Research Program in Finance
WORKING PAPER SERIES

WORKING PAPER NO. 75
PERFORMANCE MEASUREMENT
AND
PERFORMANCE ATTRIBUTION
by
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PERFORMANCE MEASUREMENT AND PERFORMANCE ATTRIBUTION

Barr Rosenberg

Investment performance may be attributed to three major components: systematic return obtained from a passive mixture of the risk-free asset and the market portfolio of risky assets; return from market timing due to variation of the investment proportion in the market portfolio, and return from asset selection, due to risky asset holdings that differ from market proportions.

For simplicity, the risk characteristics of the portfolio (beta and residual standard deviation) are assumed known. Five problems are then addressed. First, how is each component of return defined in a given period? Second, what adjustment should be made for the disutility of the risk in the investment strategies? Third, what measures of the manager's ability should be used? Fourth, how should these measures be compounded over a series of investment periods? Fifth, what are the statistical properties of the accumulated performance measures, as the performance history lengthens?

I. INTRODUCTION AND SUMMARY

In "Components of Investment Performance" [1972], Fama considered the problem of "distinguishing the part of an observed return that is due to ability to pick the best securities of a given level of risk (selectivity) from the part that is due to prediction of general market price movements (timing)." The chosen exposure

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1 Professor of Business Administration, University of California, Berkeley. Michel Houklet's contributions to clarifying and programming the methods developed here and Ellen McGibbon's help in preparing the manuscript are gratefully acknowledged. I would also like to thank Hal Arbit for encouraging me to explore this topic. The material in this paper was part of the talk, "Performance Measurement and Performance Attribution," given at the Seminar on the Analysis of Security Prices, University of Chicago, May 1978.
to market price movements was established as the sum of a normal or "target" exposure, chosen by the investor, and an active shift due to the manager's market timing. Thus, three components of return emerged: the systematic return due to the investor's normal exposure, the return due to the manager's market timing, and the return due to the manager's asset selection. Fama further broke down systematic return into expected return, plus the surprise due to "market conditions." These aspects of Fama's decomposition are retained in this study.

Fama also suggested "methods for measuring the effects of foregone diversification," which took the form of adjustments to experienced return for the market return that would have been obtained from a passive portfolio at the same risk level. This study continues along the same path, seeking investor's disutility resulting from the risk of each component, which can be deducted from experienced return to obtain a net "contribution to utility." Disutility adjustments are developed in the present paper, under the fairly mild assumption that the investor's preferences may be represented by a linear mean/variance utility function, in the region of the optimal passive portfolio.

The history of performance measurement began with a series of proposals for unitary performance measures by Treynor [1965], Sharpe [1966], and Jensen [1968]. This paper points out that none of these single measures is suitable for the decision problem of
the investor who considers diversifying his portfolio across more than one manager. In fact, no single performance measure can suffice. Instead, two distinct measures of performance must be employed: first, the net contribution to utility, expressed as a "certain equivalent increased return" (CEIR) from active management; second, the information ratio (IR)—the ratio of incremental expected reward from any component of active management to the standard deviation of that component. CEIR measures the amount that the manager has contributed to a particular investor, under a particular arrangement. IR summarizes the opportunities available from the active manager (the attainable active frontier), and is the correct variable for the investor to consider when deciding what fraction of his portfolio to entrust to a manager.

The next matter of concern is the extension of a single-period model to a multiperiod context, in which a series of single periods follow one another in sequence. The problem is that, whereas within a single period the components of return are additive, when successive periods are concatenated, the returns are compounded in a multiplicative fashion, so that the logarithms of successive returns (rather than the returns themselves) are additive. This could be escaped by shortening the investment period to be infinitesimal, so as to allow continuous rebalancing (Jensen [1969]). However, this strategy is rejected because it results in performance benchmarks that cannot be achieved in actual practice,
due to prohibitive transaction costs. Hence, the problem of com-
ounding finite returns must be attacked directly. A logically con-
cistent approach, which attains sensible relationships between
single-period and multiperiod components of return, involves the
transformation of the monthly components to logarithms.

Throughout the analysis, it is assumed for simplicity that
the crucial risk characteristics of the portfolio (the systematic
risk coefficient, or $\beta$ (beta) and the residual standard devia-
tion, or $\omega$ (omega) are known. For purposes of application, this
assumption is overly strong: it would be more appropriate to as-
sume that estimates of these constructs, subject to error, were
provided by a risk model. However, the assumption is a convenient
fiction, for it simplifies the presentation enormously. When risk
is known, the usefulness of a history of portfolio returns attrib-
uted to components is to estimate "true" or "underlying" values of
the performance measures, CEIR and IR, for market timing and for
asset selection. Hence, statistically efficient estimates of CEIR
and IR must be found.

The construction of these estimators depends on whether,
when the known elements of risk changed, the expected reward, or IR,
changed. Two interesting hypotheses concerning the pattern of change
are proposed, and the consequent estimators are derived, along with
their precision. The approach also yields test statistics for above-
average performance.
2. COMPONENTS OF RETURN

Consider a time interval of finite duration, such as a month. (In what follows, the interval will be referred to as a "month," although it might be longer or shorter.) The portfolio value at the beginning of the month is unambiguous, and we assume that the systematic risk coefficient, $\beta_p$, and residual standard deviation of the portfolio, $\omega_p$, are also known.

Let $i_F$ denote the risk-free rate of return for the month and $i_M$ denote the rate of return on the market portfolio of risky assets. Consider a portfolio in which the proportion $c$ is invested in the market portfolio, and $1-c$ is invested in the risk-free asset. The portfolio rate of return is, by construction, a weighted average of the market return and the risk-free rate,

$$i_P = (1-c)i_F + ci_M = i_F + c(i_M - i_F).$$

The portfolio returns the risk-free rate plus the excess return on the market portfolio $(i_M - i_F)$ multiplied by $c$. Its beta is therefore equal to $c$, and its residual standard deviation is zero. It is the "levered market portfolio" with beta equal to $c$. As $c$ varies, a family of these portfolios is obtained.

Fama developed the basic notion that this family of portfolios "provides a convenient and somewhat familiar set of naively selected or benchmark portfolios against which the investment performance of managed portfolios can be evaluated." In
other words, the performance of a managed portfolio can be assessed by reference to an appropriate levered market portfolio.

The family of benchmark returns is specified by the choice of risk-free rate and the definition of market portfolio. A natural choice for the risk-free rate is the return on a passively managed portfolio of liquid short-term obligations or "cash equivalents." The market portfolio of risky assets may be represented by a widely recognized, capitalization-weighted index, for which "index funds" exist. The chosen risk-free rate does not fulfill the requirements of capital market theory, since nominal rather than real returns are used, and the true risk-free asset provides guaranteed purchasing power. Nor do we have the true market portfolio, since all indexes and index funds are still selective within the class of financial assets, which themselves are only one category of risky assets. But these deficiencies are somewhat offset by the fact that the benchmark investment opportunities are available to most investors. Any investor can choose to divide his portfolio between positive holdings in the two funds so as to obtain any beta between zero and one, inclusive. And, although borrowing at the stipulated "risk-free rate" is not available for all investors, it is available for some. Furthermore, few among the pension sponsors and other institutions who are the major users of performance analysis systems choose to be 100 percent
invested in common stocks, where the unavailability of borrowing importantly limits the attainability of benchmark returns.  

To define the return for the month, two problems must be dealt with: the treatment of dividends and other cash flow within the month and the treatment of transactions within the month. Fama ([1972], pp. 562-565) considers the essential problem of adjusting the benchmark returns to be consistent with the experienced portfolio return, in view of the enforced transactions arising from contributions and withdrawals. For present purposes, we simply assume that the portfolio and market returns for the month are consistently defined so as to allow a fair comparison.

A second problem is to define portfolio risk ($\beta$ and $\omega$) so as to be representative of the month as a whole and to reflect correctly within-month transactions. Again, we assume that this problem has been dealt with, so that the portfolio risk measures can be assumed to be stationary within the month.

With these preambles completed, consider the decomposition of the monthly portfolio return into its three components. First, the investor's normal portfolio beta must be defined. This specifies the passive portfolio that would be held if managers' special abilities were not available. Assume that the normal portfolio is

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2 However, for investors interested in high-beta strategies and also precluded from borrowing, benchmarks reflecting the necessary residual risk of maximally diversified high-beta strategies would be preferred.
chosen from the benchmark family. The method by which the choice is made is immaterial. But it is important to assume that the choice is optimal for the investor, for some expected market return of \( i_F + \mu_M \), and variance of market return \( \sigma_M^2 \). Thus, \( \mu_M \) is the implicit investor's expectation for the excess market return (in the absence of the manager's special information that may lead to market timing), and \( \sigma_M^2 \) is the (known) level of market variance.³ Let \( \beta_N \) denote the beta of the portfolio chosen from the benchmark family, or "normal" beta.

Define "excess return" as the return in excess of the risk-free rate. Let "r" denote an excess return, so that \( r = i - i_F \). For example, \( r_M = i_M - i_F \). Since expected total market return is \( i_F + \mu_M \), expected excess market return is \( \mu_M \). The levered market portfolio's excess return always equals beta times the excess return of the market; therefore, the expected excess return of the benchmark portfolio is \( \beta_N \mu_M \). This is the first component of returns:

(1) Expected excess return at normal beta: \( \beta_N \mu_M \).

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³These expectations of mean return and variances of return can be expressed as rates per unit time, for any time period. The time period used is not essential, as long as it is used consistently. For practical use, an annual rate is most widely recognized, but monthly rates could equally be used.
The experienced market return does not ordinarily equal the expectation. The difference is the "surprise" in market return, the element which Fama calls "market conditions." By definition, this equals \( i_M - E(i_M) \). It also equals \( r_M - u_M \), the difference between experienced and expected excess market return, since the risk-free rate appears in both terms and cancels out of the difference. The normal portfolio experiences \( \beta_N r_M \) as excess return, and has \( \beta_N u_M \) as expected return, so the portfolio surprise equals \( \beta_N \) times the market surprise:

\[
(2) \quad \text{Surprise at normal beta: } \beta_N (r_M - u_M) .
\]

The portfolio beta, \( \beta_P \), elected by the manager, may not equal the normal beta. If not, then the difference is attributable either to miscommunication or to market timing. We assume the latter, so that the difference \( \beta_P - \beta_N \) is identified with the manager's active market-timing policy. The net return earned from market-timing activity in the month is simply the difference between the excess return of the chosen levered market portfolio, \( \beta_P r_M \), and the excess return of the normal portfolio, \( \beta_N r_M \):

\[
(3) \quad \text{Return from market timing: } (\beta_P - \beta_N) r_M .
\]

Finally, when the holdings of the actual portfolio differ from the levered market portfolio, experienced portfolio excess return \( r_P \) differs from the matched-beta benchmark return, \( \beta_P r_M \).
The difference, which is known as the portfolio residual return, is the consequence of asset selection:

\[(4) \quad Asset \ selection: \ r_p - \beta_p r_M.\]

This completes the decomposition of excess return. The remaining ingredient of return is the risk-free return:

\[(5) \quad Risk-free \ rate \ of \ return: \ i_F.\]

The series of equations below exhibits the decomposition of return step by step, with the order reversed from that presented in the text. The decomposition serves to separate the components of return due to active management (asset selection and market timing) from passive management (normal systematic return and risk-free rate). Fama underscores this distinction by calling the first category "manager's returns," and the second category "investor's returns." The further decomposition of passive return into the risk-free return, the expected return from risky assets, and the surprising return from risky assets underscores these distinctions.

The breakdown of portfolio return is readily extended to an investor who invests a fraction \( w \) of his aggregate portfolio with the manager. Each contribution to return is simply multiplied by \( w \), to obtain the manager's contribution to the investor's aggregate portfolio.
\[(6) \quad i_p = i_F + (i_p - i_F) + \{i_p - i_F\}\]

portfolio return risk-free rate excess return

\[
= i_F + \{(i_p - i_F) - \beta_p (i_M - i_F)\} + \{\beta_p (i_M - i_F)\}
\]

risk-free rate selectivity systematic

\[
= i_F + (r_p - \beta_p r_M) + \beta_p r_M
\]

risk-free rate

\[
= i_F + \{(r_p - \beta_p r_M) + \{(\beta_p - \beta_n) r_M\} + (\beta_n r_M)\}
\]

risk-free rate selectivity timing normal

\[
= i_F + \{(r_p - \beta_p r_M) + \{(\beta_p - \beta_n) r_M\} + \{\beta_n (r_M - \mu_M)\} + \beta_n \mu_M\}
\]

risk-free rate selectivity timing surprise expectation at normal at normal
3. COMPONENTS OF RISK

Total portfolio variance will be denoted by \( \sigma_p^2 \). One part of this is systematic variance due to market return, equal to \( \beta_P^2 \sigma_M^2 \); notice that this is proportional to beta squared. The remaining variance, not due to market return, is residual variance, \( \omega_p^2 \), due to asset selection.

Next, systematic variance is apportioned between the normal systematic risk level and the manager's market timing. At normal beta, the systematic variance of the surprise would be \( \beta_N^2 \sigma_M^2 \). The difference in systematic risk, \( \beta_P^2 \sigma_M^2 - \beta_N^2 \sigma_M^2 \), is the net effect of market timing.

The variances of the expectation at normal beta and of the risk-free return are zero, by definition. The complete decomposition of variance is shown in the equations below.

\[
(7) \quad \sigma_p^2 = \omega_p^2 + \beta_P^2 \sigma_M^2
\]

\[
= \omega_p^2 + (\beta_P^2 - \beta_N^2) \sigma_M^2 + \beta_N^2 \sigma_M^2 
= \omega_p^2 + (\beta_P^2 - \beta_N^2) \sigma_M^2 + \beta_N^2 \sigma_M^2 
\]

The variance contributions of normal systematic return and asset selection are always nonnegative, but the contribution of market timing may be either positive or negative. The negative contribution occurs when the manager reduces beta. The decision
is motivated by a pessimistic forecast of market return, but one important incidental benefit is to reduce risk during the interval when the lower beta is maintained.

These are the variance contributions to the portfolio and would also be the contributions to the investor's portfolio if he were 100 percent invested with the manager. However, when the investor's portfolio is apportioned among several managers, with proportion w invested with the manager in question, the variance contributions are reduced. The extent of reduction depends upon the diversification of the aggregate portfolio, with respect to the manager in question. If we assume that the information processes which motivate the other managers' active strategies are uncorrelated with the manager under analysis, it may be shown that

\[ \sigma_A^2 = \beta_A^2 \sigma_M^2 + \omega_A^2 = (1-w)\beta_o^2 \omega_p^2 + (1-w)^2 \omega_o^2 + 2(1-w)w\rho \omega_o \omega_p + w^2 \omega_p^2 \]

\[ = (1-w)^2 \beta_o^2 + w^2 \beta_p^2 + 2(1-w)w \beta_o \beta_p \sigma_M^2 + (1-w)^2 \omega_o^2 \]

\[ + 2(1-w)w\rho \omega_o \omega_p + w^2 \omega_p^2 . \]

When the information processes of the managers are independent, the cross-product terms between managers equal the product of their expectations, on average. The expected beta of the managers, that experienced, on average, over time, should be \( \beta_N \), since their average market forecast should be neutral. The expected correlation between residual returns is zero. When these substitutions are made, the expression simplifies (on average over time) to:
the contribution of the manager to normal systematic variance of the aggregate portfolio is reduced by the investment proportion \( w \), and that the other two contributions to the aggregate are reduced by the squared proportion, \( w^2 \).

The treatment of the manager's contribution to aggregate risk can be generalized to allow for dependency of this manager's information process with those of other managers, and for arrangements to coordinate managers' normal positions. Coordination of normal positions involves such things as assigning a below-normal beta as the norm for one manager and an above-normal beta to some other manager, so that the two actions are mutually offsetting and the correct aggregate beta is achieved.

4. THE INVESTOR'S DECISION CONTEXT: THE DISUTILITY OF RISK

The investor's choice of the normal beta allows an important inference concerning his preferences. Recall that, after considering the potential risks and rewards from alternative benchmark portfolios, he chose to invest fraction \( \beta_N \) in risky assets. The

\[
\sigma_A^2 = (1-w)^2 \beta_O^2 + w^2 \beta_P^2 + 2(1-w)w \beta_N^2 \sigma_P^2 + (1-w)^2 \omega_O^2 + w^2 \omega_P^2.
\]

Adding and subtracting appropriate terms in \( \beta_N \) and simplifying, one reaches the final expression:

\[
\sigma_A^2 = (1-w) \beta_N^2 \sigma_M^2 + (1-w)^2 (\beta_O^2 - \beta_N^2) \sigma_M^2 + (1-w)^2 \omega_O^2 + w^2 \omega_P^2 + w^2 (\beta_P^2 - \beta_N^2) \sigma_M^2 + w^2 \omega_P^2.
\]

The latter three terms are the contributions of the manager, as reported in the text.
mean and variance of excess market return, with respect to which this choice was optimal, were $\mu_M$ and $\sigma_M^2$, respectively.\(^5\)

Figure 1 portrays the situation graphically. The vertical axis is expected reward, and the horizontal axis is variance. The curving line represents the locus of reward/risk combinations that are offered by the benchmark portfolios as $\beta_N$ is varied. It is the "attainable frontier from passive investment." The chosen normal portfolio lies at point $N$ on this frontier, with mean excess return $\beta_N \mu_M$ and variance $\beta_N^2 \sigma_M^2$. The straight broken line slopes upward from the risk-free return through the normal portfolio. The segment between these two points rises by $\beta_N \mu_M$ while running $\beta_N^2 \sigma_M^2$ horizontally. Its slope is therefore the ratio of these two terms, the ratio of the mean excess return to the variance of return. It represents all points with the same excess-mean/variance ratio. The slope, or chosen mean/variance ratio, will be denoted by the Greek letter kappa, $\kappa$.

\[
\kappa = \frac{\beta_N \mu_M}{\beta_N^2 \sigma_M^2} = \frac{\mu_M}{\beta_N \sigma_M^2}.
\]

---

\(^5\)The choice may have been made without ever explicitly considering $\mu_M$ and $\sigma_M^2$. Actual decision problems for pension sponsors, for example, are often represented by the outcomes of simulations of returns drawn from a distribution with appropriate mean and variance. Nevertheless, different values for $\mu_M$ and $\sigma_M^2$ would have led to different simulated properties, to a different choice of $\beta_N$, and finally to a changed perception of the utility of the optimal portfolio. Hence, it is natural to represent the investor's preferences as being functions of $\mu_M$ and $\sigma_M^2$.\]
FIGURE 1

Expected reward

\[ \frac{1}{\beta} + \beta N M \]

\[ \frac{1}{\beta N M} \]

Mean/variance ratio

Indifference curve

Attainable passive frontier

\[ \frac{\beta^2}{N M} \]

Variance
The rectangular region on the graph is relevant in determining the risk disutility adjustments for active management. Notice that it is quite small. The smallness of the region is important, for we assume that the investor's preferences among portfolios in the region have a simple and regular structure: there is a linear mean/variance utility function, which implies that, in graphical terms, the indifference curves are parallel straight lines. This assumption might be quite restrictive and unreasonable if extended over the whole quadrant, but it is a natural approximation within so small a region.

The remaining line in figure 1 represents the set of portfolios that are indifferent to the chosen portfolio. It is an "indifference curve." It passes through the chosen portfolio, N, by definition. Also, it is tangent at N to the attainable frontier, with the same slope as the frontier at that point.\(^6\) Since the frontier represents a quadratic equation, it may be shown\(^7\) that

\(^6\) The indifference curve never passes below the attainable frontier. (If it did, the points on the frontier lying above it would offer higher expected reward at the same risk level, would therefore be preferred to portfolios on the indifference curve, and hence would have been selected instead of the chosen portfolio—a contradiction.) Since the indifference curve touches the frontier at N and never passes below the frontier, it must be tangent at N.

\(^7\) Let \( \mu \) and \( \sigma_p^2 \) denote portfolio mean and variance. The frontier begins at the point \( \mu = \bar{\mu}_F; \sigma^2 = 0 \) and includes all points satisfying \( \mu_p = \bar{\mu}_F + \beta \bar{\mu}_M, \sigma^2 = \beta^2 \sigma^2_M \) for some \( \beta \). It's
its slope at every point is one-half of the mean/variance ratio at every point, or $\frac{1}{2} \kappa$. Hence, the slope at $N$ is $\frac{1}{2} \kappa$. This demonstrates that the slope of the indifference curve at the point of tangency is also one-half of the chosen mean/variance ratio, or $\frac{1}{2} \kappa$.

The assumption that indifference curves within the rectangular region are straight lines permits the indifference curve to be represented by a solid straight line within the region. Outside the region, the straight line is extended as a dashed line; the dashes emphasize that the exact slope outside the region is immaterial.

Figure 2 expands the small rectangular region. Parallel indifference curves are drawn throughout the region with slopes equal to the tangent curve. Curves higher up in the figure are preferred. The slopes of all curves are $\frac{1}{2} \kappa$. The utility function which generates these indifference curves is therefore

$$\text{utility} = \text{mean (excess) reward} - (\frac{1}{2} \kappa) \text{ variance}.$$ 

The equation is $\mu = i_F + (\frac{\mu_M}{\sigma_M}) \sqrt{\sigma^2}$. The slope is therefore

$$\frac{du}{d\sigma^2} = \frac{1}{2} \frac{\mu_M}{\sigma_M^3} \sqrt{\sigma^2}. \quad \text{Substituting} \quad \sqrt{\sigma^2} = \sigma = \beta \sigma_M, \quad \text{we find}$$

$$\frac{d\mu}{d\sigma^2} = \frac{1}{2} \frac{\mu_M}{\beta \sigma_M^3} = \frac{1}{2} \frac{\beta \mu_M}{\beta^2 \sigma_M^2} = \frac{1}{2} \kappa.$$

$^8$Using $\mu$ and $\sigma^2$ to represent mean and variance, the equation of an indifference curve at some constant level of utility, $c$, is $c = \mu - \frac{1}{2} \kappa \sigma^2$. The slope of the indifference curve, $du/d\sigma^2$, is found by transferring $\frac{1}{2} \kappa \sigma^2$ to the left-hand side and differentiating: $\mu = c + \frac{1}{2} \kappa \sigma^2 \Rightarrow \frac{d\mu}{d\sigma^2} = \frac{1}{2} \kappa$. 


FIGURE 2

THE REGION OF ACTIVE MANAGEMENT ENLARGED

N = normal passive portfolio
O = optimal active strategy
B = another active strategy
E = situation indifferent to O, with same risk as N
NE = CEIR of active strategy O
NA = increased expected return from strategy O
NR = increased variance from strategy O
ND = CEIR of active strategy B
The term \(-\frac{1}{2} \kappa\) is seen to be the factor by which variance must be multiplied to make it comparable with expected reward. It is the "utility coefficient" for variance. Alternatively, when the minus sign is suppressed, \(\frac{1}{2} \kappa\) is the "disutility coefficient," which must be multiplied times variance to find the amount of disutility to subtract from expected reward. This coefficient will be represented by \(\lambda_v\):

\[
\text{utility} = (\text{mean excess reward}) - \lambda_v (\text{variance})
\]

\[
\lambda_v = \frac{\mu - \bar{R}_M}{2 \beta_{N} \sigma^2_M}
\]

Figure 2 also shows a curved line similar in shape to the previous attainable frontier but much reduced in size. This is a typical attainable frontier from active management, which gives the locus of opportunities that can be obtained by superimposing active management upon the normal passive portfolio. One of these active strategies, "0," is singled out, because it is the optimal choice, in that it lies on the highest indifference curve that can be reached along the frontier. Strategy 0 results in increased portfolio expected reward amounting to the height \(N_A\), and increased portfolio variance amounting to the distance \(N_R\).

The benefits obtained from strategy 0 are indifferent to point \(E\) lying on the same indifference curve. \(E\) is not an attainable strategy but is interesting for other reasons. \(E\) has the same
risk as the normal strategy \( N \) but increased expected reward amounting to the height \( NE \). When compared with the normal passive strategy, \( N \), it differs by a certain increase in reward of \( NE \). \( NE \) is the certain-equivalent increased reward (CEIR), which corresponds to (is indifferent to) the benefits provided by strategy \( O \). It also measures the increase in client utility, expressed in terms of certain reward, offered by strategy \( O \). Equivalently, \( NA - NE = EA \) is the disutility adjustment for risk that needs to be subtracted from the increase in expected return.

This is the solution to the problem of adjustment for the disutility of the added risk from an active strategy. Mathematically, height \( EA \) is equal to \( \lambda_v \) (the disutility of risk) times the distance \( NR \) (the additional risk of the active strategy):

\[
(10) \quad CEIR = \text{increased expected reward} - \lambda_v (\text{increased variance}).
\]

This is the formula for risk adjustment implied by the utility function (equation (9)).

The same adjustment principle would apply to any other suboptimal active strategy, such as \( B \). Height \( ND \) is the CEIR of strategy \( B \) and is obtained by formula (10): applied to point \( B \).

5. THE ATTAINABLE ACTIVE FRONTIER: THE INFORMATION RATIO AS A MEASURE OF ABILITY

For the two aspects of active management—market timing and asset selection—and for the total contribution from both sources,
one can define the "information ratio" as:

\[(11) \quad \text{information ratio} = \frac{\text{increase in expected return}}{\sqrt{\text{increase in variance}}} \]

This ratio is a measure of the ability or degree of superiority of the active manager.

The meaning of the ratio is clearest in the context of asset selection. Here, the numerator, or increase in expected return, is the mean residual return, or alpha, \( \alpha \). The denominator, the square root of the increase in variance, is the residual standard deviation, or omega, \( \omega \). Thus, the information ratio equals \( \alpha/\omega \).

For market timing, the interpretation is more subtle. As the manager generates a series of market forecasts over time, beta varies around an average value, presumably equal to \( \beta_N \). By matching changes in beta correctly with changes in the market forecast, an increase in expected return is obtained relative to the expectation \( \beta_{NM} \), that would apply if beta were held constant at the normal level. This increase is the numerator. Also, as beta varies, systematic variance varies in proportion to \( \beta_P^2 \). In some periods, variance is reduced; in others it is increased. It may be shown\(^9\) that, on average

\[^9\text{Average systematic variance over time equals average } \beta_P^2 \text{ times } \sigma_M^2. \text{ Average } \beta_P^2 \text{ may be shown from the definition of variance to equal } (\text{mean } \beta_P)^2 + (\text{variance of } \beta_P). \text{ The former term is just } \beta_N^2, \text{ the square of average beta and the risk level in the absence of market timing. Therefore, average } \beta_P^2 \text{ exceeds } \beta_N^2 \text{ by the variance of } \beta_P. \text{ The increase in average systematic variance}\]
over time, systematic variance is higher than it would be if there were no market timing. This is the average increase in variance due to market timing, and the square root is the denominator of the information ratio.

The information ratio is an intuitively pleasing measure of ability: it increases as expected reward (the numerator) increases, and it decreases as risk (the denominator) increases. However, its importance arises, not so much from its intuitive appeal, as because it determines the attainable frontier from active management. Consequently, it determines the opportunities for superior active management provided by the manager. It is the unique single measure of potential (ignoring trivial transformations) which serves this purpose.

To understand the importance of the information ratio, consider first the behavior of the manager's active contribution to the aggregate portfolio as the investment proportion \( w \) increases. From section 2, above, the contribution to expected reward from active management increases in proportion to \( w \). From section 3, above, the contribution to variance of reward increases in proportion to \( w^2 \). It turns out that as \( w \) increases, the contributions to expected reward and variance trace out a part of the manager's attainable frontier, of the form shown in figure 2. As \( w \) changes, neither the

is therefore \((\text{variance of } \beta'(\sigma^2_M))\). This is always positive when market timing is present. Enhanced between-period variance arises from the changing mean of market return identified by the superior market timer, which slightly increases the variance given here.
contribution to reward nor the contribution to variance remains constant; also, the mean/variance ratio steadily decreases. However, the information ratio does remain constant, for the numerator and denominator both increase in proportion to \( w \), to that the effect cancels out.

Increasing one's investment proportion with a manager can be thought of as exploiting his services with increasing aggressiveness. The more important aspect of changing aggressiveness is that the manager himself, by means of portfolio-management principles, can build portfolios that exploit his superior knowledge to variable degrees of aggressiveness, thus building a family of portfolios that trace out the active attainable frontier. If his investment is effectively unconstrained by restrictions on short-selling and maximum position sizes, the series of portfolios traces the exact quadratic attainable frontier. When investment constraints are present, the frontier falls below its ideal shape, but in most case studies known to me, it closely approximates the quadratic over the relevant range.

This point can be further clarified by reference to asset selection. As a superior manager moves out the attainable frontier, concentrating his portfolio more and more into the most superior stocks, \( \alpha \) increases, residual standard deviation, \( \omega \), increases, the mean/variance ratio of the active strategy falls, and the mean/variance ratio of the total portfolio (active+passive) first rises and then falls. All of these phenomena can be shown in figure 2.
The only portfolio measure that remains constant is the information ratio of active management.

6. MEASURES OF PERFORMANCE

In each month, the two aspects of active management produce components of return, the sum of which is the total active contribution. Once the month is finished, each of these three elements of return is, in an important sense, the retrospective measure of performance. It states the past contribution to portfolio value. However, if we seek predictive validity, a performance measure is required that will approach, on average, the true contribution to investor's utility or, alternatively, the true ability of the manager. The historical return is suitable for neither purpose, because it does not include the necessary information concerning portfolio risk.

Under the assumption that $\beta$ and $\omega$ are known, it is easy to construct unbiased estimates of the CEIRs. In each case, the experienced component of return is an unbiased estimate of the expected return—all that is needed is to subtract off the known risk disutility from this. The unbiased estimates are reported in table 1. The investment proportion $w$ appears when the investor's portfolio is multiply managed. In the simplest case, where all of the portfolio is under management by a single manager, $w$ equals unity and disappears from the formulas.

Unbiased estimates of the information ratio are also easily obtained. Each element of return is simply divided by its standard
deviation, resulting in a "standardized variate," with expected value equal to the information ratio. The formulas are reported in table 2.

**TABLE 1**

<table>
<thead>
<tr>
<th>Category</th>
<th>Experienced Return (1)</th>
<th>Variance Contribution (2)</th>
<th>Estimate of Utility Contribution (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset selection</td>
<td>( w(r_p - \beta_p r_m) )</td>
<td>( w^2 \omega_p^2 )</td>
<td>( \text{Col}(1) - \lambda_y \text{Col}(2) )</td>
</tr>
<tr>
<td>Market timing</td>
<td>( w(\beta_p - \beta_m) r_m )</td>
<td>( w^2 (\beta_p^2 - \beta_m^2) \sigma_m^2 )</td>
<td></td>
</tr>
<tr>
<td>Total active management</td>
<td>( w(r_p - \beta_m r_m) )</td>
<td>( w^2 (\omega_p^2 + (\beta_p^2 - \beta_m^2) \sigma_m^2) )</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 2**

<table>
<thead>
<tr>
<th>Category</th>
<th>Numerator (1)</th>
<th>Denominator (2)</th>
<th>Unbiased Estimate of the Information Ratio or &quot;Standardized Variate&quot; (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset selection</td>
<td>( r_p - \beta_p r_m )</td>
<td>( \omega_p )</td>
<td>( \text{Col}(1) \div \text{Col}(2) )</td>
</tr>
<tr>
<td>Market timing</td>
<td>( (\beta_p - \beta_m) r_m )</td>
<td>(</td>
<td>(\beta_p - \beta_m)</td>
</tr>
<tr>
<td>Total active management</td>
<td>( r_p - \beta_m r_m )</td>
<td>( \sqrt{\omega_p^2 + (\beta_p^2 - \beta_m^2) \sigma_m^2} )</td>
<td></td>
</tr>
</tbody>
</table>
The investment proportion \( w \) is omitted here, since it cancels from numerator and denominator. The denominator for market timing includes the absolute value of the beta difference, since either a positive or a negative difference results in an uncertain outcome.

The experienced utility contribution will be denoted by CEIR and the standardized variate, which is an estimate of the information ratio, by \( 2 \). It is instructive to compare them with performance measures that have been suggested previously. Fama's [1972] measures should be considered first, since these were defined for the same components of performance. To clarify this comparison, notice that the disutility debits in table 2, equal to \(-\lambda_v\) times Col(2), can be expressed in basic terms by substitution for \( \lambda_v \):

\[
(12) \quad \text{disutility debit for asset selection} = \frac{w^2}{28N_1} \mu_M \left( \frac{\omega_p^2}{\sigma_M^2} \right)
\]

\[
(13) \quad \text{disutility debit for market timing} = \frac{w^2}{28N_1} \mu_M (\beta_p^2 - \beta_N^2)
\]

To contrast Fama's debits with these, it is best to begin with market timing. The debit that Fama used, which he may not have intended as a "disutility" debit, was "the incremental expected return from the manager's decision to take on a nontarget level of risk," or

\[
(14) \quad \text{Fama's debit for market timing} = \mu_M (\beta_p - \beta_N).
\]
The two market-timing debits have some important similarities. The term \( w^2 \) in the present formula is the generalization that allows for multiple management. When \( w = 1 \), the case of single management considered by Fama, it disappears. When \( w^2 \) is removed from the disutility debit, the two debits, when graphed as functions of \( \beta_p \) appear in figure 3. They coincide at the origin (where \( \beta_p = \beta_N \)) and also have the same slopes at that point, provided \( \beta_N \) equals one. However, the debit proposed here increases in proportion to the change in variance (i.e., changes as \( \beta_p^2 \), while Fama's is linear.

FIGURE 3

market timing debit

\( \beta_N \)

\( \beta_p \)

disutility

Fama
The difference between the two debits grows as \( \beta_P \) moves away from \( \beta_N \). For a numerical example, suppose that \( w \) equals 1, so the formulas are comparable, that \( \beta_N = 1 \), and \( \mu_M = 6 \) (6% per annum expected excess market return). Then, if market timing results in \( \beta_P = .5 \), Fama's debit would be -3 (a 3% credit), while the disutility debit is:

\[
\frac{w^2}{2\beta_N} \mu_M (\beta_P^2 - \beta_N^2) = \frac{1^2}{2 \times 1} 6(.5^2 - 1^2) = -2.25
\]

Thus, the utility credit for reduced risk is 2.25 percent per annum.

In regard to asset selection, however, Fama clearly sought a disutility debit, which he referred to as "the additional return that would just compensate the investor for the diversifiable dispersion taken on by the manager." When expressed in the present notation, it becomes:

\[
(15) \quad \text{Fama's debit for asset selection} = \frac{\mu_M}{\sigma_M} \left( \sqrt{\frac{\beta_P^2}{\sigma_P^2} + \beta_P^2 - \beta_P^2} \right).
\]

This expression again differs from the disutility debit because of the absence of \( w^2 \). Also, the normalizing benchmark for systematic risk is taken by Fama as \( \beta_P \), rather than as \( \beta_N \). When \( w = 1 \) and \( \beta_P = \beta_N \), the situation is as graphed in figure 4. The two debits coincide at the origin and begin with the same slope, but the disutility increases more and more rapidly and slowly climbs ahead of Fama's formula.
In order for Fama's approach to be a disutility debt, preferences must be such that (i) all indifference curves are parallel, and (ii) all of the benchmark portfolios lying along the attainable passive frontier are indifferent, so that the frontier is itself an indifference curve. The logical inconsistency in the approach is evident: if all points on the frontier are indifferent, there will never be a normal benchmark portfolio chosen over others.

Regardless of the difference in approach, Fama's method and the present method reach similar results when \( w = 1 \) and \( \beta_N = \beta_P \), for reasons that emerge clearly in figure 5. Since the curves are required to have the same slope at the point of tangency of the attainable frontier, they tend to remain close together until one moves far from the normal portfolio. This explains the importance placed earlier upon the smallness of the region in which the mean/variance indifference curves are assumed to be linear.
FIGURE 5

INDIFFERENCE CURVES IN THE REGION OF
THE OPTIMAL PASSIVE PORTFOLIO "N"

mean
return

- + a

- + b

attainable
passive
frontier

variance

a: straight lines are linear mean-variance
utility.

b: curved lines correspond to Fama's debit.
They are parallel to the attainable systematic
frontier, which is itself an indifference curve
in Fama's approach, since all points on the fron-
tier are indifferent.

The problem with Fama's formula is that it does not adequately
"compensate the investor" for the residual risk from asset selec-
tion. The difficulty is most easily explained with an example.
Suppose that \( w=1 \), \( \mu_M=6 \), \( \beta_N=1 \) as before, and market variance
is \( \sigma^2_M=400 \) (an annual standard deviation of 20 percent). Then
the investor can obtain an excess return of 6 percent with a beta
of one, and an excess return of 7.5 percent with a beta of 1.25.
The investor chooses \( \beta_N=1 \). In so doing, he accepts the market
portfolio, with mean/variance ratio of 6/400, and thereby rejects the \( \beta = 1.25 \) portfolio, precisely because the increased reward does not justify the increased risk. In fact, at \( \beta = 1.25 \), the variance is \( (1.25)^2 \times 400 = 625 \), so that the mean-variance ratio falls to 6/500.

Suppose, now, that the manager, with 100 percent of the client's portfolio, undertakes asset selection with a residual standard deviation of \( \omega = 15\% \) per annum. Residual variance is 225, so that total variance is increased to 625, equaling the levered market portfolio with \( \beta = 1.25 \). Fama's formula asserts that the increase in expected return at \( \beta = 1.25 \), compared to \( \beta = 1.0 \), is sufficient compensation for the added risk. Thus, the adjustment is

\[
(1.25 - 1.0) \times 6\% = 7.5\% - 6\% = 1.5\%.
\]

But if this were sufficient compensation for the added risk, the investor would have been indifferent between \( \beta_N = 1.0 \) and \( \beta_N = 1.25 \). The fact that \( \beta = 1.25 \) was rejected demonstrates that 1.5 percent incremental expected return is insufficient compensation and that Fama's adjustment is too small.

How much larger the adjustment should be depends on the utility function. Our convenient assumption of linear and parallel indifference curves results in the aforementioned disutility adjustment, which in this case is:

\[
\frac{V^2}{2\beta_N} \mu_N \left( \frac{\omega_P^2}{\omega_N^2} \right) = \frac{1^2}{2.1} \times 6\% \left( \frac{225}{400} \right) = 1.69\%.
\]

If the utility function were different, then the adjustment would
also be slightly different. But the magnitude of the error is unlikely to be large, even for asset selection that is as unusually aggressive as in this example. 10

It is interesting that the two approaches to computing the debit for asset selection result in such similar outcomes. For residual variance of 225, the difference is 1.69%–1.50%, or .19%. The relative difference is little more than a tenth of the magnitude of the adjustments. For residual variance of 41, probably the upper bound of most aggregate sponsor portfolios, the Fama debit is .30% and the disutility debit is .3075%, with a relative difference of only 1/40. The relative difference becomes still smaller as residual variance enters the normal range for aggregate pension sponsor portfolios. The disutility debit is conceptually the more correct, but it is encouraging that the two approaches—although apparently so different—provide closely corresponding answers. This confirms that our approach is not greatly sensitive to the shape of the indifference curves of the utility function, as long as the change in total risk is small.

10 Typical residual variances of pension sponsor aggregate portfolios are typically below 20 at this time, in contrast to the value of 225 in the example.
On the other hand, the disutility debit changes greatly when the adjustments for $\beta_N$ or for $w^2$ are brought into play. These factors are critical in obtaining the correct debit, since $w$ adjusts for diversification with respect to the manager, and $\beta_N$ adjusts for investor's risk-aversion different from the norm. For example, if $w$ is .2—a common value in multiply managed sponsor portfolios—the debit is reduced by a factor of 25, and if $\beta_N = .5$, the debit is doubled. These factors bring into play the special features of the investors' situation. Their advantage is that the CEIR is correctly tailored for the investor; however, there is a consequent disadvantage, since the utility contribution cannot serve as a universal measure of the manager's ability.

The information ratio provides such a measure. Its importance as a theoretical construct in active management has been developed at length: Treynor-Black [1973], Brealey and Hodges [1973], Ambechtsheer [1974], Ferguson [1975], Rosenberg [1976]. However, it has not been widely used in performance measurement of active management.

Sharpe's reward-to-variability ratio [1966] is the information ratio for the total portfolio return, including passive and active components. It is an important construct and an essential property of the aggregate portfolio. But it is unsuitable for the case where variable aggressiveness in exploiting a manager's skill is considered. Since the use of multiple managers and the manager's decision to vary
aggressiveness by means of portfolio construction both create changing mixtures of active and passive management, Sharpe's measure becomes inadequate in these contexts. The other two pioneering performance measures, suggested by Treynor [1965] and Jensen [1968], do not capture the risk of active management and fail to distinguish between managers who have equal alphas but achieve them with different standard deviations of residual return.

7. PERFORMANCE MEASUREMENT OVER MULTIPLE PERIODS

The investment horizon is ordinarily very long, relative to any one month in which returns are observed. It follows that the compound return over the horizon is the essential output of investment and that monthly returns are important insofar as they contribute to it.

To extend the monthly analysis to an analysis of a sequence of months, a little additional notation is required. Let the months be indexed by the time subscript $t$, which varies from 1 to $T$, where $T$ is the total number of months. Let a capital letter $C$ preceding a symbol denote cumulative experience for all $T$ months; so that, for instance, $C_{1P}$ is the cumulative risk-free rate of return. Notation for the various experienced elements of return in each month is also needed. Let
\[ \begin{align*}
  i_N &= (1 - \beta_N) i_F + \beta_N i_M & \text{be the rate of return on the normal passive portfolio} \\
  i_B &= (1 - \beta_P) i_F + \beta_P i_M & \text{be the rate of return on the systematic portfolio with market timing} \\
  d_B &= (\beta_P - \beta_N)(i_M - i_F) = i_B - i_N & \text{be the component of return from market timing} \\
  i_P &= i_F & \text{be the experienced portfolio return} \\
  d_R &= i_P - i_B & \text{be the component of return from asset selection} \\
  d_A &= i_P - i_N = d_R + d_B & \text{be the component of return from active management}
\end{align*} \]

(17)

The following relationships connect cumulative to monthly returns:

Cumulative return on the risk-free asset:
\[ 1 + C_{i_F} = (1 + i_{F_1})(1 + i_{F_2}) \ldots (1 + i_{F_T}) \]

Cumulative return on the normal passive portfolio:
\[ 1 + C_{i_N} = (1 + i_{N_1})(1 + i_{N_2}) \ldots (1 + i_{N_T}) = (1 + (1 - \beta_{N_1}) i_{F_1} + \beta_{N_1} i_{M_1}) (1 + (1 - \beta_{N_2}) i_{F_2} + \beta_{N_2} i_{M_2}) \ldots (1 + (1 - \beta_{N_T}) i_{F_T} + \beta_{N_T} i_{M_T}) \]

(18)

Cumulative return on the systematic portfolio, including market timing:
\[ 1 + C_{i_B} = (1 + i_{B_1})(1 + i_{B_2}) \ldots (1 + i_{B_T}) \]

Cumulative return on the portfolio:
\[ 1 + C_{i_P} = (1 + i_{P_1})(1 + i_{P_2}) \ldots (1 + i_{P_T}) \]
Just as in the analysis of a single monthly return, it is natural to associate the contributions of market timing and asset selection with differences between cumulative rates of return:

Cumulative contribution of market timing:  \( C_{dB} = C_{B} - C_{N} \)

Cumulative contribution of asset selection:  \( C_{dR} = C_{P} - C_{B} \)

Cumulative contribution of active management:  \( C_{dA} = C_{P} - C_{N} \)

Now, if these cumulative contributions of active-management components were the sums of the monthly contributions that were defined previously (e.g., \( C_{dA} = d_{A1} + d_{A2} + \ldots + d_{AT} \)), then the extension of the monthly analysis to multiple periods would be straightforward. If the cumulative contribution were the product (e.g., \( 1 + C_{dA} = (1 + d_{A1})(1 + d_{A2}) \ldots (1 + d_{AT}) \)), little additional sophistication would be needed to carry out the analysis. Unfortunately, the cumulative outcome is neither of these. The remainder of this section is concerned with finding a means to cope with this problem.

The compound return is the product of the monthly total returns. Equivalently, the logarithm of the compound return is the sum of the logarithms of the monthly returns. Since summation is a simple process, with clear impact on the statistical properties of random variables, we are virtually forced to use the logarithms of returns (or logarithmic returns) as the descriptions of monthly events that are related to the compound event. Jensen [1969] recognized this important point and used the logarithmic rate of return.
per unit time, referred to as the "continuously compounded rate of return," in his performance measurement study. The logarithmic return uses the natural logarithm to the base "e," denoted by \( \ln \). We will identify an element of logarithmic return by a star. A complete set of constructs for monthly logarithmic return is defined as follows:

Logarithmic rates of return:

\[
\begin{align*}
    i_f^* &= \ln(1+i_f) \; ; \; i_m^* &= \ln(1+i_m) \; ; \; i_n^* &= \ln(1+i_n) \; ; \\
    i_B^* &= \ln(1+i_B) \; ; \; i_P^* &= \ln(1+i_P)
\end{align*}
\]

(19) Logarithmic excess returns:

\[
\begin{align*}
    r_M^* &= i_M^* - i_f^* \; ; \; r_N^* &= i_N^* - i_f^* \; ; \; r_B^* &= i_B^* - i_f^* \; ; \; r_P^* &= i_P^* - i_f^*
\end{align*}
\]

Logarithmic contributions of active management:

\[
\begin{align*}
    d_B^* &= i_B^* - i_N^* \; ; \; d_R^* &= i_R^* - i_B^* \; ; \; d_A^* &= i_A^* - i_B^* = d_B^* + d_R^*
\end{align*}
\]

Similarly, the cumulative logarithmic returns and contributions from active management are:

(20) Logarithmic cumulative returns:

\[
\begin{align*}
    C_{1M}^* &= \ln(1+C_{1M}) \; ; \; C_{1N}^* &= \ln(1+C_{1N}) \; ; \; C_{1B}^* &= \ln(1+C_{1B}) \; ; \\
    C_{1P}^* &= \ln(1+C_{1P})
\end{align*}
\]

Cumulative logarithmic contributions of active management:

\[
\begin{align*}
    C_{d_B}^* &= C_{B}^* - C_{N}^* \; ; \; C_{d_R}^* &= C_{P}^* - C_{B}^* \; ; \; C_{d_A}^* &= C_{P}^* - C_{N}^*
\end{align*}
\]

Now, the essential property of these logarithms is that they are additive over time. Each of the cumulative variables is the
summation of the corresponding monthly variables. This holds true for the cumulative logarithmic returns (the \( C_i^* \)), for cumulative excess returns (\( C_r^* \), which are not used here), and for cumulative contributions from active management (the \( C_d^* \)). For example:

\[
C_{d_A}^* = d_{A1}^* + d_{A2}^* + \ldots + d_{At}^*
\]

Not only do the contributions sum, but there is no interaction between categories. For example, \( d_{B1}^* \), which arises from market timing in period 1, does not change \( C_d^*_R \), the cumulative logarithmic contribution from asset selection. It seems self-evident that no such interaction should exist, but if we were working with proportional returns, \( d_{B1} \) would influence \( C_d^*_R \).

Also, the characteristics of the compound return are determined by the characteristics of the logarithmic information ratios in a direct way. Thus, it is not just that the logarithmic components are easy to analyze; they are also the essential determinants of compound return. If we were to try to determine the properties of the compound return from the information ratios of elements of proportional return, the process would be approximate and obscure. Implicitly, it would involve recourse to logarithmic returns and their properties, something which we do explicitly at the outset by the transformation to the logarithmic environment.

The advantages of the logarithmic form are thus apparent, and the starred variables will be used as the central measure of
multiperiod portfolio properties. The problem now is to reexamine the monthly context and find a reasonable framework in which the starred contributions, rather than the unstarred ones, emerge. Jensen [1969] used by far the neatest method in his performance measurement study, by assuming continuous portfolio rebalancing. It may be argued that, in a sense, the approach was unjustifiably neat.

Recall Fama's point that the essential function of the two-parameter model is to provide a set of benchmarks for performance. To be fair benchmarks, they should be feasible investment strategies. The essence of the argument to be developed below is that the monthly benchmarks proposed by Fama are feasible ones, but that the benchmarks implicit in Jensen's analysis are not.

The monthly benchmark portfolios used thus far are mixtures of the risk-free asset with the risky market portfolio, with investment proportions set at the beginning of the month and no transactions thereafter (other than possible reinvestment of dividends and coupons). Implicit in this method is the assumption that, at the end of the month, the portfolio is rebalanced to establish appropriate proportions for the subsequent month.

The monthly transactions involved in this rebalancing are feasible in practical portfolio management. Since rebalancing occurs as part of a passive strategy, the transactions can be represented as "informationless trades," unsupported by special information. Other investors are ready to complete such transactions without demanding
a large spread, so the trades can be effected at low transaction cost.

In this mode, the beta of the passive portfolio changes uncontrollably within the month as the market moves. For example, suppose that the initial beta is .6 and the market then falls 4% in the first week. The beta at the end of the week falls to just under .59, because the proportion in equities has shrunk. To restore beta to .6, about 4/10 of one percent of the total portfolio value would need to be converted from risk-free to risky assets. Such changes in beta are allowed to "ride" until the next month-end rebalancing.

Suppose that rebalancing were more frequent. More frequent rebalancing holds beta close to the normal value. In the limit, as rebalancing occurs continuously, beta remains effectively constant, and the relationship between portfolio and market return changes character. Under continuous rebalancing, the returns on the benchmark portfolios are naturally expressed in logarithms! As $\beta_N$ varies, one traces out the attainable frontier of logarithmic returns. The starred systematic returns and contributions of market timing become linear in beta:

$$
\begin{align*}
  r^*_B &= \beta_P r^*_M ; \\
  r^*_N &= \beta_N r^*_M ; \\
  d^*_t &= (\beta_P - \beta_N) i^*_t.
\end{align*}
$$

This relationship turns out to be extremely convenient, for the benchmark returns and contributions, linearly related within the period, become the same ones that sum between periods.
However, the linear relationship of logarithmic benchmark returns requires continuous rebalancing of the benchmark portfolios, or rebalancing that is effectively continuous. Is this practicable? The answer appears to be "no." As one rebalances more and more frequently, the total number of transactions per unit time increases without limit. Weekly rebalancing requires about twice as many transactions per month as does monthly rebalancing, and daily rebalancing requires more than twice as many again. In the present trading environment, even with the benefits accorded to informationless trades, rebalancing within the month seems to incur excessive transaction costs. Therefore, the linear relationship in logarithmic returns, although theoretically correct, is not now practically attainable.

Jensen also assumed that the residual return of the portfolio was additive in logarithmic terms, thus requiring that individual active holdings be continuously rebalanced relative to one another as individual assets change in value. This kind of rebalancing, to keep investment proportions in individual assets constant, involves

---

For example, at $\beta = .5$, monthly rebalancing requires turnover of about one-fourth of total portfolio value in a year; weekly rebalancing would require almost 100% turnover of total value, or 200% of the value of risky assets. Since all trades are exchanges of risk-free assets for risky assets, and informationless, the total cost of a trade is quite small, probably less than 1/2% of the value traded. Nevertheless, if 1/2% is used as the cost figure, the charges against total portfolio value for rebalancing are roughly .125% per annum for monthly rebalancing, .25% for weekly, and .50% for daily. In these circumstances, it is relatively easy to demonstrate that the gains in terms of improved risk/reward trade-off, when netted against transaction costs, do not favor rebalancing frequently within the month.
enormous turnover and will probably not be feasible even when computerized exchanges come into being.

If continuous rebalancing is rejected, the linear model of proportional returns, so familiar in one-period capital asset pricing theory, must be retained. This takes us back to the analysis of the first six sections. The solution is therefore to link the monthly unstarred proportional contributions—discussed in those sections—to the monthly starred logarithmic contributions that enter in the multiperiod compound analysis.

Some of the ingredients of this transformation are beyond the scope of this paper. (They are not so much difficult as tedious.) The formulas (19) allow us to compute the logarithmic contributions to return from the proportional ones, so there is no difficulty there. Inspection of the formulas for the various elements of return allow us to approximate the variances of the logarithmic contributions, given the variances of the proportional contributions. The investor's utility function in proportional returns can be transformed into a slightly different utility function for logarithmic returns, which is essentially equivalent (in the sense of implying similar decisions concerning active management).

When these transformations have all been made, we can compute the monthly \( \hat{CEIR}^* \) and \( \hat{z}^* \), just as in section 6, except that these now apply to the contributions to logarithmic return. In the concluding section, the use of these to prepare cumulative measures will be analyzed.
8. EFFICIENT ESTIMATES OF THE CUMULATIVE UTILITY
CONTRIBUTION AND INFORMATION RATIO

In this section, all constructs are assumed to be transformed so as to relate to logarithmic returns. The portfolio risk characteristics, $\omega^*_t$ and $\omega^*_t$, are assumed known in every month $t$. The problem is to estimate the mean reward from each of the components of active management and, based upon this estimate, to estimate the average monthly utility contribution, $\overline{CEIR}^*$, and information ratio, $\overline{z}^*$. Of course, these estimates are to be derived from the monthly values of the two performance measures. The question is: how are the cumulative averages to be estimated most accurately, and how precise are these efficient estimates?

The approach developed below applies equally to each component of active management, and to the total. Any of these three elements can be chosen as the case for analysis. Asset selection is the natural choice, because it is the easiest in terms of notation: $\omega^*_t$ represents the contribution to expected return in period $t$, $\omega^*_t$ is the standard deviation of residual return from asset selection, and $\omega^*_t^2$ is the contribution to total variance. The analysis for any other component is obtained by substituting for these terms the corresponding terms in tables 1 and 2.

A critical preliminary question is that of possible changes in $\omega^*_t$ over time, in the cases where $\omega^*_t$ shifts. Our understanding of the portfolio-construction problem provides guidance concerning this. For simplicity, assume that the disutility coefficient $\lambda^*_v$
(the parameter of the transformed utility of logarithmic return) remains constant. Then, the optimal strategy for the manager is to maintain the mean/variance ratio at some constant value. In other words, if he chooses to adjust $\omega_t^2$, it must be that $\alpha_t^*$ is adjusting in the same proportion. This implies that, according to his best judgment, there is some constant $k$, such that in every period

(23) \quad \text{Constant mean/variance ratio assumption: } \frac{\alpha_t^*}{\omega_t^2} = k.

The mean/variance ratio equals $k$. In this case, the unknown contribution to return is summarized by the mean/variance ratio $k$—this is what must be estimated.

There is, however, another interesting possibility. The changes in $\omega_t^2$ may be unrelated to changes in the information ratio; in fact, the information ratio or intrinsic ability may remain constant over time, while variance is changing for some other reason. In this case, we have the

(24) \quad \text{Constant information ratio assumption: } \frac{\alpha_t^*}{\omega_t^2} = z.

Here, the contribution to expected return is proportional to its standard deviation, and the constant of proportionality $z$, equal to the information ratio, is the essential item to be estimated.

A third case would have $\alpha^*$ itself constant as $\omega^*$ changes. This at first seems reasonable, but, in fact, it implies very irrational behavior: when the information ratio falls, the only way to retain
\( \alpha^* \) constant is to increase aggressiveness and \( \omega^* \). Thus, the less that is known by the manager, the more risk must be taken to "lever" exposure to that smaller amount of knowledge and keep expected total return constant. Not only is this behavior destructive to the investor's welfare, but it is clearly imprudent. Therefore, when \( \omega^* \) is changing, one hopes that \( \alpha^* \) is changing also. Of course, when \( \omega^* \) remains constant, all three of these possible assumptions are equivalent to one another. But when \( \omega^* \) changes over time, Constancy of the Information Ratio and Constancy of the Mean/Variance Ratio are the natural competitors for description of the manager's behavior.

With these two assumptions in mind, consider the cumulative utility contribution from asset selection. The contribution in each month \( t \) is:

\[
CEIR_t^* = \alpha_t^* - \lambda_t^* \omega_t^2.
\]

where \( \lambda_t^* \) is assumed constant over time. The average monthly contribution is therefore:

\[
\overline{CEIR}^* = \frac{1}{T} \sum_{t=1}^{T} \alpha_t^* - \lambda_t^* \left( \frac{1}{T} \sum_{t=1}^{T} \omega_t^2 \right)
\]

The second term is known from the known values of \( \lambda_t^* \) and \( \omega_t^* \). The problem, therefore, is to estimate the first term, or average \( \alpha_t^* \). The data from which to estimate this are the observed contributions to return, \( \alpha_t^* \), for each of the months.
Under the Constant Mean/Variance Ratio Assumption, the most precise estimate of the average alpha is the average of $\frac{d^*}{R_t}$. It is an unbiased estimator, and its standard error is $\frac{1}{T} \sqrt{\sum_{t=1}^{T} \frac{\omega^2}{T}}$. When this formula is substituted back into the formula for $\overline{CEIR^*}$, we find that the most precise estimate of $\overline{CEIR^*}$ is just the average of the monthly estimated utility contributions:

Estimation of the average utility contribution, with constant mean/variance ratio:

$$\overline{CEIR^*} = \frac{1}{T} \sum_{t=1}^{T} \overline{CEIR^*}_t,$$

where

$$\overline{CEIR^*}_t = d^*_t - \frac{\lambda \omega^2}{\nu T}$$

and

$$\text{standard error} = \frac{1}{\sqrt{T}} \sqrt{\sum_{t=1}^{T} \frac{\omega^2}{T}}$$

Under the Constant Information Ratio Assumption, the most precise estimator is a roundabout one. First, the most precise estimator of the average information ratio is found: this is simply

$$12\text{The efficient estimator is derived as follows: Average alpha equals a known factor } \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\omega^2}{T} \right) \text{ times the unknown mean/variance ratio, } k. \text{ Hence, the efficient estimator for average alpha is obtained by efficiently estimating } k, \text{ and applying the known factor. The efficient estimator } \hat{k} \text{ is found by generalized least squares for the regression model:}$$

$$d^*_R = k \omega^2 + u_t \text{, } t=1,\ldots,T, \text{ where } E(u_t) = 0, \text{ VAR}(u_t) = \omega^2 T.$$  

Appropriate substitutions result in the simple result given in the text.

$$13\text{The efficient estimator is obtained as a known multiple of the unknown parameter } z. \text{ The efficient estimator } \hat{z} \text{ is obtained}$$
the average of the standardized variates, \( \hat{z}_t^* = \frac{1}{T} \sum_{t=1}^{T} \frac{d_{Rt}^*}{\omega_t^*} \). This is an unbiased estimator with standard error equal to \( \frac{1}{\sqrt{T}} \). The estimator is then multiplied by the average \( \omega_t^* \) to obtain the average alpha. The resulting estimate for the average utility contribution is:

Estimation of the average utility contribution, with constant information ratio:

\[
\begin{align*}
\text{CEIR}^* &= \left( \frac{1}{T} \sum_{t=1}^{T} \frac{d_{Rt}^*}{\omega_t^*} \right) \left( \frac{1}{T} \sum_{t=1}^{T} \omega_t^* \right) - \lambda^* \left( \frac{1}{T} \sum_{t=1}^{T} \omega_t^{2*} \right) \\
\text{standard error} &= \frac{1}{\sqrt{T}} \left( \frac{1}{T} \sum_{t=1}^{T} \omega_t^* \right) 
\end{align*}
\]

Next, consider estimation of the average information ratio. Use as the definition of "average" the equal-weighted average of the information ratios in all months. Then, for the constant/mean variance assumption, the estimator is quite complicated:

Estimation of the average information ratio, with constant mean/variance ratio:

by generalized least squares for the regression model:

\[
d_{Rt}^* = z_{\omega_t}^* + u_t, \ t=1, \ldots, T, \ \text{where} \ \mathbb{E}(u_t) = 0, \ \text{VAR}(u_t) = \omega_t^*.
\]

\[14\] The efficient estimator is obtained from the efficient estimator for the mean/variance ratio \( k \), presented before.
\[ z^* = \left( \sum_{t=1}^{T} \frac{\omega^*_t}{T} \right) \left( \sum_{t=1}^{T} \frac{d^*_R_t}{T} \right) \]

\[
\text{standard error} = \left( \frac{T}{\sum_{t=1}^{T} \frac{\omega^*_t}{T}} \right) \left( \frac{1}{\sqrt{\sum_{t=1}^{T} \frac{\omega^*_t}{T}}} \right) \]

In contrast, for the case of a constant information ratio, the estimator is simply the average standardized variate, i.e., the average of the estimates of the information ratios from the individual months.

Estimation of the information ratio, when assumed constant:\(^{15}\)

\[ z^* = \sum_{t=1}^{T} \frac{\omega^*_t}{T} = \sum_{t=1}^{T} \frac{d^*_R_t}{T} \]

\[
\text{standard error} = \frac{1}{\sqrt{T}} \]

One crucial statistic, often sought in performance studies, is the test statistic for expected mean reward (alpha) being different than zero. This is the basis for a test of whether the manager's performance is significantly different from zero. Under both assumptions, the test statistic is the ratio of the average alpha to its standard error (or—which is the same thing—the ratio of the estimated average information ratio to its standard error):

\(^{15}\) The efficient estimator was explained in footnote 13.
Test statistics for significant superiority:

\[ \frac{\sum_{t=1}^{T} \frac{d^*_R}{R_t}}{\sqrt{\sum_{t=1}^{T} \frac{\omega^*_t}{t}}} \]

Assuming constant mean/variance ratio:

Assuming constant information ratio: \[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{d^*_R}{R_t} \frac{\omega^*_t}{t} \]

In both cases, since the variance has been assumed known, the test statistic is normally distributed, with standard deviation equal to one and mean equal to zero under the hypothesis that performance is null.
REFERENCES


