Information and the Sequential Valuation of Assets in Arbitrage-Free Economies

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Abstract. State variables in an economic environment have informational content for investment decisions; therefore, such information affects asset values. This paper gives a rigorous development of the relationship of informational variables to asset prices in a general economic setting, that of no riskless arbitrage possibilities available. Connections with more specific equilibrium settings, such as the CAPM and the Ross (1976b) model, are also explored.

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1. Introduction

Considerable empirical research in accounting has been directed towards the relationship of the equity price levels and the accounting numbers (or other information) associated with a firm. Earnings and dividends, for example, have been extensively investigated as possible determinants of equity share price levels. One reason for studying such accounting measurements is the notion that such measures reflect "true" underlying values which investors employ in determining their demand schedules for risky assets. More recent is the recognition that such "true" underlying variables are unobservable to investors and researchers alike, and thus the more pragmatic approach is to deal with accounting numbers as they are operationally measured, i.e. directly via "generally accepted accounting procedures."

This paper extends the existing work on information (accounting)-based valuation models to a systematic and complete treatment of both general and partial equilibrium. Previous work in this area is due to LeRoy (1973), Ohlson (1977,79), and to some extent Rubinstein (1976). Whereas these papers either have been preference based, with the attendant strong assumptions, or have presumed a priori that equilibrium is given by the Capital Asset Pricing Model (CAPM), the current paper requires only weak equilibrium conditions which amount to the absence of riskless arbitrage opportunities. Investors here are permitted diverse utility functions or no utility functions at all; however, it is convenient to retain the identical investor beliefs assumption employed by the previously-mentioned works.

Our objective is to describe and derive valuation models, i.e. functions which map accounting variables (or other forms of information) into equilibrium market prices; as a related result, an (endogenous) characterization of ex ante returns can always be obtained. There are many ways to rationalize the existence of valuation functions, the simplest being the observation that investors act upon whatever information they possess, and accounting information is one the most prevalent forms of such information. For example, certain variables calculated from income and balance statements are felt to be related to the riskiness of equity shares; indeed, if the CAPM is adopted, the corresponding systematic risk measure (betas) have been shown empirically related to such accounting quantities.

The explicit accounting valuation models that we derive here are, like their predecessors, linear mappings from accounting information to equilibrium prices. The analysis is restricted to this class of valuation functions only because of their simplicity; in principle, nonlinear valuation functions could be developed via the same methodology. (However, the valuation functions are obtained by trial solutions to finite difference equations and would be much more difficult to find in the nonlinear case.) The price that is paid for linearity is that stationarity and continuity in the state variables must be imposed. This means that the results obtained herein apply to the valuation of equity-type assets that do not have fixed maturity dates, unlike bonds or warrants. Moreover, the stationarity requirement imposed rules out the analysis of time-dependent shifts in capital structure or environmental parameters. These are limitations of the the particular valuation models described herein, but not limitations of the general methodology. It should also be emphasized that while our analysis is largely phrased in terms accounting variables, any other numerical forms of information may be treated equally well by the ensuing methodology.

2. Some basic assumptions.

In the remainder of the paper we adopt the following assumptions:

A1. Perfect markets. There are no transactions costs or restrictions on short sales. No individual investor may affect the prices of financial assets.

1. Beaver et al. (1970)
A2. Concordant expectations. All investors agree upon the prices and dividends that will obtain for any given (realized) state at every point in time. All investors agree upon the current state and the set of possible future states.

A3. Homogeneous beliefs. All investors possess identical beliefs.

A4. Markovian environment. The stochastic state evolution is generated by a time-independent Markovian process. Hence, if \( Z_t \) is the state at time \( t \), then the distribution of the state space random variable \( Z_{t+1} \) is given by \( \Pr[Z_{t+1} \leq z_{t+1} | z_t, z_{t-1}, \ldots, z_1] = \Pr[Z_{t+1} \leq z_{t+1} | z_t] = F(z_{t+1} | z_t) \). Given assumption A3, \( F \) is thus common to all agents in the economy.

A5. Arbitrage-free economy. There exist no riskless arbitrage opportunities.

A6. Stationarity. The valuation of financial assets depends only on state and not upon time. Hence, given A2, one can always write \( P(z_t) \) as the valuation function for asset \( z \).

A7. Riskfree asset. There exists a riskfree asset in all states and dates. The total return on this asset will be denoted by \( 1 + \nu_t(z_t) \equiv R(z_t) \).

3. Comments on the assumptions.

Perfect markets. This assumption is standard in much of equilibrium theory. It is a stringent assumption; for example, one of its implications is that available information is costlessly analyzed.

Concordant expectations. The first sentence in this assumption more commonly goes by the name "rational expectations." However, we consider the latter a poorer choice of terminology, since it connotes aspects of individual choice behavior. It is known that the concordant-expectations assumption is equivalent to the existence of a corresponding set of contemporaneous forward markets in all assets at all future times. However, this equivalency does not assure market completeness in a welfare sense; indeed, the economies treated here are generally incomplete. (That is, not all investors' marginal rates of substitution across wealth in different states will be necessarily equal.) The second sentence in this assumption is somewhat redundant, given A3. However, we wish to emphasize that A3 may be independently relaxed, as explained next.

Homogeneous beliefs. This assumption is much less restrictive in an arbitrage framework than it typically is in a utility-based framework. In the latter, belief heterogeneity contributes to the difficulties of aggregation over investors. In the arbitrage approach, aggregate investor beliefs will be seen to be impounded in a "summary statistic" describing the entire market, while the beliefs of a market observer (who might also be an investor) are treated explicitly. When the homogeneous-beliefs assumption is dropped, the second part of the concordant-expectations assumption (i.e. agreement on the set of possible states) comes into active play. In fact, the relaxation of the homogeneous beliefs assumption would be an easy matter here; but since it would also involve notational complication, we forego this modest increase in generality.

Markovian environment. The state variable \( Z \) may be thought of as a vector of \( n \) random variables: \( Z = (Z_1, \ldots, Z_n) \). It should be noted that we do not require linear independence amongst the variables \( Z_1, Z_2, \ldots, Z_n \), so that there is no loss (or gain) of generality in assuming that one state variable may be a function of another. Likewise, we do not require that the distribution of \( Z_{t+1} \) depends upon every (or any) component of the vector of realizations \( z_t \). Thus, it is not required that \( \text{Var}(Z_{t+1} | z_t) > 0 \) for all (or any) variables \( i \), and variables might be identical across some states and dates. In the same vein we note that nothing precludes the inclusion of lagged variables in the state-description at time \( t \). For example, \( z_{t-1} \) may be earnings per share disclosed at some time prior to \( t \) (say \( t-4 \)). (Furthermore, the number of variables \( n \) may be infinite; this will be of some significance later.)

Of course the state variables employed must represent a complete description of the states involved; all relevant forms of uncertainty must be captured in order to sensibly define a valuation function. In particular we will wish to focus on those variables which are intuitively...
important for valuation, e.g. earnings per share, dividends, etc.

Arbitrage-free economy. This is the crucial assumption that will permit us to characterize the set of admissible valuation functions in a general yet meaningful fashion. Essentially, the assumption states that it is impossible to find a portfolio such that no money is invested and yet the portfolio returns a positive amount in every state until some future date. The development of the exact nature of this assumption is given later.

Stationary valuation. This assumption requires assets to have valuation functions that are independent of time, depending only on states. Taken at face value, this assumption then seems to rule out the valuation of bonds, warrants, and options, all of which have fixed maturities. Moreover, if the total value of the firm is stationary and the firm issues fixed maturity instruments, then the valuation of the residual equity will generally be nonstationary, even though it has no fixed maturity. This apparently rules out any time-dependent shifts in any aspect of capital structure within the firm. However, time-independence is a practical problem and not a theoretical one; we can always include time as another Markovian state variable, the only cost being increased complication in the state space. The assumption is therefore made solely in the interests of notational simplicity.

Riskfree asset. The inclusion of a riskfree asset is a standard simplifying assumption in equilibrium analysis, but it is of no great significance. If such an asset does not exist, then the valuation equations described later will take on a more complicated form without altering their essential structure. The absence of a riskfree asset certainly might give rise to some identification problems in the empirical exploration of our development (as usual), but has no particular theoretical impact.

General. In reviewing our assumptions, then, we find them quite general: A3, A4, A6, and A7 are not critical and could be relaxed without damaging the essential theory herein; A5 is quite weak (what economies could admit riskless arbitrage?); A1 and A2 are somewhat stringent, but are also quite standard -- indeed, few theories of economic equilibrium have not used them.


The no-riskless arbitrage concept tells us about a characteristic that does not describe the economies studied here. This is a very negative form of information, however, and we would wish something more positive about what does characterize our economies. Theorems of the alternative provide just such a vehicle: they convert the negative information into a positive form.

Riskless arbitrage may be accomplished if it is possible to issue (sell) a portfolio for a positive price in the present that creates no next-period payment obligations; thus we wish to find a portfolio \( \{y_i\} \) which satisfies

\[
\sum y_i p_i(z_i) < 0 \tag{A}
\]

and

\[
\sum y_i p_i(z_i) + d_j(z_{i+1}) \geq 0 \quad \text{for all states } z_{i+1}
\]

where the \( y_i \)'s range over any financial assets available in the economy. Of course, here \( d_j(z_{i+1}) \) denotes the (cash) dividends paid to the holders of asset \( j \) in state \( z_{i+1} \).

The problem (A) will be termed an arbitrage problem and a solution \( \{y_i\} \) would be an arbitrage portfolio. The no-riskless arbitrage assumption states that no arbitrage portfolio exists for (A). A "theorem of the alternative" known as Farkas’ Lemma\(^2\) then implies the following:

\(^2\) For an introduction to Farkas’ Lemma, see for example Gale (1960), p. 44.
Either (A) has a solution or the problem

\[ P_j(z) = \sum_{i=1}^{n} k(z_{i+1}, z) \left[ P_i(z_{i+1}) + d_j(z_{i+1}) \right] \quad \text{for all assets } j \]  

\[ k(z_{i+1}, z) \geq 0 \quad \text{for all } z_{i+1} \]  

has a solution \( k(z_{i+1}, z) \), but not both. \( \blacksquare \)

Therefore, since no solution can exist for (A) (due to the fact that arbitrage is impossible), it must be the case that the solution to (NA) does exist. There are two important aspects of this conclusion. First, \( k(z_{i+1}, z) \) does not depend on the security \( j \) in question, but is the same for all existing securities. Second, the solution \( k(z_{i+1}, z) \) constitutes a nonnegative linear operator mapping future contingent prices and dividends into the current price. An obvious corollary is that if the sum of conditional price and conditional dividend is the same for two securities across all states, they must have the same current price.

5. Fundamental difference equation.

The problem (A) is only one of several possible arbitrage situations ruled out by the no-arbitrage assumption. It is a convenient choice, however, since it deals with two successive dates which our Markovian assumption will assure is sufficient. From its Farkas’ Lemma alternative (NA), we have the following

**Theorem 1**. Assumptions A1-A7 imply that there exists a single random variable \( \tilde{Q}_{i+1} \) such that the valuation function \( P_j \) of every asset \( j \) must satisfy the difference equation

\[ P_j(z) = g_j(z) + R^{-1}(z)E[P_j(\tilde{Z}_{i+1})|z] + \text{Cov}(\tilde{Q}_{i+1}, P_j(\tilde{Z}_{i+1})|z) \]  

for every \( z \), which can occur, \( g_j(z) \) is given by

\[ g_j(z) \equiv R^{-1}(z)E[d_j(\tilde{Z}_{i+1})|z] + \text{Cov}(\tilde{Q}_{i+1}, d_j(\tilde{Z}_{i+1})|z). \]

Furthermore, \( \tilde{Q}_{i+1} \) is nonnegative with probability one and \( E[\tilde{Q}_{i+1} | z] = R^{-1}(z) \). \( \blacksquare \)

**Proof.** From (NA), nonnegative \( k(z_{i+1}, z) \) exists. Let

\[ \tilde{Q}_{i+1} \equiv \begin{cases} k(z_{i+1}, z)/f(z_{i+1}, z), & f(z_{i+1}, z) > 0, \\ 0 & \text{otherwise}, \end{cases} \]

where \( f(z_{i+1}, z) \) is the probability density (mass) function corresponding to \( F(z_{i+1}, z) \). By (NA) and the nonnegativity of \( f, \tilde{Q}_{i+1} \) is nonnegative with probability one. The riskfree asset has current price \( R^{-1}(z) \) and total value one in every future state, hence

\[ R^{-1}(z) = \sum_{i=1}^{n} k(z_{i+1}, z)1 = E[\tilde{Q}_{i+1} | z]. \]

Finally, for an arbitrary asset \( j \), (NA) implies

\[ P_j(z) = E[\tilde{Q}_{i+1}(\tilde{P}_{p+1} + \bar{D}_{p+1})|z] \]

\[ - E[\tilde{Q}_{i+1}|z]E[\tilde{P}_{p+1} + \bar{D}_{p+1}|z] + \text{Cov}(\tilde{Q}_{i+1}, \tilde{P}_{p+1} + \bar{D}_{p+1}|z), \]

from which (1) follows immediately when \( \bar{D}_{p+1} \) is exogenously specified as \( D_{p+1} = d_j(\tilde{Z}_{i+1}) \). \( \blacksquare \)

For purposes of pure exchange analysis, the stochastic behavior of \( d \) must be taken as given; in partial equilibrium analysis (the valuation of subsets of assets) it must also be assumed that the distribution \( \tilde{Q} \) is completely exogenous. \( P_j(z) \) therefore will be sought as

3. Theorem 1 is more or less similar to the results found in Bejra (1967), Ross (1976a), Rubinstein (1976), and Garman (1978).
a valuation solution to the fundamental difference equation (1). Our objective, of course, is to include accounting (and other) forms of information within the state variable $z_i$, in which case the solution $P_i(z_i)$ is an information-valuation function, i.e. maps information into price levels.

Note that the solution to the difference equation (1) is not necessarily unique; there may be many linear and other forms of solutions. (When there is a complete market and well-specified boundary conditions, then valuations will indeed be unique.) Note that if $g_i(z_i)$ is uniformly zero, meaning no dividends will be paid in any state, then a zero price for $P_i(z_i)$ is always a solution consistent with (1). This is, of course, what one should expect. Additionally, if $g_i(z_i)$ depends upon some component of the state variable $z_i$, then in general $P_i(z_i)$ will also depend upon this component. In other words, if certain state variables are important for forecasting dividends (e.g. earnings), then these state variables will also be important for forecasting prices. Again, this has intuitive appeal.

It should be noted that linearity of the equation (NA) gives rise to the linear form of (1). This implies that if we find solutions of (1) for two assets, any linear combination (i.e. portfolio) of the two assets also yields a solution. This means the "packaging" of assets into portfolios does not affect asset valuation, or vice versa.


Explicit solutions of the difference equation (1) can be obtained in some cases. It will be assumed that $R_i(z_i)$ is independent of $z_i$, that is, the interest rate on the risk-free asset is constant. Given this assumption, a broad class of linear valuation functions can be combined with the Markovian nature of the state dynamics.

We now perform a partial equilibrium analysis for a single asset; hence, the asset subscripts for the price, dividend, and valuation parameters will be suppressed below. In particular, $D_i$ in the following specification refers to the dividends of a particular asset. Let $Z_t = (X_t, X_{t-1}, \ldots, X_{t-n}, D_t)'$ be the relevant state description and suppose that the state dynamics of the vector $Z_t$ is described by the general linear process

$$
\tilde{Z}_{t+1} = \eta + \delta_i + \theta_i + \tilde{\epsilon}_{i,t+1} \tilde{z}_i + \tilde{v}_{t+1},
$$

where

- $\eta$ = a vector of $n$ constants;
- $\delta_i = 1$ if $i = j$, 0 otherwise;
- $[\theta_i]$ = a matrix of constants, $i_j = 1, 2, \ldots, n$;
- $\tilde{v}_{t+1}$ = a vector of $n$ random variables;
- $[\tilde{\epsilon}_{i,t+1}]$ = a matrix of random variables.

Suppose further that the random variables $\tilde{v}_{t+1}$ and $\tilde{\epsilon}_{i,t+1}$ all have zero means (for every $z_i$) and covariances with $\tilde{Q}_{i,t+1}$ which are exogenous and independent of $z_i$:

$$
R \text{ Cov} [\tilde{v}_{t+1}, \tilde{Q}_{i,t+1}] \equiv -\sigma_{kv},
$$

$$
R \text{ Cov} [\tilde{\epsilon}_{i,t+1}, \tilde{Q}_{i,t+1}] \equiv -\sigma_{kz}, \quad \text{for all } z_i.
$$

(Note that it is not precluded that the R.H.S. variables of (2) be correlated.)

The result below shows that there is a solution of the fundamental difference equation (1) which is linear in the information $z_i$ available at time $t$:

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4. See Olson (1979) for more detail on the no-dividends case.
5. Ibid.
Theorem 2. Under assumptions A1-A7, a state-independent risk-free interest rate, and the state dynamics of equation (2), there exist constants $B = (B_0, B_1, \ldots, B_n)'$ such that

$$P(z_t) = B_0 + B_1 x_{1t} + \cdots + B_{n-1} x_{nt-1} + B_n d_t$$

is a solution of (1), i.e., a linear information valuation function. Moreover, $B$ is solved for by an explicit system of simultaneous linear equations; under mild regularly conditions the solution of this system will be unique.

Proof. The solution to the valuation-coefficients $B_0, \ldots, B_n$, as a function of the parameters governing the state-dynamics process, is solved by "plugging in" the linear valuation function on both sides in equation (1). The L.H.S. is linear in the state-variables, and, as it turns out, so is the R.H.S. Equation (1) will therefore be satisfied if, and only if, the two sets of coefficients for the state-variables (and intercepts) are equated. Further examination of the obtained expressions will reveal that $B$ is solved for by an $(n+1) \times (n+1)$ system of simultaneous equations. Specifically,

$$
\begin{bmatrix}
  m_{0j} & f_j \\
  \vdots & \vdots \\
  m_{nj} & f_j \\
\end{bmatrix}
= 
\begin{bmatrix}
  B_0 \\
  \vdots \\
  B_n \\
\end{bmatrix}
$$

where $m_{ij}, f_j, (ij = 0, 1, \ldots, n)$ are explicit functions of the exogenous parameters $\eta, [\theta_{ij}], \sigma_{\nu_i}, \sigma_{\epsilon_{ij}}$ and $R$:

- $m_{00} = r_f$
- $m_{0j} = 0$ \hspace{1cm} (j = 1, \ldots, n)
- $m_{0j} = -(\eta_j - \sigma_{\epsilon_{ij}})$ \hspace{1cm} (j = 1, \ldots, n)
- $m_{ij} = r_f - (\theta_{ji} - \sigma_{\epsilon_{ij}})$ \hspace{1cm} (j = 1, \ldots, n)
- $m_{ij} = -(\theta_{ji} - \sigma_{\epsilon_{ij}})$ \hspace{1cm} (i, j = 1, \ldots, n; i \neq j)
- $f_j = m_{jm}$ \hspace{1cm} (j = 0, \ldots, n-1)
- $f_n = 1 + r_f - m_{nn}$

If the rank of $[m_{ij}]$ is $n+1$ then a unique solution of $B_0, \ldots, B_n$ is ensured.

Comment. Of course, many of the $B_i$'s may be zero. Under such circumstances, it suffices to analyze subsets of $z_t$. For example, suppose $z_i = (d_{i1}, \ldots, d_{in})$ and $\tilde{D}_i = (1 + \tilde{\theta}_i + \tilde{\epsilon}_i) d_t$ for all $i$. Then it can be shown that, generally, $P_i(z_t) = P_i(d_t)$, i.e., $d_t$ is valuation-sufficient for asset $i$. In the examples below, it is implicitly assumed that the pertinent state variables are valuation-sufficient.

We shall refer to the above model as the General Linear Model (GLM).

Example 1. As a specific illustration, consider the case in which $n = 2$ and $\eta_i = \tilde{\nu}_i = \tilde{\epsilon}_{i1} = \tilde{\epsilon}_{i21} = 0$ for all $i$ and $t$. In this case, the valuation function will imply $B_0 = 0$ and explicit expressions for $B_1$ and $B_2$ are easily derived (see Ohlson (1979)):

$$B_1 = \theta_{21} R \ Det^{-1}$$
$$B_2 = [(r_f - \tilde{\theta}_{11})(1 + \tilde{\theta}_{22}) + \theta_{21}\theta_{12}] \ Det^{-1}$$

where

$$\tilde{\theta}_\nu = \eta - \sigma_{\epsilon_i}$$
$$Det = m_{11}m_{22} - m_{12}m_{21}$$
Example 2. As a second special case,\(^6\) assume
\[
\tilde{D}_{i+1} = (1 + \theta + \epsilon_{i+1})d_i
\]
and
\[
R \text{ Cov}(\tilde{\epsilon}_{i+1}, \tilde{D}_{i+1}|d_i) = -\sigma_{\epsilon}.
\]
Then \(P_i = Bd_i\), where
\[
B = (1 + \theta - \sigma_{\epsilon})/(r_p - (\theta - \sigma_{\epsilon})�).
\]
This model is actually equivalent to the expected dividend-capitalization model
\[
P_i = \sum_{\delta = 1}^{\infty} \rho^{(\gamma - 1)} \text{E}[\tilde{D}_\delta|d_i]
\]
provided that \(\rho = R(1 + \theta)(1 + \theta - \sigma_{\epsilon})^{-1}\).

7. Relationship of \(\hat{Q}_{t+1}\) to other equilibrium models.

The "summary statistic" variable \(\hat{Q}_{t+1}\) will be unfamiliar to some readers, so it is worthwhile to summarize here its relationship to other equilibrium models. It should be emphasized that these other models are simply specializations of the variable \(\hat{Q}_{t+1}\), and are in no sense "alternative" models; all arbitrage-free economies must obey the general relationships set forth in (1). The most familiar model specializing \(\hat{Q}_{t+1}\) is the CAPM. An alternative to the CAPM is Ross's linear-returns model, which is based on an asymptotic concept of arbitrage. Both of these models will be given exposition below, in order to relate their conclusions to the arbitrage-based valuation theory developed herein.

**CAPM.** It is easily demonstrated\(^7\) that the CAPM holds if and only if
\[
\frac{\text{Cov}(\tilde{R}_{t+1}, \hat{Q}_{t+1}|z_t)}{\text{Var}(\tilde{R}_{t+1}|z_t)} = \frac{\text{Cov}(\tilde{R}_{t+1}, \tilde{R}_{t+1}|z_t)}{\text{Var}(\tilde{R}_{t+1}|z_t)}.
\]
There are two known cases (in discrete time) where equation (3) is satisfied. One occurs when the \(\tilde{R}_{t+1}\) are all jointly normally distributed and \(\hat{Q}_{t+1} = \psi(\tilde{R}_{t+1}, z_t)\) is differentiable in the first argument; this case has been explored by Rubinstein (1976). The more readily apparent case occurs when \(\tilde{Q}_{t+1} = R(z_t)\lambda(z_t)\tilde{R}_{t+1} + \mu(z_t)\). In this circumstance, \(\lambda(z_t)\) is some (negative) constant, and \(-\lambda(z_t)\) is the "price of risk" in the usual sense. Substituting into (1), we have
\[
P_j(z_t) = R^{-1}(z_t)[E[\tilde{P}_{j+1} + D_{j+1}|z_t] + \lambda(z_t) \text{Cov}[\tilde{P}_{j+1} + \tilde{D}_{j+1}, \tilde{R}_{t+1}|z_t]]
\]
where
\[
-\lambda(z_t) = \frac{E[\tilde{R}_{t+1}|z_t] - R(z_t)}{\text{Var}[\tilde{R}_{t+1}|z_t]}.
\]

**Ross's model.** The one-factor version of Ross's "large markets" asymptotic arbitrage analysis assumes a linear (one plus) returns process:
\[
\tilde{R}_{j+1} = \alpha_j + \beta_j \tilde{\delta}_{j+1} + \tilde{U}_{j+1} \quad (j = 1, 2, \ldots)
\]
where \(\alpha_j = \alpha_j(z_t), \beta_j = \beta_j(z_t), \text{Var}(\tilde{U}_{j+1}|z_t) = \sigma_j^2(z_t), \text{Cov}(\tilde{U}_{j+1}, \tilde{U}_{j+1}|z_t) = 0\), and the sum of the \(\tilde{U}_{j+1}\) variables over sufficiently "large" arbitrage portfolios vanishes with probability one. Here \(\tilde{\delta}_{j+1}\) is the "commonality factor" of the securities \(j = 1, 2, \ldots\) (Note: \(j = 1, 2, \ldots\) may be a subset of the universe of all securities). It is apparent that the variables \(\tilde{U}_{j+1}\) should not enter into the valuation of assets in a Ross-type economy, since any "risk premia" associated with these variables may be arbitraged away via the law of large numbers. Hence it would seem that

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6. See also Rubinstein (1976) and Ohlson (1979) for similar developments of this model.
7. We translate (1) from price space to return space and sum over all assets. For details, see Garman (1978).
\( \hat{Q}_{t+1} = \phi(\delta_{t+1}) \), and we have from (1) the valuation relation

\[ P_r(z_t) = R^{-1}E[\tilde{P}_{r+1} + \tilde{D}_{r+1} | z_t] + Cov[\tilde{P}_{r+1} + \tilde{D}_{r+1}, \phi(\delta_{t+1}) | z_t]. \]

Indeed we shall later show that combining the assertion \( \hat{Q}_{t+1} = \phi(\delta_{t+1}) \) with the appropriate Markovian state dynamics allows us to derive the Ross model.

The above can be generalized into a multi-factor model:

\[ \tilde{R}_{r+1} = \alpha_r + \sum_{1}^{K} \beta_{rj} \hat{\delta}_{r+1} + \tilde{U}_{r+1} \quad (j = 1, 2, \ldots). \]

Then \( \phi(\delta_{t+1}) \) is replaced above by \( \phi(\delta_{t+1}, \hat{\delta}_{t+1}, \ldots, \hat{\delta}_{K+1}) \).

Comments. One might object that in the foregoing we have made assumptions regarding the distribution of returns although this function is endogenous. Therefore the system might end up being over-determined. However, the objection is not really valid. The basic problem has merely been slightly reformulated: given the return-process, is it possible to find some distribution \( F \) and valuation function \( P_r \) such that (1) is satisfied and \( \{ P_r(z_t), F(z_{t+1} | z_t) \} \) is consistent with the assumed return-processes? The point will be illustrated below; it will be shown that consistent linear solutions can in fact be developed.

In a very general sense, it is impossible to prove the existence of truly unique solutions to (1). Suppose that for any given \( F, P_r \) solves (1) uniquely. This uniqueness is spurious: \( F \) may be a proper marginal distribution of some distribution \( F \), and nothing precludes the possibility that there exists some \( \tilde{P} \) and \( \tilde{F} \) which also solves (1). In other words, the valuation proposition will not imply the existence of a unique state-space. All our analysis will therefore presume that \( z_t \) is a sufficient description of the state prevailing at time \( t \).

Unless \( \hat{Q}_{t+1} \) is specified in rather precise terms, such as an observable or identifiable quantity, the valuation proposition is without any real empirical content. This ought not to be surprising since the assumptions A1-A7 are very general indeed. An explicit endogenous derivation of \( \hat{Q}_{t+1} \) requires either a specification of preferences, wealth-distribution across investors, etc., or some restrictions on return distributions (e.g., normally distributed returns, or an infinite number of securities governed by a Ross type of \( K \)-factor generating process), or both.\(^8\)

In a finite security space, it can be shown that \( \hat{Q}_{t+1} \) is generally simultaneously determined with the \( \tilde{P}_{r+1} \) and \( P_r \) for all assets. This may seem to suggest (1) may be over-determined if the distribution of \( \hat{Q}_{t+1} \) is specified exogenously. Again, this is not the case once the proper perspective is invoked. In partial equilibrium analysis, complete specificity with respect to the distribution of returns (and prices) of assets not subject to valuation analysis is unnecessary. This follows simply because \( \hat{Q}_{t+1} \) is a "summary statistic" which depends upon the outcome of all assets in the economy. Thus, if for example \( \hat{Q}_{t+1} \) is some transform of the return on the market-portfolio, then it is still perfectly logical and consistent to assume that \( E[\tilde{R}_{m+1} | z_t] \) is state-independent (state-dependent), although it may turn out that the endogenous distribution of asset \( i (i \neq m) \) has a state-dependent (state-independent) mean. There is no inconsistency simply because nothing specific has been about the price-determination of any individual security other than asset \( i \).

The comments made above are no different from those which are usually made in distinguishing between partial and general equilibrium analysis. For example, consider the problem of deriving the value of a warrant, with positive net supply, on the market-portfolio. In partial equilibrium analysis this is done by ruling out arbitrage and specifying some exogenous market-return distribution and, say, a constant risk-free rate. In general equilibrium, the problem is quite different since previously exogenous distributions and variables now become endogenous.

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8. See the discussion by Rubinstein (1976).
The valuation theorem need not be restricted to partial equilibrium analysis, and it is just as valid in general equilibrium analysis. Implicitly the theorem was used by LeRoy (1973) to derive the value of the market-portfolio in a Markovian environment; the exogenous state-dynamics pertained to dividends. Another implicit application of the valuation theorem is an extension of LeRoy’s work by Ohlson (1977). In the papers mentioned preferences were exogenous, as well as the state-dynamics; thus \( \tilde{Q}_{t+1} \) and the return on the market portfolio were jointly determined.

At a subsequent point in this paper it will be shown that a complete dynamic general equilibrium can be constructed. All assets are priced endogenously, and the aggregation of asset-returns into market-returns is solved without the creation of inconsistencies. Furthermore, it is also demonstrated that under appropriate state-dynamics assumptions the endogenous return-distributions are state-independent. (Note that our example 2 induces such a distribution in partial equilibrium).

For a somewhat different reason, it is always possible to make some parameters and functions exogenous in the price-determination process. Ross (1976a) has shown that there will always exist preferences which are consistent with prevailing equilibrium prices as long as there are no arbitrage opportunities. But such arbitrage opportunities are ruled out in the present analysis if \( Q_t \geq 0 \) with probability one. As a practical matter, this means for instance that if one assumes that \( \tilde{Q}_{t+1} = \phi(\tilde{R}_{t+1}|z_t) \) for some nonnegative exogenous function \( \phi() \), then there will always exist preferences consistent with the implied equilibrium model. Thus, although different functions \( \phi() \) will have different implications with respect to \( E(\tilde{R}_{t+1}|z_t) \) and \( \text{Var}(\tilde{R}_{t+1}|z_t) \) this need not concern us overly. No matter what these two functions look like, preferences will always exist such that these functions are exactly those that must prevail in equilibrium. This point is illustrated in the latter part of the next section where we develop a complete, consistent dynamic equilibrium.

8. A complete dynamic equilibrium

Suppose that for all assets \( j = 1, 2, \ldots \), the state-dynamics is given by
\[
\tilde{D}_{t+1} = (1 + \theta_j + \omega \tilde{\delta}_{t+1} + \bar{\epsilon}_{t+1}) d_{ij}
\]
where \( \tilde{\delta}_{t+1}, \tilde{\delta}_{t+1}, \tilde{\delta}_{t+1}, \ldots \) are serially and contemporaneously independent with zero means. Without loss of generality, put \( \text{Var}(\tilde{\delta}_{t+1}) = 1 \).

Suppose next that \( \phi \) is some nonnegative function such that
\[
R \text{Cov}(\tilde{\delta}_{t+1}, \tilde{\delta}_{t+1}) = -\gamma < 0.
\]
It will also be assumed that \( \tilde{Q}_{t+1} = \phi(\tilde{\delta}_{t+1}) \), \( E\tilde{Q}_{t+1} = R^{-1} \) independent of \( z_t \).

The state at time \( i \) is given by \( z_t = (d_{1}, d_{2}, \ldots) \). A general linear solution \( P_\theta \) can then be shown via the GLM to be reduced to \( P_{\theta}(z_t) = B_{\theta} d_{ij} \) (all \( j \)) with
\[
B_{\theta} = (1 + \theta_j + \gamma \omega)/(\theta_j + \gamma \omega).
\]
Hence, it is convenient to add to the regularity condition
\[
(\theta_j + \gamma \omega) >> 0 \text{ for all } j.
\]
The solution will imply that returns are governed by a one-factor model:
\[
\tilde{R}_{t+1} \equiv (\tilde{P}_{t+1} + \tilde{D}_{t+1})/P_t = (B_{\theta} + 1)\tilde{D}_{t+1}/P_t
\]
\[
= (B_{\theta} + 1)(1 + \theta_j + \omega \tilde{\delta}_{t+1} + \bar{\epsilon}_{t+1})/B_{\theta}.
\]

9. In LeRoy (1973) \( \tilde{Q}, Q_t \) is proportional to the payoff on the market portfolio; in Ohlson (1977) it is proportional to the return on the market portfolio. Stated somewhat differently, the equilibrium conditions are, respectively,
\[
|E(\tilde{R}_{t+1}|z_t)| - R|/\text{Var}(\tilde{R}_{t+1}|z_t) = \text{constant}_1 P_m,
\]
\[
|E(\tilde{R}_{t+1}|z_t)| - R|/\text{Var}(\tilde{R}_{t+1}|z_t) = \text{constant}_2 P_m.
\]
The constants are preference-dependent; \( P_m \) is the value of the market portfolio.

10. Cf. Example 2 of Section 6. Note that \( \epsilon_{t+1} \) in that example corresponds to \( \omega \tilde{\delta}_{t+1} + \bar{\epsilon}_{t+1} \) in the state-dynamics model described here.
or

$$\hat{R}_{t+1} = \alpha_j + \beta_j \hat{\delta}_{t+1} + \hat{U}_{t+1}, \quad (j=1,2,...)$$

where

$$\alpha_j = (1 + \theta_j) R (1 + \theta_j - \gamma \omega_j)^{-1} \{ (B_j + 1) B_j^{-1} (1 + \theta_j) \}$$

$$\beta_j = \omega_j R (1 + \theta_j - \gamma \omega_j)^{-1} \{ (B_j + 1) B_j^{-1} \omega_j \}$$

and

$$\hat{U}_j = R (1 + \theta_j - \gamma \omega_j)^{-1} \hat{\epsilon}_j$$

so that

$$\text{Var} (\hat{U}_j) = R^2 (1 + \theta_j - \gamma \omega_j)^{-2} \text{Var} (\hat{\epsilon}_j) = \{(B_j + 1)^2 B_j^{-2} \text{Var} (\hat{\epsilon}_j)\}.$$

The return-distribution is state-independent for all assets; in this particular example one has \(\alpha_j, \beta_j\) and \(\text{Var} (\hat{U}_{t+1})\) state-independent.

One can further show without difficulty that the above implies that

$$\alpha_j = R, \quad (j = 1,2,...).$$

The residual variance is therefore irrelevant in equilibrium. This is, of course, precisely what one would expect since any other relationship would be inconsistent with Ross's analysis.

It is important to note that the one-factor model was derived from the exogenous state dynamics and the function \(Q_i = \phi(\delta_{t+1})\). It is therefore of some theoretical interest to note that if \(\gamma = 0\) then \(\alpha_j = R\).

One can verify that the CAPM also obtains, i.e.,

$$\frac{\text{Cov} (\phi(\hat{\delta}_{t+1}), \hat{R}_{t+1}|z_i)}{\text{Cov} (\phi(\hat{\delta}_{t+1}), \hat{R}_{t+1}|z_i)} = \frac{\text{Cov} (\hat{R}_{t+1}, \hat{R}_{t+1}|z_i)}{\text{Var} (\hat{R}_{t+1}|z_i)}.$$

Proof is as follows: Let \(w_i(z_i), i = 1,2,...\), be the relative value of asset \(i\) at time \(t\). The law of large numbers will imply that\(^{11}\)

$$\sum_{i=1}^{\infty} w_i \hat{U}_{i+1} = \sum_{i=1}^{\infty} w_i \hat{\epsilon}_{i+1} = 0.$$

It follows that

$$\frac{R \text{Cov} (\phi(\hat{\delta}_{t+1}), \hat{R}_{t+1}|z_i)}{R \text{Cov} (\phi(\hat{\delta}_{t+1}), \hat{R}_{t+1}|z_i)} = \frac{\gamma \beta_j}{\gamma \sum_i w_i \beta_j} = \frac{\beta \sum_i w_i \beta_j}{(\sum_i w_i \beta_j)^2} \frac{\text{Cov} (\hat{R}_{t+1}, \hat{R}_{t+1}|z_i)}{\text{Var} (\hat{R}_{t+1}|z_i)},$$

where the last and first equality use the law of large numbers.

It is also easily shown that the "capital market line" is constant across different states and dates:

$$E[\hat{R}_{t+1}|z_i] = R = \gamma \sqrt{\text{Var} (\hat{R}_{t+1}|z_i)}.$$

However, note that \(E[\hat{R}_{t+1}|z_i]\) and \(\text{Var} (\hat{R}_{t+1}|z_i)\) are state-dependent; it follows because the market portfolio weighs (relative to market values) are state-dependent.

The above development can also be related to preference-based theories of equilibrium. The "consensus" investor whose demands for risky assets are consistent with a constant capital

11. Using more precise notation we should write \(w_i(z_i), \ i = 1, \cdots, n, \ n = 1,2,\ldots\), where, thus, \(w_i\) is the relative market value of the \(i\)th asset given that there are \(n\) assets in the economy. The law of large numbers then implies that \(\lim_{n \to \infty} \sum_{i=1}^{n} w_i \hat{U}_{i+1} = 0\) with probability one.
market line, given \( R(z) = R \), has the following state-dependent "derived" expected utility function at time \( t \):

\[
\max_p \mathbb{E}[V(R_p(\hat{Z}_{t+1}); \hat{Z}_{t+1}) | \bar{z}] = \max_p \mathbb{E}[E(\hat{R}_{t+1}) - \frac{1}{2} \text{Var}(\hat{R}_{t+1}) \{ \text{Var} \{ \hat{R}_{t+1} | \bar{z} \} \}^{-1/2}] 
\]

where \( G \) is some monotonically increasing function and \( p \) denotes a portfolio. Due to state-independence, the distribution of any portfolio \( p \) does not depend upon \( z \). However, the distribution of \( \hat{R}_{t+1} \) does depend upon \( z \), since the equilibrium weights will depend upon \( z \).

Aside from the state-independence result, the above model can be generalized into a \( K \)-factor model with (or without) additional state variables. Most of the results will carry through after appropriate modifications. Thus, in the GLM, suppose the random components can be written as

\[
\tilde{e}_{it} = \underbrace{\sum_{k=1}^K \omega_{ik} \delta_{it}}_{\text{pure noise}_{it}} + \underbrace{\sum_{j=1}^n \omega'_{ij} \delta_{jt}}_{\text{not pure noise}_{it}}
\]

where \( \omega_{ik} \) and \( \omega'_{ij} \) are fixed constants and \( i \) is an index for asset \( i \); set \( \text{Var}(\delta_{it}) = 1 \). Suppose further \( \phi(\delta_{1t+1}, \ldots, \delta_{Kt+1}) = Q_{t+1} \), let \( R \text{ Cov}(Q_{t+1}, \delta_{it}) = -\gamma_i < 0 \). The state at time \( t \) is now conveniently written as

\[
z_t = (z_{1t}, z_{2t}, \ldots)
\]

where

\[
z_t = (x_{1t}, \ldots, x_{K-1t}, d_t).
\]

Again, \( I \) indexes the assets. Given these assumptions and some mild regularity conditions it can be shown that in equilibrium

\[
\hat{R}_{t+1} = \alpha_y + \sum_{k=1}^K \beta_{yt} \tilde{z}_{t+1} + \hat{U}_{t+1}.
\]

the coefficients \( \alpha_y, \beta_{yt} \) and \( \text{Var}(\hat{U}_{t+1}) \) are all state-dependent; the distribution of returns is thus no longer necessarily state-independent. It will also follow that \( P_t(z_t) \) is linear in \( z_t \). The functions \( \alpha_y(z_t) \) and \( \beta_{yt}(z_t) \) are ratios of linear functions.

It is again straightforward to show that

\[
\alpha_y = -R - \sum_{k=1}^K \gamma_k \beta_{yt} \quad (t = 1, 2, \ldots)
\]

for all \( z_t \). This merely verifies that the equilibrium conditions are consistent with Ross's result.

Finally, a constant capital-market hyperplane can be derived, and the induced utility function of the consensus investor can be stated. This is done below.

If the returns follow a one factor model, then the capital market line remains the same from period to period; this holds regardless of whether the dimension of the vector \( z_t \) is one or greater. State-independence of returns is not essential. To see this, simply note that

\[
-R \text{ Cov}(\phi(\delta_{1t}), \hat{R}_{t+1} | z_t) = \gamma \sum_{j=1}^n w_j(z_t) \beta_j(z_t) = \gamma \sqrt{\text{Var}(\hat{R}_{t+1} | z_t)},
\]

and where the far L.H.S. also equals \( E[\hat{R}_{t+1} | z_t] - R \) as a result of (1).

A similar result, entailing an intemporal constant capital market hyperplane, can be derived for the general \( K \)-factor \( n \)-variable model. Let \( \hat{R}_{k+1} \) be the return on \( k = 1, 2, \ldots, K \) "fully diversified" portfolios such that

\[
\text{Cov}(\hat{R}_{k+1}, \hat{R}_{k' + 1} | z_t) = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}
\]
for all \( k, k' = 1,...,K \). Then it follows that

\[
E[\tilde{R}_{kt+1} | z] - R = -R \text{ Cov}(\tilde{Q}_{it+1}, \tilde{R}_{kt+1} | z)
\]

\[
= -R \text{ Cov}(\phi(\tilde{\delta}_{1t+1}, \ldots, \tilde{\delta}_{Kt+1}), (Var[\tilde{R}_{kt+1} | z])^{-1/2} \tilde{R}_{kt+1} | z) \sqrt{Var[\tilde{R}_{kt+1} | z]}
\]

\[
= \gamma_k \sqrt{Var[\tilde{R}_{kt+1} | z]},
\]

since \( Var(\tilde{\delta}_{kt+1}) = 1 \) and \( \tilde{R}_{kt+1} \sqrt{Var(\tilde{R}_{kt+1} | z}) = \tilde{\delta}_{kt+1} \); \( \gamma_k \) is as defined above and, thus state-independent. The ex ante investment opportunity set, characterized in terms of the expected return on a portfolio and the standard deviations of the \( K \) uncorrelated and fully diversified portfolios, remains unchanged from one period to the next. The result is obvious if the joint return distribution is state-independent; the more remarkable fact is that it does not require or imply state-independence.

The demands for risky assets, or, in this case, basis portfolios, of the consensus investor can be obtained from derived expected utility functions given by (at time \( t \)):

\[
\max_{\rho} E[V(R_p(\bar{Z}_{it+1}); \bar{Z}_{it+1}) | z_t]
\]

\[
= \max_{\rho} G[\sum w_k E[\tilde{R}_{kt+1} | z] - \frac{1}{2} \sum_{k=1}^{K} \gamma_k w_k^{-1}(Var(\tilde{R}_{kt+1} | z))^{-1/2} Var(w_k \tilde{R}_{kt+1} | z)]
\]

where \( w_k \) are portfolio-weights and \( \sum w_k(z_t) = 1 \).

To summarize this section, then, we have demonstrated the following

**Theorem 3.** Suppose the number of assets is countably infinite. Let \( z_t \equiv (z_{1t}, z_{2t}, \ldots) \) where \( z_t \equiv (x_{1t}, \ldots, x_{nt-1}, d_t) \). Assume that for all (assets) \( l \), the vectors \( z_t \) are generated by the linear dynamics (2). Suppose that the random components in (2) have \( K \) "commonality factors", \( (\tilde{\delta}_{1t}, \ldots, \tilde{\delta}_{Kt}) \) across all assets. Let \( \tilde{Q}_{it+1} \equiv \phi(\tilde{\delta}_{1t}, \ldots, \tilde{\delta}_{Kt}) \) for some nonnegative function \( \phi(.) \) with \( R \text{ Cov}(\tilde{\delta}_{kt+1}, \tilde{Q}_{it+1} | z) = -\gamma_k < 0 \) independent of \( z_t \); \( R \) is assumed state-independent.

Then there exists a general equilibrium solution such that prices are linear in the state variables with \( P_j(z_t) = P_j(z) \). The returns will be governed by the \( K \)-factor model

\[
\tilde{R}_{yt+1} = \alpha_y + \sum_{k=1}^{K} \beta_k \tilde{\delta}_{kt+1} + \bar{U}_{yt+1}
\]

for all \( l \) and \( z_t \). Also, \( \alpha_y = \alpha_j(z_t) = \alpha_j(z) \), \( \beta_{yt} = \beta_{yt}(z) = \beta_{yt}(z) \), and these functions can be solved for explicitly. Similarly, the distribution of \( \bar{U}_{yt+1} \) can be derived from the state dynamics and the state at time \( t \).

In equilibrium we have

\[
\alpha_y - R = \sum_{k=1}^{K} \gamma_k \beta_{yt} \quad \text{(for all } z_t \text{ and assets)}.
\]

Moreover, the capital-market hyperplane will be constant and independent of \( z_t \). A state-induced utility function of a consensus investor consistent with the equilibrium conditions exists. Finally, the equilibrium conditions themselves neither imply nor require that the investment opportunity set be state-independent.

**9. Summary and Implications for Empirical Research**

It is important to understand that the structure of the valuation equations, as stated in Sections 5 and 6, have little if any empirical content until the "economy-wide" variable \( \tilde{Q}_{it+1} \) variable is identified or specified. Hence, the first two theorems reveal something about the logical structure of valuation functions, but they do not, as such, yield empirically testable
implications. The third theorem specified (exogenously) $\hat{Q}_{t+1}$ in terms of identifiable variables of the economy. It was then seen that the analysis yielded a fairly rich set of implications. Theorems 1 and 2 therefore illustrate the considerable generality of our information/valuation methodology, while Theorem 3 illustrates the significance of an identifiable $\hat{Q}_{t+1}$ variable.

More generally, from the point of view of empirical research it is probably most fruitful to regard the specification of $\hat{Q}_{t+1}$ as a "maintained hypothesis". One such maintained hypothesis which has been employed by a sizable body of research in accounting is the CAPM. As was shown, this case is a special one of the general theory developed. Using the CAPM as an equilibrium condition, some of the more well-known empirical results can be deduced as analytical propositions. In Ohlson (1979) it is shown that the model described in Example 1, combined with the CAPM, allows for a derivation of the following hypotheses:

- Returns of equity assets are affected by the information effects of unexpected changes in dividends. This is in conformity with empirical results of Fama-Fisher-Jensen-Roll.
- The "market-beta" can be derived from the information dynamics in such a fashion that the market-beta equals the "accounting-beta". In the notation of Example 1, $\sigma_{\mu t}$ (exogenous) will be equal to $\text{Cov}(\hat{R}_t, \hat{R}_m) / \text{Var}(\hat{R}_m)$ (endogenous).
- Assets with high expected earnings growth will also have high market risk. That is, the market-beta is increasing in the parameter $\theta_{1t}$.

Whether other empirically testable propositions can be derived is an inviting question. The question is intrinsically related not only to the $\hat{Q}_{t+1}$ variable, but also to the identification and enumeration of the state variables and the prior assumptions about their dynamics. The previously stated propositions required assumptions above and beyond those employed in Example 1. In general, it appears quite necessary to make additional assumptions: (primitive) state variables would need some kind of empirical "labeling" (such as "accounting-earnings") if plausible restrictions are to be imposed on their stochastic process. Also, of course, the state variables themselves will need such empirical labeling if the implied model is to have a sufficiently rich empirical interpretation.

The above discussion raises the issue as to whether there are solutions to (1) other than the linear ones developed in Theorem 2. In Ohlson (1978a) it is shown that the solutions of Theorem 2 are special cases of a broader class of stochastic processes describing the information dynamics. Empirical studies will therefore have to deal with the difficult problem of selecting among alternative information-dynamics models. (Of course, this has always been the case.) Unfortunately, there is no reason to suspect the valuation functions to be robust with respect to these: it is perfectly possible that small changes in the information dynamics (or a slight empirical miss-specification) might lead to significant changes in the structure of asset prices. The last point is fairly obvious if one simply recalls that the solutions may not even be unique.

Another way of viewing the preceding discussion is that the valuation theory developed here is one of considerable generality. In fact, we would argue that all existing theories must be, and are, consistent with Theorem 1. The latter point can be given a pertinent illustration. A paper by Beaver and Morse (1978), presented at the same conference as this paper, considers valuation equations on the basis of a "permanent earnings" concept and combines this with alternative ARIMA ("Box-Jenkins") processes characterizing the time-series behavior of accounting earnings. It can be demonstrated (Ohlson (1978b)) that this approach is a special case of Theorem 2; the Beaver-Morse valuation equations follow by solving the appropriate linear equations. Indeed all ARIMA processes (univariate or multivariate) can be given a representation in a form consistent with the linear processes (2). All these empirically popular processes will therefore have linear valuation functions, and so are subsumed by the results of the present paper.
REFERENCES


