OPTIMAL DURATION OF GROWTH
INVESTMENTS AND SEARCH

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Abstract

This paper deals with the optimal duration of growth investments (e.g., when to harvest a forest, when to bottle an aging wine, etc.) under uncertainty, when the decision to terminate the investment is made sequentially. That is, at each point in time the investor inspects the economic conditions, the physical conditions (e.g., the height of the tree) and the prospects for the investment, and then decides whether to sell or retain the investment. We show that when uncertainty increases, a risk-neutral investor will wait for higher offers to come along before he sells his investment. However, the expected duration will not necessarily increase. Under the traditional definitions of increasing risk these results will not apply to risk averters. We show, however, that under some recent definitions of increasing risk, the above results apply to risk averters too.

The structure of our problem resembles a search problem. Thus, our results are also applicable (for the search interpretation the words "selling the investment" should be replaced by "stopping the search and accepting the offer") to the case of search, where the prospects of the object of search are stochastically growing over time and are subject to stochastic cyclical fluctuations.
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I. Introduction

Normative theories in finance center around investment decisions under certainty and under uncertainty. These theories provide rules for accepting or rejecting investment projects which may yield cash proceeds in the future. However, the question of optimal duration of an investment, i.e., when to terminate a given investment, has received very little attention in the literature. Works dealing with the duration problem\(^1\) usually refer to the classical examples: when to harvest a forest (cut a tree), when to bottle an aging wine, etc. With the exception of Kaplan [1972], the few published works dealing with this problem (the most recent of which are Bierman [1968]; Hartman [1976], and Hirshleifer [1970]) assume certainty.\(^2\) Kaplan [1972], following Karlin [1962], relaxed the assumption that the increase in the value of the growth asset is deterministic. He used dynamic programming techniques to compute the optimal policy when the increase in value is a random variable whose distribution depends either on the value or on the age of the asset. His framework of analysis, however, differs considerably from ours. He does not derive his solutions using the expected utility criterion, and he does not analyze the effects of changes in key parameters (such as risk) on the optimal policy and on the expected duration of the investment.\(^3\)

In this paper we investigate the optimal duration problem under uncertainty in the framework of expected utility maximization. We
provide optimal rules for the termination of investments and derive the properties of these rules, and the resulting outcomes. Our analysis is carried out under the assumption that the investor makes his decision sequentially, i.e., at each point in time he reviews the available information and decides whether to continue or terminate the investment. We show that the optimal policy of the investor is characterized by a series of critical values which are functions of the age of the investment. At each time, the investor compares the value of the investment with the critical value corresponding to that age and makes his decision accordingly. In this case the duration is not a specific number, as it is in the case of certainty, but rather, it is a random variable depending upon the outcomes at each point in time. We show that an increase in uncertainty will result in larger critical values for every age of investment, but the expected value of the duration does not necessarily grow with uncertainty. We also show that the value of the investment grows with uncertainty. The basic framework is presented in section II. Under our framework the problem of determining the optimal duration of a growth investment resembles a search problem. Thus, our results are applicable also to the case of search, where the prospects of the object of search are stochastically growing over time, and are subject to stochastic cyclical fluctuations. For the search interpretation we must replace the words "selling the investment" with "accept the offer and terminate the search." In sections III and IV we investigate the effect of increased risk on the optimal strategy of risk-neutral and risk-averse investors,
respectively. In section V we provide a numerical example supporting the propositions stated and proved in the paper.

II. A Sequential Determination of the Optimal Duration of the Investment

The optimal duration of an investment under certainty has been derived in the literature and, as expected, corresponds to the point where the proportionate increase in the investment proceeds equals the cost of money (the interest rate). The analysis of the same question under uncertainty requires additional assumptions regarding risk and preferences. Also, under certainty there is no need for a sequential decision scheme, because no new information is obtained with time (as in the case of uncertainty). Under uncertainty, however, there is no reason for the investor to commit himself in advance to deliver the product at any specific date. We assume, therefore, that the investor is involved in a series of 0, 1 decisions, i.e., at each period of time he has the option to sell or retain the investment. This option is taken into account in the determination of the optimal duration. Since this scenario is complicated, it is necessary to establish a framework that helps in deriving meaningful, yet rather general, results. The analysis is thus first carried out for the case where the investment is used only once, and the investor is risk neutral. These simplifying assumptions are made for ease of exposition, but provide some interesting results. Later, we show how the results alter for risk-averse investors.

Let \( X_t \), a random variable, denote the proceeds from the investment at age \( t \), and let the infinite vector
\[(x_1, x_2, \ldots, x_T, \ldots)\]

denote the proceeds from this investment at all periods of time. Suppose
the preferences of the investor over such infinite vectors are described
by the following utility function:

\[U(x_1, \ldots, x_T, \ldots) = \sum_{t=1}^{\infty} e^{-1T}u(x_t), \quad (2.1)\]

where \(u\) is a von Neumann-Morgenstern (VNM) utility function and \(i\) is
the interest rate.\(^5\) Under the assumption that the only proceeds \(x_T\)
from the investment are obtained at the time of termination, \(T\), the in-
vestor's infinite horizon receipts are given by the vector

\[(0, 0, \ldots, x_T, \ldots) .\]

Since \(u\) is a VNM utility function, it may be so calibrated that \(u(0) = 0\).
Thus, the utility of the proceeds is given by

\[U(0, \ldots, x_T, 0, \ldots) = e^{-1T}u(x_T). \quad (2.2)\]

Therefore, if the objective of the investor under uncertainty is to max-
imize his expected utility and his decision variable is the termination
time \(T\), then he will choose \(T\) to maximize \(e^{-1T}E[u(x_T)]\). The objective
of a risk-neutral investor is to maximize \(\beta^{T-1}E[X_T]\) (to simplify nota-
tions we assume that the proceeds are obtained at the beginning of the
period and use \(\beta^{T-1}\) instead of \(e^{-1T}\)).
We assume that there are two sources for the uncertainty in the future proceeds: (1) uncertainty regarding the general economic conditions and the physical properties of the investment; (2) conditional upon these, uncertainty regarding the future prospects of the investment. We next describe the stochastic process that generates the proceeds of the investment.

Assume that at any period of time, the economy may be in one of \( K_1 \) states: \( s'_1, s'_2, \ldots, s'_{K_1} \), where a larger indexed state represents improved economic conditions. For example \( s'_1 \) may represent a depression, \( s'_2 \) a mild depression, etc. Suppose also that the physical characteristics of the investment (e.g., the height of the tree) can be characterized by one of \( K_2 \) possible states \( s''_1, \ldots, s''_{K_2} \). Thus, there are \( K = K_1 \cdot K_2 \) possible combinations of state of the economy and physical conditions of the investment. Henceforth each such combination will be called the state of the investment and the states will be labeled by \( s_1, \ldots, s_K \). The transition from state to state follow a constant Markov chain. In other words, the probability, \( \pi_{ij} \), that the investment is in state \( s_j \) at time \( t \), given that it has been in state \( s_i \) at time \( t-1 \), is the same for all \( t \). Conditional upon \( s_i \) and \( t \), the age of the investment, the value of the investment (the proceeds) is a random variable \( X^i_t \) with a given Cumulative Distribution Function (CDF) \( F^i(x,t) \). We assume that for all \( i \) and \( t \) the expectations \( \mu^i_t = \mathbb{E}[X^i_t] \) are finite and satisfy:

\[
\mu^i_t \leq \mu^i_{t+1} \leq \ldots \leq \mu^i \leq \infty \quad i = 1, \ldots, K, \tag{2.3}
\]
and that the $X^i_t$'s converge in distribution to the random variable $X^i$, possessing the CDF $F^i(X)$. Since the $\mu^i_t$'s are bounded and monotonic, they converge. Let $\mu^i = \lim_{t \to \infty} \mu^i_t$. Assumption (2.3) states that the investment, on the average, appreciates with time (see footnote 3).

At each period $t$ the investor must decide whether to terminate the investment and get an income $x$ from selling it, or to wait for $t+1$. To arrive at his optimal strategy we introduce the following definitions:

$V_{it}(x)$ is the expected discounted proceeds from the optimal decision at age $t$, given state $s_i$ and the current offer $x$.

$W_{it} = E_{it}[V_{it}(x)]$, where $E_{it}$ is the expectation operator with respect to the CDF $F^i(x,t)$.

The expected discounted proceeds $V_{it}(x)$ from the optimal decision must be the maximum of the proceeds $x$ from terminating the investment and the discounted expected income

$$n_{it} = \beta \sum_{k=1}^{K} \pi_{ik} W_{ik, t+1}$$  \hspace{1cm} (2.4)

from continuing. Hence

$$V_{it}(x) = \max\{x, n_{it}\}. \hspace{1cm} (2.5)$$

Therefore, the optimal strategy of the investor is to inspect, at each $t$, the state $s_i$ and the current value of $x$. If $x \geq n_{it}$, he should terminate the investment. If $x < n_{it}$, he should retain it until time $t+1$. 
when he will reconsider his decision. Thus, the \( \eta_{it} \)'s are the critical
values that guide the investor in making his decision. The properties of
\( \eta_{it} \) and \( W_{it} \) are derived below.

Our problem is in fact a variation of the wage search problem
(see, e.g., Lippman and McCall [1976a, b] and Kohn and Shavell [1975]),
where the prospects of the object of search are stochastically growing
with time and are subject also to uncertainty concerning future economic
conditions. For the search interpretation the \( X_t \)'s should be inter-
preted as the price offers for a seller of some product, or wage offers
to a worker in the job market. Also the words "selling the investment"
should be replaced by "accepting the offer and stopping the search."
The physical conditions are not relevant, of course, under this inter-
pretation. For the case of job search where the \( X_t \)'s represent job offers
it is assumed that the worker stays in the same job indefinitely once he
accepts it. In this case his expected utility is \( \beta^T(1-\beta)^{-1}E[u(X_T)] \) and
the problem is equivalent to ours. We also note that our results apply
(with the appropriate modifications) to the case of a buyer searching for
a product whose price is stochastically decreasing over time (for example:
the prices of some products such as transistor radios, calculators, and
digital watches, decreased considerably close to their introduction to
the market, and leveled after a while).

III. The Effect of Uncertainty on \( \eta_{it} \) and \( W_{it} \)

We assume that the random process that creates changes in the
states of the investment remains fixed, and we investigate the effect of
increased uncertainty (dispersion) in the $X_t^i$'s, where increased uncertainty means an increase in the dispersion of each $X_t^i$ for all $i = 1, \ldots, K$, $t = 1, 2, \ldots$.

An increase in dispersion is defined in the sense of Rothschild and Stiglitz [1970]: let $\tau$ be the shift parameter, denoting dispersion, and rewrite the CDF of $X_t^i$ as $F^i(s, t, \tau)$. Define

$$F^i_{\tau}(x, t, \tau) = \frac{\partial F^i(s, t, \tau)}{\partial \tau}.$$ 

Then, an increase in $\tau$ means an increase in uncertainty if

$$\int_{-\infty}^{\gamma} F^i_{\tau}(x, t, \tau) dx > 0 \quad \text{for all } \gamma > -\infty, \quad i = 1, \ldots, K \quad (3.1)$$

$$\int_{-\infty}^{\infty} F^i_{\tau}(x, t, \tau) dx = 0 \quad \text{for all } i = 1, \ldots, K . \quad (3.2)$$

To prove the forthcoming Proposition 3.1 we use the variables $W_{it}$, $n_{it}$, and $V_{it}(x)$ for the case $\tau = \tau_1 > \tau_0$, analogous to $W_{it}$, $n_{it}$, and $V_{it}(x)$ defined for the case $\tau = \tau_0$. In the sequel we need the following lemma. (This lemma has earlier been proved by Rothschild [1974]. We provide here the proof for completeness.)

Lemma 3.1: $\frac{\partial H(s)}{\partial \tau} \geq 0$, where $H(s) = \int_{s}^{\infty} (x-s)dF(x,t,\tau)$.

Proof: By the definition of $H(s)$
\[ H(s) = \int_{s}^{\infty} (x-s) dF(x,t,r) = (\mu-s) - \int_{-\infty}^{s} (x-s) dF(x,t,r). \]

Integrating the last integral by parts yields (assuming that \[ \lim_{x \to -\infty} xF(x,t,r) = 0 \]):

\[ H(s) = (\mu-s) + \int_{-\infty}^{s} F(x,t,r) dx. \]

Thus, by the definition of increasing risk in (3.1) and (3.2), it follows that:

\[ \frac{\partial H(s)}{\partial r} = \int_{-\infty}^{s} F_r(x,t,r) dx \geq 0. \]

Q.E.D.

The main result of this section is Proposition 3.1 below. In its proof the following lemma is used.

**Lemma 3.2:** If there exists a time period \( T \), such that \( W_{it}^i > W_{it} \) for \( i = 1, \ldots, K \), then \( W_{it}^i \geq W_{it} \) for all \( t \leq T \).

**Proof:** By induction. Suppose \( W_{it}^i > W_{it} \) for \( i = 1, \ldots, K \), for some \( t \leq T \). By the definition of \( W_{it} \),

\[ W_{it} = E_{i,t,r_0}[\max(x, \eta_{it})], \tag{3.3} \]

where \( E_{i,t,r_0} \) is defined with respect to \( F(x,t,r_0) \). Hence it follows (for the proof see DeGroot [1970], p. 333) that
\[ W_{it} = \eta_{it} + H_{i,t,r}(\eta_{it}), \] (3.4)

where the function \( H_{i,t,r}(s) \) is defined by

\[ H_{i,t,r}(s) = \int_s^\infty (x-s)dF^i(x,t,r). \] (3.5)

Then

\begin{align*}
W_{i,t-1} &\leq \eta_{i,t-1} + H_{i,t,r}(\eta_{i,t-1}) \\
&\equiv E_{i,t-1,r}(\max\{\eta_{i,t-1}\}) \\
&\leq E_{i,t-1,r}(\max\{\eta'_{i,t-1}\}) = W'_{i,t-1}.
\end{align*}

The first inequality follows from Lemma 3.1, which shows that an increase in uncertainty (i.e., from \( r_0 \) to \( r_1 \)) increases the function \( H \). The existence of the identity was argued in (3.3) and (3.4). The last inequality follows from the induction assumption, since

\[ \eta_{i,t-1} = \beta \sum_{k=1}^K \pi_{ik} W_{kt} \leq \beta \sum_{k=1}^K \pi_{ik} W'_{kt} = \eta'_{i,t-1}. \]

This completes the proof. \( \quad \) Q.E.D.

**Proposition 3.1:** If for some very large \( t \), say \( T^m, \mu_t^i = \mu^i \) for all \( t \geq T^m, i = 1, \ldots, K \), then \( W'_{it} \geq W_{it} \) and \( \eta'_{it} \geq \eta_{it} \) for all \( t \).

**Proof:** We show in Lemma A.1 that the \( W_{it} \)'s and the \( \eta_{it} \)'s converge and we denote by \( W_i, W'_i, \eta_i, \) and \( \eta'_i \), the limits of \( W_{it}, W'_{it}, \eta_{it}, \) and
\( \eta_i^t \), respectively. It follows then from the assumption of this proposition that for some \( \bar{T} > T^m \), \( W_{i \bar{T}} = W_i \), \( W_{i \bar{T}}' = W_i' \), \( \eta_{i \bar{T}}' = \eta_i' \), and \( \eta_{i \bar{T}} = \eta_i \). In the process of proving Lemma 3.2 it has been shown that if \( W_i' \geq W_i \), \( i = 1, \ldots, K \), then also \( W_{i,t-1}^i \geq W_{i,t-1} \), \( i = 1, \ldots, K \). It then follows, by applying backwards the recursion relations (3.4) and (2.4) starting from \( \bar{T} \) (for the computation of \( W_{i \bar{T}}', r_i \) should replace \( r_0 \) in 3.4), and using the relation \( W_{i \bar{T}}' \geq W_{i \bar{T}} \), that at each period \( t \leq \bar{T} \), \( W_{i,t} \geq W_{i,t}^i \), and \( \eta_{i,t} \geq \eta_{i,t}^i \). Since \( \bar{T} \) can be chosen arbitrarily large, these inequalities hold for all \( t \).

Q.E.D.

Under increased uncertainty the critical numbers \( \eta_i^t \) assume larger values at all \( t \). Since the strategy of the investor is determined by \( \eta_i^t \), it follows that he will wait for larger proceeds before terminating his investment. This does not imply, however, that under increased uncertainty he should wait longer. Under the sequential scheme described here the duration of the investment is not fixed; rather it is a random variable depending on the obtainable proceeds \( x \). The expected value of the duration, \( E(T^*) \), does not necessarily increase with an increase in uncertainty, since with increased dispersion the probability of obtaining larger \( x \) is increased and this may offset the effect of larger \( \eta_i^t \).

In section V we present a numerical example showing that \( E(T^*) \) is not a monotone function of dispersion.

We note also that the increase in risk causes an increase in the expected discounted proceeds, \( W_i^t \). This is a result of the investor's
decision process; if he observes low prices he may wait until a "better" price comes along. The larger the dispersion, the larger the probability of obtaining a high price.

It should also be noted that the effect of uncertainty on the \( \eta_{it} \)'s and \( w_{it} \)'s does not depend on the property that the investment is growing. The same effect of increased uncertainty also obtains when the investment is shrinking (in this case, however, the \( w_{it} \)'s and \( \eta_{it} \)'s are nonincreasing in \( t \)). The crucial assumption is that the \( \eta_{it} \)'s converge, so that the \( w_{it} \)'s and \( \eta_{it} \)'s can be computed by backward induction from their steady state limits, using the recursion relations (3.4) and (2.4).

Thus far we have assumed risk neutrality. Below we extend the analysis to a risk-averse investor.

IV. The Optimal Strategy for a Risk-Averse Investor

The analysis here is analogous to the previous analysis, except that the actual monetary values are replaced by utilities. From section II it follows that the objective of a risk-averse investor is to maximize \( \beta^{T-1}E[u(X_T)] \), where \( u \) is an increasing concave function. The only function to be redefined is \( V_{it}(x) \), i.e.: \( V_{it}(x) \) is the expected utility from the optimal decision at age \( t \), given state \( s_t \) and the current offer \( x \). The relation between \( w_{it} \) and \( \eta_{it} \) is given by:

\[
w_{it} = \eta_{it} + u'(\eta_{it}),
\]

where
 Proposition 3.1, established under risk neutrality, does not hold under risk aversion since, with risk aversion, we have two offsetting effects. One effect, discussed earlier, is the rise in expected discounted proceeds with increased risk. The second effect, pertinent to risk averters, is the adverse effect of uncertainty on the expected utility of the investment. Unless we make further assumptions about the utility function and/or the distribution function of $X^*_c$, it is not possible to predict which effect will dominate. If the investor's risk aversion is rather small, he may benefit (i.e., enjoy larger values of expected utility) from the increase in risk. This result is contrary to the result, obtained in Venezia and Brenner [1976] for the case where the optimal duration is determined in advance, and does not subsequently change. There we have shown that, for all investors with decreasing (or constant) risk aversion, no matter how small, an increase in risk reduces the value of the expected utility. The reason for these different results is the following. When the investor makes his decision in advance, the first effect, of increased expected discounted proceeds, is not existent (he cannot benefit from larger dispersion) but the second one remains unchanged.
We show below that the results obtained under risk neutrality are preserved under risk aversion if we replace the definition of increasing risk, used above, with the Diamond-Stiglitz (DS) definition. According to DS an increase in \( r \) represents an increase in risk if:\(^\text{12}\)

\[
\int_0^y u_x(x) F_r^i(x, t, r) dx \geq 0 \quad \text{for all} \quad 0 \leq y \leq 1, \quad i = 1, \ldots, K \quad (4.3)
\]

\[
\int_0^1 u_x(x) F_r^i(x, t, r) dx = 0 \quad \text{for all} \quad i = 1, \ldots, K, \quad (4.4)
\]

where

\[
u_x(x) = \frac{du(x)}{dx} .
\]

We prove the above statement in the proposition below. In its proof we use:

**Lemma 4.1:** Suppose an increase in \( r \) denotes an increase in risk in the DS sense. Then

\[
\frac{\partial H_{i, t, r}^u(s)}{\partial r} \geq 0 .
\]

**Proof:** From (4.2) it follows that

\[
H_{i, t, r}^u(s) = \int_{x_s}^1 [u(x) - s] dF_r^i(x, t, r)
\]

\[
= E[u(x)] - s - \int_0^{x_s} [u(x) - s] dF_r^i(x, t, r) .
\]
Integration by parts yields

\[ H_{1,t,r}^u(s) = E[u(x)] - s + \int_0^x u_x(x) F_t^i(x,t,r) dx. \]

Since by the definition of increasing risk in the sense of DS
\[ \partial E[u(x)]/\partial r = 0, \]
it follows then from (4.3) that

\[ \partial H_{1,t,r}^u(s)/\partial r = \int_0^x u_x(x) F_t^i(x,t,r) dx \geq 0. \]

Q.E.D.

**Proposition 4.1:** If \( W_{it} \) and \( W_{it}' \) correspond to \( r = r_0 \) and \( r = r_1 > r_0 \), respectively, where \( r \) is a shift parameter denoting risk in the DS sense, then \( W_{it}' > W_{it} \) and \( \eta_{it}' > \eta_{it} \), for all \( i = 1, \ldots, K \), and all \( t \).

**Proof:** The proof here is analogous to the proof of proposition 3.1. Here, however, we use Lemma 4.1 in conjunction with (4.1), instead of using Lemma 3.1 in conjunction with (3.4), as we did in proposition 3.1. It can also be verified (see Lemma A.3), that Lemma A.1 and A.2 hold when \( H \) is replaced by \( H^u \).

Q.E.D.

**V. Example**

The purpose of this section is twofold. First, we derive the expected value of the duration, \( E(T^*) \), as a function of uncertainty in \( X_t^i \). Second, we show numerically how the various parameters and functions, described in section III, relate.
We assume two possible states for the economy. The transition probabilities are given by the matrix:

\[ \Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} \]

and the discount factor is 0.9. For each state \( s_i \), the series \( \{\mu_t^i\} \) is S-shaped.\(^{14}\) For each \( i \) and \( t \), \( X_t^i \) may assume the seven values:

\[
\begin{align*}
\mu_t^i & - v_0 \frac{v_t^i}{\sqrt{p_3}}, \\
\mu_t^i & - v_0 \frac{v_t^i}{\sqrt{p_2}}, \\
\mu_t^i & - v_0 \frac{v_t^i}{\sqrt{p_1}}, \\
\mu_t^i & + v_0 \frac{v_t^i}{\sqrt{p_1}}, \\
\mu_t^i & + v_0 \frac{v_t^i}{\sqrt{p_2}}, \\
\mu_t^i & + v_0 \frac{v_t^i}{\sqrt{p_3}},
\end{align*}
\]

with probabilities \( p_3, p_2, p_1, p_0, p_1, p_2, p_3 \), respectively, where \( p_0 > p_1 > p_2 > p_3 \). Hence the distribution of the \( X_t^i \)'s has a Gaussian-like shape with:

\[
E[X_t^i] = \mu_t^i, \quad \text{Var}(X_t^i) = v_0^2(v_t^i)^2.
\]

We assumed at first that \( v_t^i = 1 \), \( v_0 = 7 \) and the \( p_i \)'s are: \( p_0 = .5, p_1 = .156, p_2 = .063, p_3 = .031 \). Based on these assumptions, we present in table 1 the \( \mu_t^i \)'s and the functions \( W_{it} \) and \( \mu_{it} \), for \( t = 1, \ldots, 20 \) and \( i = 1, 2 \). In addition, we present the following functions: \( \phi_{it} \)--the probability that the investment will be terminated at time \( t \), given that it was not terminated before and the state is \( s_i \), i.e.,
\[ \phi_{it} = \Pr[x^i_t \geq \eta_{it}] ; \]

\( \alpha_{it} \) -- the probability of terminating the investment at time \( t \), as evaluated at the starting point, if the economy is in state \( s_i \) at that time. Because of the finite nature of our example, we assume that the investment must be terminated by \( t = 20 \). Hence \( \phi_{i,20} = 1 \) by definition, for \( i = 1,2 \). \( W_{i,20} \) is given by

\[ W_{i,20} = \sum_{k=1}^{2} \pi_{ik} \sum_{j=1}^{7} x^k_{tj} q_j , \]

where \( j \) runs over the seven values assumed by \( x^k_{tj} \) and \( q_j \) represents their respective probabilities. For \( t < 20 \), \( W_{it} \) is given by:

\[ W_{it} = \sum_{j=1}^{7} q_j v_{it}(x^i_{tj}) , \]

where

\[ v_{it}(x^i_{tj}) = \max\{ x^i_{tj}, \beta \sum_{k=1}^{K} \pi_{ik} W_{k,t+1} \} . \]

The probability distribution of the optimal duration, as evaluated at the starting point, is presented in table 1. An interesting statistic based on this distribution is \( E(T^*_i) \), the expected optimal duration, at the starting point, given \( s_i \). Thus \( E(T^*_i) \) is given by

\[ E(T^*_i) = \sum_{t=1}^{20} t \alpha_{it} . \]
and it is a function of the variances of the $X_t^i$'s. This relation is presented in table 2 for the case where these variances are stationary. For the series $\{\mu_t^i\} i = 1,2$, presented in table 1, for $\nu_t^1 = 1$ and for $\nu_0$ ranging from 0.1 to 10, we show $E(T^*_i)$ and $W_{11}$ as functions of $\nu_0$.

In table 3 we present $E(T^*_i)$ and $W_{11}$ for $i = 1,2$, for the case where the variances are nonstationary. We use series of $\{\mu_t^i\}$ which are slightly different than those presented in table 1; we use the same series for $\mu_t^1$, but for $\mu_t^2$ we assume $\mu_t^2 = 1.2 \mu_t^1$ and the following $p_i$'s: $p_3 = .063$, $p_2 = 0.94$, $p_1 = .187$, $p_0 = .312$. We also assume that $\nu_t^i = (\mu_t^i)^{1/2}$ and present the parameters for $\nu_0$ ranging from 0 to 3.

The numerical example presented in tables 2 and 3 supports our assertion, given in section III, that the expected value of the duration is not a monotone function of uncertainty.
APPENDIX

Lemma A.1: The series \( \{W_{it}\}, \{W'_{it}\} \) converge to \( W_i, W'_i \), respectively, for \( i = 1, \ldots, K \). These limits satisfy:

\[
\infty > W'_i > W_i, \quad i = 1, \ldots, K.
\]

Proof: From the definitions of \( W_{it} \), and from (2.3) it follows that for each \( i = 1, \ldots, K \), the series \( \{W_{it}\} \) is monotonically nondecreasing.

We next show that these series are also bounded. Consider a situation where the only possible state of the economy is \( K \) (the superior state), and that for each period of time \( X^i_t \) is distributed as the random variable \( X^K \), with an expected value \( \mu^K \geq u^i_t \) for all \( i = 1, \ldots, K \), and for all \( t \). Let the expected value of the investment for such a situation be denoted by \( W^* \). Then a finite \( W^* \) may be obtained by solving the following equation (for the proof, see DeGroot [1970], p. 382):

\[
W^* = \frac{1}{1-\beta} H^K(BW^*),
\]

where \( H^K \) is the function \( H \) corresponding to \( X^K \). However, for all \( i = 1, \ldots, K \), and for all \( t \), \( W_{it} \leq W^* < \infty \). Hence, since the series \( \{W_{it}\} \) are bounded and monotonic, they converge. From the definition of \( n_{it} \), it follows that these series also converge. Thus we denote:

\[
\begin{align*}
W_i &= \lim_{t \to \infty} W_{it} \quad &i = 1, \ldots, K, \\
n_i &= \lim_{t \to \infty} n_{it} \quad &k = 1, \ldots, K,
\end{align*}
\]

(A.1)
where

\[ \eta_i = \beta \sum_{k=1}^{K} \pi_{ik} W_k. \]

From (3.4) it follows that the relation between \( W_i \) and \( \eta_i \) is given by:

(A.2) \[ W_i = \eta_i + H^i(\eta_i), \]

where (note that we use superscripts in \( H \) to denote states)

(A.3) \[ H^i(s) = \int_s^\infty (x-s) dF^i(x,r). \]

\( W_i \) in (A.2) is defined for the case \( r = r_0 \). Consider now a small change \( dr \) in \( r \). Let the corresponding changes in \( W_i \) be \( dW_i \), for \( i = 1, \ldots, K \). Then by taking the total differential of (A.2) we obtain:

(A.4) \[ dW_i = \beta \sum_{k=1}^{K} \pi_{ik} dW_k + \beta H^i_{\eta} \sum_{k=1}^{K} \pi_{ik} dW_k + H^i_r dr, \]

where

\[ H^i_{\eta} = \frac{\partial H^i}{\partial \eta_i}, \]

\[ H^i_r = \frac{\partial H^i}{\partial r}. \]
The system of equations (A.4) may be written in matrix form as follows:

\[
\begin{bmatrix}
1 - \beta \pi_{11}(1 + H_{r}^{-1}), \\
\vdots \\
\vdots \\
1 - \beta \pi_{i}(1 + H_{r}^{-1}) \\
1 - \beta \pi_{KK}(1 + H_{r}^{-1})
\end{bmatrix}
\begin{bmatrix}
dW_{1} \\
\vdots \\
dW_{K}
\end{bmatrix}
= \begin{bmatrix}
H_{r}^{-1} \\
\vdots \\
H_{r}^{-1}
\end{bmatrix}
\]

or

(A.6) \quad A dW = a,

where \(dW\) is the vector of \(dW_{i}\)'s, \(A\) is the matrix premultiplying it in (A.5), and \(a\) is the vector in the right-hand side of (A.5).

We show in Lemma A.2 that the inverse \(A^{-1}\) of \(A\) exists, and contains nonnegative elements only. In Lemma A.2 we show that \(H_{r}^{-1} > 0\) for all \(i = 1, \ldots, K\). Hence it follows that all the elements of \(dW\), given by \(dW = A^{-1}a\), are nonnegative, that is, \(dW_{i} \geq 0\), \(i = 1, \ldots, K\). Since for small changes in \(r\), \(W_{i}' = W_{i} + dW_{i}\), it follows that \(W_{i}' \geq W_{i}\) for \(i = 1, \ldots, K\).

Q.E.D.

Lemma A.2: The matrix \(A\) defined in (A.5) has an inverse \(A^{-1}\) which contains nonnegative elements only.
Proof: In Bellman ([1970], p. 305), it is shown that the lemma holds for all matrices satisfying the following three conditions: (1) the diagonal elements are positive; (2) the off-diagonal elements are negative; (3) the row sums are positive.

Hence, to prove the lemma we show that our matrix \( A \) indeed satisfies these conditions. A typical diagonal element of \( A \), \( A_{ii} \), is given by:

\[
A_{ii} = 1 - \beta \pi_{ii} (1 + H^i_\eta).
\]

However,

\[
(A.7) \quad H^i_\eta = - \int_{\eta_i}^{\infty} dF(x,r).
\]

Hence \( -1 \leq H^i_\eta \leq 0 \), and thus \( A_{ii} > 0 \).

A typical off-diagonal element of \( A \), \( A_{ij} \), is given by:

\[
A_{ij} = - \beta \pi_{ij} (1 + H^i_\eta).
\]

Hence it follows from (A.7) that \( A_{ij} < 0 \).

A typical row sum of \( A \) is given by:

\[
R_i = 1 - \sum_{j=1}^{K} \beta \pi_{ij} (1 + H^i_\eta) = 1 - \beta (1 + H^i_\eta) > 0.
\]

Thus our matrix \( A \) satisfies the above-mentioned condition.

Q.E.D.
Lemma A.3: Lemmas A.1 and A.2 hold when \( H \) is replaced by \( H^u \).

Proof: This may be shown by first replacing, in lemmas A.1 and A.2, \( H \) by \( H^u \). Then our claim is proved if we can show that

(i) \[ H^u_{r_1} > 0 \]

(ii) \[ -1 \leq H^u_\eta \leq 0 \]

where

\[ H^u_{r_1} = \partial H^u(\eta_1)/\partial \eta_1 \]

\[ H^u_\eta = \partial H^u(\eta_1)/\partial r. \]

However (i) has been proved in Lemma 4.1. (ii) may be verified by noting that

\[ H^u_{r_1} = \partial \left[ \int_{-\infty}^{\infty} (u(x) - \eta_1) dF^i(x,t,r) \right] / \partial \eta_1 \]

\[ = - \int_{\eta_1}^{\infty} dF^i(x,t,r) = - F^i(X \geq \eta_1). \]

Q.E.D.
FOOTNOTES

1Hirshleifer [1970] notes that this problem was treated as early as 1891 by Bohm-Bawerk.

2It is interesting to note the existence of a vast literature in Agricultural Economics dealing with the issue of optimal duration of timber under certainty (a famous one is Gaffney [1957]). Some works, however, deal with uncertainty (e.g., Lembersky and Johnson [1975]), but they use a different framework and do not derive the properties of the solution.

3Also, in his analysis the value of the asset must necessarily grow with time, thus he does not allow for cyclical effects on the value of the asset.

4For simplicity, we restrict our attention to investments which provide cash flows only when terminated (e.g., timber, wine, etc.) and disregard maintenance costs.

5In the economic literature, the additivity assumption is common; for example, in financial theory see Neave [1971], in growth theory see Arrow and Kurz [1970], and in environmental policy theory see Keeler, Spence, and Zeckhauser [1972]. Conditions under which the assumption is allowed are given by Koopmans [1972]. These conditions differ from the usual VNM axioms by the separability axiom.

6Some restrictions must be imposed on the $\pi_{ij}$'s. For example, if in the tree-chopping problem $s_i$ corresponds to a higher tree than $s_j$, then it is appropriate to assume that $\pi_{ij} = 0$. 
A similar definition of increased risk for finite multiperiod investments is presented in Levy and Paroush [1974].

For a definition of increased risk which does not require the assumption of equal expectations (3.2), see Levy [1977].

To avoid technical difficulties, Rothschild and Stiglitz [1970] restricted themselves to random variables defined on closed and bounded intervals. The other two equivalent definitions of increasing risk, which they have considered, second-order stochastic dominance and the addition of an independent random noise, do not require this restriction, and therefore do not exclude distributions such as normal, lognormal, etc.

This assumption was made to avoid technical difficulties. Actually all we need is that for some \( \bar{T} \), \( W_{it} = \bar{W}_i \) for all \( t > \bar{T} \). In practice, computations of \( W_{it} \) and \( \eta_{it} \) are performed by setting \( W_{iT} = \bar{W}_i \) for some large \( \bar{T} \) (for this we only need convergence of the \( W_{it} \)'s) and then working backwards as shown in the following proof. A stronger proposition, namely \( W'_{it} > W_{it} \) and \( \eta'_{it} > \eta_{it} \) for all \( t \), can be proved if \( W'_i > W_i \), \( i = 1, \ldots, K \). Sufficient conditions for the existence of these strong inequalities can be obtained from (A.5) and (A.6) in the appendix. For example, strict inequalities \( W'_i > W_i \) exist if all \( H^i_t > 0 \) (see (A.7)), and this occurs for almost all "smooth" distribution functions. Other sufficient conditions can also be obtained from (A.5) and (A.6), but this is outside the scope of this paper.

We assume that the investor is risk averse if each period's utility function is concave. Neave [1971] has shown in the context of a
slightly different model that if each period's utility has decreasing risk aversion, then the multiperiod utility function exhibits decreasing risk aversion.

12Note that here the random variable $X_t^1$ is defined on a closed bounded interval (without loss of generality we may use the [0,1] interval).

13Since this relation cannot be determined analytically it is important to derive it numerically.

14The $\mu_t^i$ series were obtained as follows. We have constructed a series of $\mu_t$ by: $\mu_t = 100 \Phi[-2.5 + 5(t-1)]$, where $\Phi$ is the CDF of the standard normal distribution. In tables 1 and 2, $\mu_t^1 = \mu_t$ and $\mu_t^2 = \mu_t + 5$. In table 3, $\mu_t^1 = \mu_t$ and $\mu_t^2 = 1.2 \mu_t$.

15We note that the series \{W_{it}\}, in table 1, decreases for the last values of $t$, contrary to our argument in section III of monotonically nondecreasing $W_{it}$'s. It is due, however, to the fact that here we use a finite time horizon, whereas in section III it is infinite.
REFERENCES


## TABLE 1

VALUES OF $\mu_1^i$, $W_{it}$, $\eta_{it}$, $\alpha_{it}$, and $\phi_{it}$

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<th>$s_2$</th>
</tr>
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*The numbers "0.00" denote zero, the numbers ".00" denote numbers less than .01.*
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**TABLE 3**

$E(T^*_i), \ W_{11}$, AS FUNCTION OF $v_0$, FOR $i = 1,2$, WHEN

THE VARIANCES OF THE $X^i_E$'s ARE NONSTATIONARY

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