payoff functions at the right-hand and left-hand limits of the \( b_i \) and also into the payoff functions' slope at \( b_{N+1} = \infty \). (Positivity of linear combinations of all such quantities is equivalent to the existence of dominance profits.)
Note that condition (1b) is equivalent to the \( N \) conditions
\[
\sum_{i=1}^{n} \frac{k_{ip}}{b_i} \leq 1 \quad \text{for} \quad 1 \leq n \leq N.
\]

because of the assumed nonnegativity of terms in the sum. Suppose instead that condition \( n \) is violated. Then the arbitrageur makes a sure profit by selling \( \beta k_{i}/b_i \) (\( \beta > 0 \)) units of the first \( n \) supershares and receiving \( \beta \sum_{i=1}^{n} (k_{ip}/b_i) \) for his trouble. Since condition \( n \) is assumed violated, this amount exceeds \( \beta d \). He thereupon lends the proceeds of sale, so that at date 1 his return from the loan exceeds \( \beta \). His obligations at date 1 are to the holders of the supershares he sold; but the worst that can happen occurs when \( s \) is slightly less than \( b_i \), in which case he must pay an amount \( (b_i/k_{i}) \) per winning supershare sold. Since he sold each in the amount \( \beta k_{i}/b_i \), this means his total obligation is no more than \( \beta \). But he received more than \( \beta \) in the loan repayment, so he pockets the difference while maintaining zero net investment. Clearly, these arbitrage profits are limited only by how large he can make \( \beta \).

Violation of any of the conditions leads to similar arbitrage tactics, and so they are necessary for rational supershare pricing. Moreover, the conditions are also sufficient: their satisfaction implies that no one-period arbitrage scheme can assure a certain profit.

3. THE EXACT VALUATION OF SUPERSHARES

We now introduce additional assumptions to derive specific valuation formulae for supershares. Specifically, we assume:

\( (A6) \) The random variable \( s \) follows a geometric Brownian motion process such that \( s/(k_0p_0) \) has mean \( \mu ' \) and \( \ln(s/(k_0p_0)) \) is normally distributed with mean \( \mu \) and variance \( \sigma^2 \tau \), where \( \tau \) is the time period between dates 0 and 1.

\( (A7) \) Investors engage in continuous trading.

Under these conditions, riskless continuous hedges may be formed and Ito's lemma can be applied [Merton, 1974] to show that the values of the supershares, \( p_i(p_0, \tau, b_i, b_{i+1}), i = 1, 2, ..., N \) must satisfy the differential equation
\[
\frac{1}{2} \sigma^2 \rho^2 \frac{\partial^2 p_i}{\partial \rho^2} + \alpha \rho \frac{\partial p_i}{\partial \rho} - \alpha p_i = \frac{\partial p_i}{\partial \tau}
\]

with the initial conditions
\[ p_i(0, \tau, b_i, b_{i+1}) = 0 \]

and
\[ p_i(p_0,0,b_i,b_{i+1}) = \begin{cases} p_0 & \text{if } b_i \leq k_0p_0 < b_{i+1} \\ 0 & \text{otherwise} \end{cases} \]

Following the arguments of Cox and Ross [1975], we observe that the solution of (2)-(3) is independent of investor preferences, hence we may employ any preference structure consistent with attainment of market equilibrium. Universal risk neutrality is computationally the simplest. We therefore apply any risk-neutral valuation method to yield the following valuation formulae:

\[
T_1: \quad p_1 = \frac{k_0p_0}{k_1} (1 - \phi(q_2))
\]

\[
T_2: \quad p_2 = \frac{k_0p_0}{k_2} \phi(q_2) - \phi(q_3)
\]

\[
T_3: \quad p_3 = \frac{k_0p_0}{k_3} \phi(q_3) - \phi(q_4)
\]
\[ T_{N-1} \quad p_{N-1} = \frac{k_0 p_0}{k_{N-1}} \phi(q_{N-1}) - \phi(q_N) \]  
\[ T_N \quad p_N = \frac{k_0 p_0}{k_N} \phi(q_N) \]  
(4.N-1)  
(4.N)

where \( \phi(\cdot) \) is the cumulative distribution function\(^6\) of the standardized normal variate and

\[ q_i = \frac{\ln(k_0 p_0/b_i) + (\alpha + \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}, \quad i = 2, 3, \ldots, N. \]  
(5)

We note that the valuation equations (4.n) must (a) hold independently of investor preferences; (b) satisfy the rational pricing constraints (1); and (c) satisfy the differential equation (2) with boundary conditions (3).

4. SOME COMPARATIVE STATICS

The (inverse) hedge ratio, \(-\partial p/\partial p_0\), is the number of underlying portfolio units \( T_0 \) that should be held for every supershare \( T_i \) held in order to create a riskless hedge. This quantity is given by

\[ \eta \equiv -\frac{\partial p_i}{\partial p_0} = -\frac{k_0}{k_i} \left[ \phi(q_i) - \phi(q_{i+1}) \right] + \frac{1}{2 \sqrt{\pi \tau}} \left[ e^{-q_i^2/2} - e^{-q_{i+1}^2/2} \right]. \]  
(6)

Evidently, \( \eta \) may take on values both greater or less than zero, depending on the values of \( \sigma, \tau, \) and \( q_i, q_{i+1} \). When \( b_i \leq k_0 p_0 < b_{i+1} \) let us term the supershare \( T_i \) "in-the-money," otherwise we shall call it "out-of-the-money." Out-of-the-money supershares may further be classified as being "bear" supershares \( (b_{i+1} \leq k_0 p_0) \) or "bull" supershares \( (k_0 p_0 < b_i) \). In general, the investor who is risklessly hedged will then be long in both (or short in both) bear supershares and portfolio units; with regard to bull supershare and portfolio units, he will be long in one and short in the other.

The reaction of supershare prices to changing interest rates is given by

\[ \frac{\partial p_i}{\partial \alpha} = \frac{k_0 p_0 \sqrt{\tau}}{k_i \sigma} \left[ e^{-q_i^2/2} - e^{-q_{i+1}^2/2} \right]. \]  
(7)

It can be seen that \( \partial p_i/\partial \alpha > 0 \) for all bear supershares. Moreover, in-the-money and bull supershares will have \( \partial p_i/\partial \alpha > 0 \) whenever

\[ k_0 p_0 < \sqrt{b_i b_{i+1}} e^{-\left(\alpha + \sigma^2/2\right) \tau}, \]

and \( \partial p_i/\partial \alpha \leq 0 \) otherwise.

Thus the more "bullish" these supershares, the longer their maturity, or the greater the volatility of the interest rate, the less positively they will react to interest rate increases.

The reaction of the supershare pricing formulae to changes in maturity and volatility are more complicated, but only because the precise boundaries for sign changes of \( \partial \eta/\partial \alpha, \partial \eta/\partial \tau \) are difficult to compute. Intuitively, it is quite clear that the signs of these partial derivatives will tend to be negative for the in-the-money supershares, positive for out-of-the-money supershares.

\(^6\) One way to interpret the information given in the observed prices of supershares is that they allow one to deduce the implied density function of states assuming a risk-neutral market. Note that (4.n) involves only differences in cumulative distributions, scaled by price units; these differences approximate the implied density function to an exactness limited only by the fineness of supershares on the price line.
supershares. Figure 1 supports this general conclusion, and illustrates the overall shape of \( p_i(p_0, \tau, b, b_{i+1}) \) for typical parameter values. Note that for large \( \tau \) the shape of \( p_i \) is rather flat and somewhat asymmetric; as \( \tau \) approaches 0, the function becomes more peaked, eventually converging to the dotted lines which represent the boundary condition \( \tau = 0 \) in Figure 1.

5. CONCLUSIONS

Supershare rational pricing conditions, which do not depend upon any distributional assumption, have been presented. When the distributional assumption of geometric Brownian motion is added, supershares are shown to take on explicit values. Some comparative statics of supershare valuation have also been explored.
REFERENCES


Ross, S., 1976, Options and efficiency, Quarterly Journal of Economics 40, 75-89.