Optimal Continuous Speculation
With Information Extracted
From Price History

by

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Abstract

In this paper, I investigate the intertemporal optimization problem of a speculator who optimally extracts value information from the following:
- price history
- public information
- private information

The foreign exchange desk of a major commercial bank or a market maker in a commodity market are examples of this kind of speculator.

To deal with the problem as realistically as possible this model has the following features:
- The speculator trades continuously in order to optimize his objective intertemporally.
- The speculator takes the price impact of his trading into account.
- Dynamic information unfolds and resolves continuously.
- The speculator observes, incorporates and updates information continuously.
- Adjustment costs and carrying costs are included.
- The speculator will optimally acquire costly private information.

Comments welcome

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1. Introduction

In this paper, I investigate the intertemporal optimization problem of a speculator who optimally extracts value information from the following:

- price history
- public information
- private information

The foreign exchange desk of a major commercial bank or a market maker in a commodity market are examples of this kind of speculator.¹

In this model, price movements are driven by two sources. On one hand, market prices roughly reflect the continuously unfolding economic environment. On the other hand, market prices are continuously "kicked" by random noises caused by temporary order imbalances, liquidity trading activity, etc. This second source creates a profit opportunity for a speculator who can distinguish price movements caused by the second source from those caused by the first source.

An analogy might help to clarify the meaning of the last paragraph. Imagine that there are two butterflies flying in the air, one chasing the other. Fig.1.1 illustrates this chasing game. There are three attributes of the butterflies' motion worth noticing. First, the "chasee" moves up and down in a manner somewhat like a random walk. Second, the chaser follows the chasee. Third, the chaser's movements do not exactly mirror those of the chasee.

Now, how does this story relate to economic markets? The

¹For each specific commodity, there are generally a very large number of sporadic individual market participants (e.g., farmers, businesses, etc.) whose buying and selling orders are generally not synchronized. The lack of synchronization of these order flows creates profit opportunities for a small number of big dealers or market makers who specialize in that market and are able to get speculative profits by taking temporary positions against order imbalances.
Fig. 1.1 The analogy of two butterflies

The instantaneous position of the chaser is analogous to an instantaneous price. The instantaneous position of the chasee is analogous to what the price would be if there were no noise in the market. We call this the benchmark value. Formally, the benchmark value, \( v \), of a good is defined as the market clearing price of a good in the absence of either speculative or noise trading.

If the market is absolutely perfect -- in other words there is no noise caused by order imbalances, there are no liquidity trading activities, and buyers are perfectly synchronized with sellers -- the price should exactly equal the benchmark value. But in the real world, there is noise-trading in the market. This is because there is liquidity trading activity, and buyers and sellers are not perfectly synchronized. This noise will kick the price away from the benchmark value.

We have portrayed a picture in which the benchmark value moves up and down somewhat randomly because unexpected events of the economic environment bombard the market continuously. This
resembles the somewhat random up and down movements of the chassee butterfly. At the same time, the price roughly follows the benchmark value just as the chaser butterfly roughly follows the chassee butterfly.

This mean-reverting tendency of prices around the benchmark value creates speculative opportunities to buy-low-sell-high. That is, buying when the price is lower than the benchmark value and selling when the price is higher than the benchmark value.

But things are not so simple because the benchmark value is not directly observable, and it is changing continuously.

One way the speculator might deal with this imperfect information problem is to first make a best estimate of where the benchmark value is at each instant, and then conduct the buy-low-sell-high scheme according to the estimate. The following discussion shows that this method is in fact optimal.²

The more precisely a speculator can estimate where the benchmark value is, the less likely he will make mistakes. That is, the less likely he will buy (sell) according to his estimates when he should in fact sell (buy) according to the real benchmark value.

This is where information comes into the picture, because the precision of the estimate depends on the quality of the speculator's information.

What are the sources of information for a speculator? Take the example of a market maker in an agricultural commodity market. He is paying close attention to the price tick by tick. At the same time he is observing the public information contained in the news media and in government statistical releases. Furthermore, he receives private information from experts about such things as weather and politics. And he might even receive a few tips from his friends. In summary, there are three sources of information:

-- information from price history
-- public information

²See the discussion about the separation principle in Thm.A2.3
-- private information

The objective of this paper is to capture the optimization problem of the speculator using a mathematical model. To deal with the problem as realistically as possible, this model has the following features:

-- The speculator trades continuously in order to optimize his objective intertemporally.
-- The speculator takes the price impact of his trading into account.
-- Dynamic information unfolds and resolves continuously.
-- The speculator observes, incorporates and updates information continuously.
-- Adjustment costs and carrying costs are included.
-- The speculator will optimally acquire costly private information.

This model is closely related with models presented in Lu (1990) and Lu (1991). The concept of benchmark value\(^3\) in continuous time is first introduced in Lu (1990) with a model in which the only source of information is the price. The current paper extends this model to include information from public and private sources. As a companion of the current paper, Lu (1991) further extends the model presented here to include N speculators with heterogeneous information and also considers the nature of equilibrium price behavior. In this sense, the analysis in Lu (1991) makes use of the model developed in this paper.

While these models are not similar to the models in previous financial research, some ingredients are not new. Before getting into the modeling, I will briefly summarize the relationship of this paper with other literature.

Garman (1976) investigates a model in which a single market maker faces an order flow which comes as a Poisson process. The rate of the Poisson process resembles a downward sloping residual

\(^{3}\text{This concept was called fundamental value when I introduced it in Lu (1990).} \)
demand curve. In Garman's model, information will not enter into the picture because, in the language introduced above, the benchmark value is perfectly known.

The concept that a trader should trade smoothly in time in order to reduce the market impact of his trading is captured by Kyle (1985a). In Kyle's model, there is a market maker who sets the price equal to the expectation conditioned on the aggregate order flow.

The three information sources as stated above are used by Kyle (1985b) as the information set of an informed speculator. He argues that prices should not fully reveal private information because that would take away any incentive to acquire costly private information.

Detemple & Kihlstrom (1987) presents a model in which a price-taking representative agent optimally acquires costly information continuously. Because their model assumes there is a representative agent and price-taking behavior, it could be thought of as a decision problem of how much expenditure should be devoted to collecting public information.

Leland (1990) presents a two period model in which a large trader, who needs to trade a considerable amount because he has inside information and who faces a downward sloping residual demand curve, will take the price impact of his trading into account.

The plan of this paper is as follows:

Section 2 will present close form solutions for two single commodity examples. The example is simplified with the assumption of basic but still sensible dynamics. Section 3 presents a much more comprehensive model: a multi-commodity market with any dynamics as long as they can be characterized by linear state space representations.4 The cost of the comprehensiveness is that a closed form solution does not exist. To make the paper more readable, most of the technical detail has been included in Appendices 1 & 2.

4Section 3 is not included in this version.
Lu (1990) introduced a network method of studying market structure which is called AGFP (Agent-Good-Flow-Price) network method. Just as in a hydraulic controlled network, where devices are connected by tubes and signal lines with fluid flowing around at equilibrium pressures, the AGFP network method takes the perspective that agents are connected by market and information channels with goods flowing around at equilibrium prices. The network method makes it tractable to study complex multi-agent multi-market economic problems without losing intuition. In fact, the original motivation of Lu (1990) was to facilitate the model-building process for problems like the one in this paper. Although this paper is written with those who have not read Lu (1990) in mind, readers who have read Lu (1990) might find that glancing at the AGFP networks in Appendix 3 before reading the next section will save time and provide more insight into the models.
2. Examples of Single-Good Speculation

The objective of this section is to explicitly solve simple examples of the generalized model which will be introduced in section 3. The emphasis here is tractability and intuition. We will see how each parameter affects the outcome in the close form solutions derived in this section.

Two models will be introduced in this section. In Model 1, we temporarily assume that the benchmark value, \( v \), is perfectly observable to the speculator. This assumption will be relaxed in Model 2, where the speculator makes imperfect observations of \( v \) based on price history and information sources.

2.1 Continuous speculation with perfect information, Model 1\(^5\)

In order to concentrate first on intertemporal dynamics, we will temporarily assume away the effect of imperfect information by considering a model in which the benchmark value, \( v \), is perfectly observable.

OBJECTIVES OF THE SPECULATOR:
Discounted future profit flow (= capital gain - adjustment cost - carrying cost)

\[
J = \max_{f(\cdot, \mathcal{F})} \int_0^{\infty} e^{-rt} (gdp - \delta f^2 dt - \lambda g^2 dt) dF
\]  
(2.1.1)

Or equivalently, discounted future cash flow (= trading cash flow - adjustment cost - carrying cost)

\[
J = \max_{f(\cdot, \mathcal{F})} \int_0^{\infty} e^{-rt} (pf - \delta f^2 - \lambda g^2) dt - p(0)g(0)
\]  
(2.1.2)

\(^{5}\)See Appendix 3 for AGFP network.
where \( \delta \geq 0 \), and \( \lambda > 0 \).

In general, a speculator likes speculative profits, dislikes adjustment, and dislikes net inventory positions, which are a measure of his risk.

The \( q \) is the inventory held by the speculator. The \( p \) is the instantaneous price. So, the first term in the parentheses, \( gdp \), represents the instantaneous speculative profit flow made by the speculator.

The second term in the parentheses, \( \delta f^2 \), represents the adjustment cost, which is assumed to be in quadratic form\(^6\). Here, the "adjustment cost" includes all costs related to the adjustment of the speculator's inventory position. If a speculator is in a hurry to change his position, he might have to search more aggressively for trading opportunities and will thus be in a disadvantageous position in terms of adjustment costs (e.g., transaction costs, telephone bills, utilization of valuable time of traders, hurried result of bargaining). The case of zero adjustment cost is included because \( \delta \) is allowed to be zero.

The third term in the parentheses, \( \lambda g^2 \), represents the carrying cost which could be physical costs, like those associated with storage, or the equivalent risk premium the speculator would require in order to be willing to hold the un-hedged position, \( g \). To capture the general fact that disagreeableness of the inventory is increasing when the inventory gets larger, the cost is assumed to be quadratic. In other words, the marginal carrying cost increases linearly with the absolute value of the inventory. Note that I assume the speculator's optimal holding of inventory is zero. But if it is not, all we need to do is to substitute \( g \) with \( g-g_0 \) in which \( g_0 \) is the optimal inventory.

Here the speculator is only speculating on the spot price. Generally, a forward or future position is equivalent to a spot position plus an arbitrage position. If we assume that any

\(^6\)The quadratic form is chosen mainly to increase the tractability of the model.
arbitrage opportunity will be taken advantage of until it is so small that we can assume it is zero, then speculating on forward or future prices is equivalent to speculating on the spot price. This observation is in fact supported by empirical data which shows that forward and future prices move almost together with the spot price. This assumption allows us to aggregate the speculative activities of the forward and spot markets into the spot price speculation only.

The equivalence of Eq.2.1.1 and 2.1.2 is an immediate result of the equivalent representation theorem, Thm.A2.4.

**DYNAMICS**

**Relationship of inventory and trading flow:**

\[ dg = -fdt + γgdt \]  \hspace{1cm} (2.1.3)

The derivative of the inventory, \( g \), is equal to trading flow, \( f \), plus the appreciation rate of the inventory, \( γ \).

**Dynamic of noise trading flow:**

\[ du = -φ_u u dt + σ_u db_u, \quad φ_u > 0 \]  \hspace{1cm} (2.1.4)

Where \( db_u \) is a standard Brownian Motion.

Here the noise trading flow, \( u \), is assumed to be an Ornstein-Uhlenbeck process which captures the following facts:

-- The mean of \( u \) is zero.
-- \( u \) is mean-reverting.

**Dynamic of benchmark value:**

\[ dv = -ε(ν - m) dt + σ_ν db_ν, \quad ε > 0 \]  \hspace{1cm} (2.1.6)

where \( db_ν \) is a standard Brownian Motion independent of \( db_u \).

In the long run, the benchmark value should be mean-reverting. If there is a draught inflating this year's wheat price, the price
is likely to be lower next year. But in the short run there should be little predictability in $v$. In other words, if there is a drift in $v$, it should be very small. To capture these two points at the same time, the $v$ is assumed follow an Ornstein-Uhlenbeck process which has a very small drift term. $\varepsilon$ is used to emphasize that the drift term is small. Especially, as $\varepsilon$ approaches zero, $v$ approaches a random walk (Brownian Motion).

To introduce the market clearing condition, let's first consider the case without the speculator. Fig.1.2 gives a graphical demonstration of the following concepts:

-- The unfolding (i.e. continuous formation) of the benchmark value by the continuous interaction of the ever changing demand and supply schedules.

-- The continuous formation of the price by adding the effect of noise trading on the benchmark value.

![Diagram](image)

**Fig. 2.1 Unfolding and Resolving of dynamic information**

We should distinguish between stock concepts, which are measured in terms of units of goods, and flow concepts, which are measured in terms of units of goods per unit of time. In
traditional static models, demand, supply and noise trading are considered stock concepts. But in reality, demand, supply and noise trading come continuously and are in fact flow concepts. This is why static models do not work well for modeling intertemporal dynamics. It is worth emphasizing that the concepts in Fig.1.2 are flow concepts.

Note the inverse of the slope of the residual demand curve at point E is \( \varphi_m \) which reflects the responsiveness of the price to noise trading. It will be defined as the liquidity of the market. If we also include the contribution of speculative trading, we get Market clearing law:

\[
\varphi_m (p - v) + f = u \tag{2.1.5}
\]

or in words:

residual demand + speculator's trading = noise trading

The first term in market clearing law reflects the residual demand of the good. Note, if both noise trading, \( u \), and speculator's trading, \( f \), are zero, then \( p = v \). In other words, if there were neither noise trading nor speculative trading in the market, the market clearing price would equal the benchmark value, \( v \). This is exactly consistent with the definition of benchmark value given in the last section.

INFORMATION STRUCTURE

Information set:

\[
\mathcal{F}(t) = \{v(\tau), p(\tau); \tau \leq t\} \tag{2.1.7}
\]

The information of the speculator includes the whole history (including the current value) of \( p \) and \( v \). In this model, it is temporarily assumed that \( v \) is observed perfectly by the speculator. This is what we mean by perfect information.

The central characteristic of speculative activities is that
of buying (selling short) something with the intention of selling (buying) it at higher (lower) price in the future. To concentrate my analysis on this characteristic, I will basically assume away other issues like convenience yields, depreciation, delivery, etc. Speculation on the exchange rate of a foreign currency (yen for example) provides a good case in which these side issues either do not exist or are not important. In this case, inventory, \( g \), is the balance of the yen account; the appreciation rate of yen, \( \gamma \), is the interest rate of yen; and the price of yen is the exchange rate of yen in terms of dollars.

In order to isolate the speculation on the price of yen (i.e. exchange rate) from the speculation on the interest rate, the interest rate of yen, \( \gamma \), is assumed to be equal to the interest rate of dollars, \( r \), which is assumed to be constant.

In the derivation of the solution, the optimal trading policy is first represented as a function of state variables which include \( u \). Although \( u \) is not directly observable, it is inferable from the market clearing law (Eq.2.1.5) in which the only un-observable term is \( u \). In other words, \( u \) can be represented as a linear combination of directly observable variables. If we substitute this relation into the optimal trading policy, we will get an equivalent representation of the policy which includes only directly observable variables. In the following, we will present both of these two equivalent representations of the solution, the one in terms of \( u \) is more intuitive, while the one in terms of directly observable variables is what the speculator will follow in practice.

**Solution of Model 1**

1. Noise trading flow representation

Optimal trading policy for the speculator:

\[
\hat{f} = a_1 (v - m) + a_2 u - a_3 g
\]

(2.1.8)

where
\[ \alpha_1 = \frac{\varepsilon}{2\theta (\alpha_3 + \varepsilon)} \]  

(2.1.9)

\[ \alpha_2 = \frac{\varphi_v}{2\theta \varphi_m (\alpha_3 + \varphi_u)} \]  

(2.1.9a)

\[ \alpha_3 = \frac{r}{2} + \sqrt{\left(\frac{r}{2}\right)^2 + \frac{\lambda}{\theta}} \]  

(2.1.10)

\[ \theta = \delta + \varphi_m^{-1} \]  

(2.1.11)

Net present value (NPV) of the policy:

\[ J = x^T(0) S x(0) + \frac{1}{r} [S_{11} \sigma_v^2 + S_{22} \sigma_u^2] \]  

(2.1.13)

= term related with initial condition + perpetuity of future profit opportunities.

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3)^T \), \( X = (v - m, u, g)^T \) and

\[
S = S^T = \begin{bmatrix}
\frac{\theta \alpha_1^2}{r + 2\varepsilon} & \frac{\theta \alpha_1 \alpha_2}{r + \varphi_v + \varepsilon} & \frac{\alpha_2}{2(\alpha_3 + \varepsilon)} - \frac{1 - \varphi_m^{-1} \alpha_2}{2} \\
\frac{\theta \alpha_2^2}{r + 2 \varphi_v} & \frac{\alpha_1}{2 \varphi_m (\alpha_3 - \varphi_u)} & \frac{\varphi_m^{-1} (1 - \alpha_2)}{2} \\
\frac{-\theta \alpha_3}{\alpha_3 + \varphi_m^{-1} \alpha_3}
\end{bmatrix}
\]  

(2.1.14)

2. Price deviation representation

Optimal trading policy for the speculator:

\[ f^* = \frac{\alpha_1}{1 - \alpha_2} (v - m) + \frac{\alpha_2 \varphi_v}{1 - \alpha_2} (p - v) + \frac{\alpha_3}{1 - \alpha_2} g \vartriangle a \gamma \]  

(2.1.15)
i.e.

\[ f^* = \alpha Ty, \quad x = Ty \]  \hspace{1cm} (2.1.16)

where

\[
T = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{1-\alpha_2} & \frac{\varphi_m}{1-\alpha_2} & \frac{\alpha_3}{1-\alpha_2} \\
0 & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (2.1.17)

NPV of the optimal policy:

\[ J = y^T(0) T^T S Ty(0) + \frac{1}{\gamma} [s_{11}\sigma^2_v + s_{22}\sigma^2_u] \]  \hspace{1cm} (2.1.18)

= the term related with the initial condition + the perpetuity of the future profit opportunities.

where \( y = (v - m, p - v, q)^T \).

Proof: The thrust of the proof is the transformation of the problem into the standard form of Thm.A2.1 (Discounted-Linear-Quadratic-Gaussian Stochastic Control) by using the equivalent representation theorem, Thm.A.2.4. See Appendix A1.1 for the formal proof.

Discussions of the solution of Model 1:

The optimal trading policy, Eq.2.1.8, is a flow. This is apparent if we take a close look at the market clearing law, Eq.2.1.5. As soon as the speculator buys or sells, he bids the price against his favor. He must buy or sell smoothly in order to minimize the price impact of his trading. This is captured by the concept of flow.

The three terms of Eq.2.1.15 have definite intuitive meanings. The first term represents speculation on the long term trend of the benchmark value, \( v \). The second term represents speculation on the
short term mean-reverting fluctuation of price, \( p \), around \( v \). The third term represents the activity to reduce the size of the inventory in order to lessen carrying cost.

The fact that we have a closed form solution both in optimal trading policy and in the value function, Eq.2.1.18, makes it easy to conduct comparative static analysis. We will leave it to readers to do comparative static analysis that might of interest.

As an example of interesting comparative static analysis, note, if \( \varepsilon \rightarrow 0 \), then \( \alpha_i \rightarrow 0 \). In other words, as \( v \) approaches a random walk, the first term of the trading policy (Eq.2.1.9), approaches zero. This is because the first term reflects the speculation on the trend of the benchmark value, \( v \), and the random walk has no trend -- or, in more mathematical words, the random walk is a martingale.

As we mentioned earlier, there should be little predictability in the movement of \( v \) in the short term. This is why we use \( \varepsilon \) to emphasize that the mean-reverting drift is very small.

This phenomenon could also be explained from the perspective of the NPV of the optimal policy (Eq.2.1.18). Note the first term in Eq.2.1.18 reflects the effect of the initial condition \( y(0) \) on \( J \), and the second term in Eq.2.1.18 reflects the perpetuity of future daily speculative profits. Here \( s_{ij} \) reflects the contribution of the trend of \( v \) on the average daily profit flow. If \( \varepsilon \) is very small, then \( s_{ij} \) is also very small. In other words, almost no profit could be made by speculating on the trend of the benchmark value, \( v \).

To summarize the above analysis, we have:

**Prop.2.1.1:** When \( \varepsilon \) is very small, we can pretend that the benchmark value is a random walk with almost no effect on the optimal trading policy and value function, \( J \).

Note, if \( \varepsilon = 0 \), the \( v \) will be a random walk which has the tendency to drift to infinity. This makes it seemingly unreasonable to assume exogenously that \( v \) follows a random walk in
a model with infinite horizon. But Prop.2.1.1 says that it is a fine assumption if the purpose of the analysis is to derive optimal trading policy and value function. Intuitively, it is the difference between \( p \) and \( v \) rather than the absolute value of \( v \) that matters.

As we will see in Model 2, this assumption allows us to reduce the dimension of the state variables by one.

### 2.2 Speculation with imperfect information, Model 2

In Model 2, we will relax the assumption in Model 1 that the benchmark value, \( v \), is perfectly observable. Instead, we introduce information into the picture by assuming the speculator has \( M \) information sources, which are noisy observations of the benchmark value.

Most of the set up is about same as that in the Model 1, and so we can concisely state the model as follows:

**OBJECTIVE OF THE SPECULATOR**

discounted future profit flow (= capital gain - adjustment cost - carrying cost)

\[
J = \max_{\{f^t, \tilde{v}^t\}} \max_{\tilde{v}^0} E \left[ \int_0^\infty e^{-rt} (gdp - \delta f^2 dt - \lambda g^2 dt) \right] 
\]

(2.2.1)

Or equivalently, discounted future cash flow (= trading cash flow - adjustment cost - carrying cost)

\[
J = \max_{\{f^t, \tilde{v}^t\}} \max_{\tilde{v}^0} E \left[ \int_0^\infty e^{-rt} (pf^2 - \delta f^2 - \lambda g^2) dt - p(0) g(0) \right] 
\]

(2.1.2)

where \( \delta > 0 \), and \( \lambda > 0 \).

**DYNAMICS**

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\(^7\)See Appendix 3 for AGFP network.
relationship of inventory and trading flow:

\[ dg = -fdt + rgd \]

\[ (2.2.2) \]

dynamic of noise trading flow:

\[ du = -\varphi_u u dt + \sigma_u db_u, \quad \varphi_u > 0 \]

\[ (2.2.3) \]

dynamic of benchmark value:

\[ dv = \sigma_v db_v \]

\[ (2.2.5) \]

market clearing law:

\[ \varphi_m (p - v) + f = u \]

\[ (2.2.4) \]

**INFORMATION STRUCTURE**

Information set:

\[ \mathcal{F}(t) = \{ p(\tau), w_1(\tau), \ldots, w_M(\tau); -\infty < \tau < t \} \]

\[ (2.2.7) \]

Where \( w_m, m = 1, 2, \ldots, M \) are M information sources of \( v \) with following specification\(^6\):

\[ dw_m = v dt + \sigma_m db_m, \quad w = 1, 2, \ldots, M \]

\[ (2.2.6) \]

In the set up of Model 2, \( b_u, b_v, b_1, \ldots, b_m \) are assumed to be independent standard Brownian Motions.

In order to present the solution of Model 2, I need first present two propositions. The first proposition concerns the

\(^6\)This kind of observation is a continuous time counterpart of a discrete time observation, \( e_t = v_t + n_t \), with \( n_t \) i.i.d. normally distributed.
unconditional distribution of $u$ and $v$.

**Prop.2.2.1 (unconditional variance of $u$ and $v$)**

Given the set up of Model 2, the unconditional variance of $u$ is:

$$V_u = \frac{\sigma_u^2}{2\phi_u^2}$$ (2.2.8)

and the variance of $v$ conditioned on the current price only (hereafter referred to as the unconditional variance of $v$) is:

$$V_v = \phi_m^{-2} \frac{\sigma_u^2}{2\phi_u^2}$$ (2.2.9)

**Proof:**

Apply Thm.A2.5 on the dynamics of $u$, i.e. Eq.2.2.3. For any initial condition, $(m_0, V_0)$, solving the differential equation Eq.A2.5.3, we get:

$$m = m_0 e^{-\phi_m t} \to 0, \text{ as } t \to \infty$$

From Eq.A2.5.4, the stationary $V_u$ satisfies the differential equation:

$$0 = \dot{V}_u = -2\phi V_u + \zeta_u^2$$

Solving it, we get Eq.2.2.8.

To derive Eq.2.2.9, rewrite Eq.2.2.4 into:

$$v - Z_1 = -\phi_m^{-1} u$$

of which the unconditional variance of both sides should be same because

$$Z_1 = p + \phi_m^{-1} f$$

is related with the current price only.\[\]
The next proposition is about the expression of the conditional distribution of \( v \) conditioned on the information set of the speculator. Note we have a huge amount of data in the information set which includes the whole continuous history of \( p \) and \( w_1 \ldots w_m \) from the current instant to \(-\infty\). Fortunately, we can find a sufficient statistic which is the conditional mean of \( v \) given \( \mathcal{F}(t) \). The existence of the sufficient statistic makes it possible to do the calculation recursively applying Kalman Filter.

Prop. 2.2.2 (Filtering problem of Model 2)

Given the set up of Model 2, the conditional distribution

\[
\begin{bmatrix}
  v(t) \\
  u(t)
\end{bmatrix} \mid \mathcal{F}(t) 
\sim \text{Normal}
\begin{bmatrix}
  \hat{v}(t) \\
  \hat{u}(t)
\end{bmatrix}
\begin{bmatrix}
  \Omega_v & -\Phi_m \Omega_v \\
  -\phi_m \Omega_v & \phi_m^2 \Omega_v
\end{bmatrix}
\]

(2.2.10)

which are perfectly negatively correlated. And

\[
\frac{\Omega_v}{\nu_v} = \frac{\Omega_u}{\nu_u} = \frac{2}{1 + \sqrt{1 + \frac{\phi_m^2 \sigma_u^2 + \phi_m^2 \sigma_u (\phi_m^2 \sigma_u^2 + \sigma_v^2)}{\phi_u^2 \sigma_v^2} \sum_{m=1}^{M} \frac{1}{\sigma_m^2}}}
\]

(2.2.11)

\( \hat{v} \) satisfies the following differential equation:

\[
d\hat{v} = \frac{\Omega_v \phi_v + \sigma_v^2}{\phi_m^2 \sigma_u^2 + \sigma_v^2} \left( dz_1 - (\phi_u \hat{v} - \phi_v z_1) dt \right) + \Omega_v \sum_{m=1}^{M} \left[ \frac{1}{\sigma_m^2} (dw_m - \hat{v} dt) \right]
\]

\[
\hat{v}(-\infty) = p(-\infty)
\]

(2.2.12)

(2.2.13)

where
\[ z_1 = p + \varphi_m^{-1} \hat{\epsilon} \]  

and

\[ \varphi_m^{-1} \hat{u} + \hat{\nu} = \varphi_m^{-1} u + \nu = z_1 \]  

\[ (2.2.14) \]  

\[ (2.2.15) \]

**Proof:** See Appendix 1.2.[9]

Note the second term in the "\( \sqrt{\cdot} \)" of Eq.2.2.11 is the contribution of the price information, and the third term is the contribution of the non-price information. The effect of these contributions is the reduction of the conditional variance. In other words, the information makes the estimate more precise.

The Eq.2.2.12 gives the scheme to continuously incorporate new information from the price and the information sources into the conditional mean recursively. The first bracket is the innovation\(^9\) from p (note Eq.2.2.14), and the second bracket is the innovation from the M information sources.

Armed with the two propositions above, we are ready to state the solution of Model 2. For the same argument as in Model 1, two equivalent representations will be presented.

**Solution of Model 2:**

Noise trading flow representation

Optimal trading policy for the speculator:

\[ f^* = a_1 \hat{u} - a_2 g \]  

\[ (2.2.17) \]

where

\[ \text{"Innovation" means the difference between the observation and estimation.} \]
\[
\alpha_1 = \frac{\varphi_u}{2\theta \varphi_m (\alpha_2 + \varphi_u)}
\]
(2.2.18)

\[
\alpha_2 = \frac{r}{2} + \sqrt{\left(\frac{r}{2}\right)^2 + \frac{\lambda}{\theta}}
\]
(2.2.19)

\[
\theta = \delta + \varphi_m^{-1}
\]
(2.2.20)

NPV of the optimal trading policy:

\[
J = X^T(0) S X(0) + \frac{1}{r} s_{11} \sigma_u^2 (1 - \frac{\Omega_u}{\nu_u})
\]
(2.2.21)

= term related with initial condition + perpetuity of future profit opportunities

where \( \alpha = (\alpha_1, \alpha_2)^T \), \( X = (u, g)^T \) and

\[
S = S^T = \begin{bmatrix}
\frac{\theta \alpha_1^2}{r + 2\varphi_u} & \frac{\alpha_2}{2\varphi_m (\varphi_u + \alpha_2)} - \frac{1 - \alpha_1}{2\varphi_m} \\
\frac{\alpha_2}{2\varphi_m (\varphi_u + \alpha_2)} - \frac{1 - \alpha_1}{2\varphi_m} & -\theta \alpha_2 + \varphi_m^{-1} \alpha_2
\end{bmatrix}
\]
(2.2.22)

and \( s_{ij} \) are the items of \( S \).

Price deviation representation

Optimal trading policy of the speculator:

\[
f^* = \frac{\alpha_1 \varphi_m}{1 - \alpha_1} (P - \varphi) + \frac{\alpha_2}{1 - \alpha_1} g
\]
(2.2.23)
If we define \( y = (p - v, g)^\tau \), then:

\[
f^\tau = \alpha T^\tau y, \quad x = Ty
\]

(2.2.24)

where

\[
T = \begin{bmatrix}
\varphi_m & \alpha_2 \\
1 - \alpha_1 & 1 - \alpha_1 \\
0 & 1
\end{bmatrix}
\]

(2.2.25)

NPV of the policy:

\[
J = y^\tau T(0)T^\tau S_T y(0) + \frac{1}{\tau} s_{11} \sigma_u^2 (1 - \frac{\Omega_y}{V_v})
\]

(2.2.26)

= term related with initial condition + perpetuity of future profit opportunities

Proof of the solution of the Model 2
Sketch of the proof:

Similar to the proof of Model 1, the thrust of this proof is to transform the optimization problem into the standard form of Thm.A2.3 (Discount-Linear-Quadratic-Gaussian stochastic control with imperfect information) by using the Equivalent Representation Theorem, Thm.A.2.4.

Proof 1:

The solution can be derived the same way as the solution of Model 1 (except that Thm.A2.3 is used instead of Thm.A2.1) and then let \( \varepsilon \rightarrow 0 \).

Proof 2:

Note, \( v \) is a martingale, according to Thm A2.4, we can rewrite the objective of the speculator, Eq.2.2.1, into discounted potential profit flow form:
\[ J = E\left[ \int_0^T e^{-rt} \left[ (p-v)f - \delta f^2 - \lambda g^2 \right] dt \right] + E\left[ -g(0) [p(0) - v(0)] \right] \]  
(2.2.27)

The rest of the proof is moved to Appendix 2 because it is purely technical.

An interesting point can be made by comparing Eq. 2.2.7 with Eq. 2.2.1. Note the "gdp" term in Eq. 2.2.1 is the realized capital gain while the "(p - v)f" term in Eq. 2.2.27 is the potential capital gain. It is called the potential capital gain because the benchmark value, \( v \), is the "center of the gravity" around which the price, \( p \), is mean-reverting. In this sense, \( (p - v)f \) is the expected future capital gain. The fact that expected realized capital gain is equal to the expected future capital gain is an important insight from the Equivalent Representation Theorem, Thm. A2.4.

**Discussion of the solution of Model 2:**

The optimal trading policy, Eq. 2.2.23, has two terms. The first term represents the speculative activity of buy-low-sell-high and is called the speculative term. The second term in Eq. 2.2.23 represents the speculator's activity to reduce the size of his inventory and is called the inventory correction term. The optimal trading policy says that the optimal trading process is as follows:

- The speculator updates his estimate of \( v \) continuously by incorporating the new data of the price and the observations.
- He conducts the buy-low-sell-high strategy continuously using the estimate of \( v \).
- At the same time, he will try to prevent his inventory from becoming too large because that will mean too much carrying costs.

There are two reasons which discourage the speculator from
adjusting his holding. First, there is an explicit adjustment cost, $\delta f^2$. Second, there is an implicit adjustment cost caused by the price impact of trading. This is because the speculator's trading bids the price against his favor (see market clearing condition, Eq.2.2.4).

The sum of these two costs is captured in the parameter $\theta$ as defined in Eq.2.2.20. This parameter is called the generalized adjustment cost. The lower the generalized adjustment cost, $\theta$, the more aggressively the speculator will correct his inventory (see Eq.2.2.19).

Note that the price impact cost is equal to the inverse of the market liquidity, $\varphi_m$. In other words, the more liquid the market the less cost of price impact incurred by the speculator.

The ratio $\lambda/\theta$ in Eq.2.2.19 shows that the higher the carrying cost is in relation to the adjustment cost, the more aggressively the speculator will correct his inventory. At the extreme, when $\lambda \ll \theta$, $\alpha_2$ reaches the lowest value, $r$, which corresponds to the continuous conversion of the interest flow generated by the inventory of yen into dollars.

The NPV of the optimal policy (Eq.2.2.26) is made up of two terms. The first term is related to the initial condition of $y(0)$. The second term is the perpetuity of future profit flow. The ratio $\Omega_v/V_v$ in the parentheses represents the quality of the speculator's information. Recall that $\Omega_v$ is the conditional variance of $v$ and $V_v$ is the unconditional variance of $v$. So the ratio is some number in $[0, 1]$. If the quality of the information is infinitely good, then $\Omega_v$ is zero and so is the ratio. In that case, the speculator's profit will reach the same level as in Model 1. This is not surprising considering $\Omega_v = 0$ is another way of saying that $v$ is perfectly observable. At the other extreme, if the quality of the speculator's information is very bad, the conditional variance $\Omega_v$ is hardly smaller than the unconditional variance, $V_v$ (i.e., possession of the information will hardly improve the estimate of $v$). In that case, the ratio will be almost as large as one. In that case, the expected profit flow is near zero.
Because, in fact, there should be some information in the price history observable to the speculator, the ratio should never be as large as one. This argument could be made precise by taking a look at the explicit expression of this ratio given in Eq.2.2.11 in which we could see how each piece of information will contribute to the NPV. Even if he does not have any information other than the price (i.e. the third term is zero), the second term in the square root (which is the contribution of the price information) will make the ratio smaller than one. In actual markets, the speculator will get public information and private information in addition to the price information. In that case, the third term in the square root, which is the contribution of the non-price information, will make the ratio even smaller.

Following are some other results of comparative static analysis which are apparent if we take a look at the optimal trading policy and the NPV:

1) The speculator will trade more aggressively if:
   -- The adjustment costs of the speculator are lower
   -- The market is more liquid
   -- The deviation of the price from the speculator's estimates of the benchmark value is larger

2) The profit of the speculator will be higher if:
   -- The market is less liquid
   -- The noise order activity of the market is stronger

Because the optimal trading policy and NPV is in close form, the comparative static analysis is straightforward. Readers are encouraged to do any comparative static analysis which interests them.

Model 2a: Optimal acquisition of private information

As an application of Model 2, let's look at an example where the speculator will balance the cost and benefit of private

\[\text{See Appendix 3 for AGFP network.}\]
information and endogenously acquire private information optimally. Assume that there are two sources of non-price information. One is public information which is free. Another is private information which costs \( c \) dollars per unit time to get an observation with precision \( c/\sigma_p^2 \). In other words, private information technology exhibits constant returns to scale.\(^{11}\)

private information technology:

If the speculator pays \( c \) dollars per unit time, he will get a private observation \( dw_p \).

\[
dw_p = vdt + \frac{\sigma_p}{\sqrt{c}} db_p
\]  

(2.2.28)

Public information:

\[
dw_g = vdt + \sigma_g db_g
\]  

(2.2.29)

Information set:

\[
\mathcal{F}(t) = \{ p(\tau), w_p(\tau), w_g(\tau); -\infty < \tau < t \}
\]  

(2.2.30)

where \( b_\nu, b_\nu, b_p, b_g \) are independent standard Brownian motions.

Using the solution of Model 2, we can get an explicit expression of the net gain:

\[
\Delta J(c) = J \text{ with } w_c - J \text{ without } w_c
\]  

(2.2.31)

\(^{11}\)One way to show without using the concept of precision that private information exhibits constant returns to scale is as follows. Assume that there is an information technology which costs \( c \) dollars to get \( c \) observations of \( v \):

\[
dw_i = vdt + \sigma_i db_i, \quad i = 1, 2, \ldots, c,
\]

with \( b_i \)'s independent of each other. A sufficient statistic of this set of observations is the average of \( w_i \)'s which is defined as \( \bar{w}_p \). It is then straightforward to show that \( \bar{w}_p \) satisfies Eq.2.2.29.
\[
\frac{1}{r} s_{i,1} \sigma_u^2 \left( \frac{2}{1 + \frac{\varphi_m^2 \sigma_u^2}{\sigma_v^2} + \frac{\varphi_m^2 \sigma_u^2 \left( \varphi_m^2 \sigma_u^2 + \sigma_v^2 \right)}{\varphi_u^2} \frac{1}{\sigma_g^2}} \right) = \frac{1}{r} \left( \frac{1}{1 + \frac{\varphi_m^2 \sigma_u^2}{\sigma_v^2} + \frac{\varphi_m^2 \sigma_u^2 \left( \varphi_m^2 \sigma_u^2 + \sigma_v^2 \right)}{\varphi_u^2} \left( \frac{1}{\sigma_g^2} + \frac{c}{\sigma_p^2} \right)} \right) \]
\]

= gain from the private information \( w_p \)

- cost of the private information \( w_p \)

(2.2.31")

Here, the cost of private information equals \( c/r \) which is a perpetuity of cost \( c \) per unit time.

Fig.2.2.1 shows two cases of optimal information acquisition by plotting the cost-benefit relationship of Eq.2.2.31". In case 1, the speculator's optimal acquisition for private information is \( c' \) which comes from first order condition of Eq.2.2.31.

But in case 2, the optimal information acquisition is zero because it will not pay to get even one unit of private information. This is because the first dollar spent on collecting private information will result in less than one dollar gain in average profit.

**Summary**

In this paper, I investigate the intertemporal optimization problem of a speculator who optimally extracts value information from the following:

- price history
- public information
- private information

To deal with the problem as realistically as possible this
model has the following features:

-- The speculator trades continuously in order to optimize his objective intertemporally.
-- The speculator takes the price impact of his trading into account.
-- Dynamic information unfolds and resolves continuously.
-- The speculator observes, incorporates and updates information continuously.
-- Adjustment costs and carrying costs are included.
-- The speculator will optimally acquire costly private information.

A notion of benchmark value is introduced to capture the "intrinsic value" of a good in continuous time.

For a simple single-market example, the closed form solution
of the optimal trading policy and the optimal value function have been presented.
Appendix 1 Details of Calculations

This appendix deals with the technical details of the calculations in the main text.

A1.1 Proof of solution of model 1:

This proof can be sketched as:
(a) Transform the problem into the standard form of Thm.A2.1.
(b) Apply Thm.A2.1 to get noise trading flow representation.
(c) Use variable transformation to get price deviation representation.

First, use Thm.A2.4 to rewrite 2.1.1 into:

\[ J = \max_{f(s)} E \left[ \int_0^\infty e^{-zt} \left[ (p-m) f - \delta f^2 - \lambda g^2 \right] dt - [p(0)-m]g(0) \right] \]  \hspace{1cm} (A1.1.1)

rewrite 2.1.5

\[ p-m = (v-m) + \phi_m^{-1} u - \phi_m^{-1} f \]  \hspace{1cm} (A1.1.2)

A1.1.2 \rightarrow A1.1.1, and put it into the standard form of Thm.A2.1

\[ J = \max_{f(s)} E \left[ \int_0^\infty e^{-zt} \left( v-m \begin{bmatrix} u \\ g \end{bmatrix} f \right) \right] - \left[ \begin{bmatrix} v-m \\ u \\ g \\ f \end{bmatrix} \right] \]  \hspace{1cm} (A1.1.3)

\[ \Delta J' - [p(0)-m]g(0) \]  \hspace{1cm} (A1.1.4)

Rewrite the dynamics equations into standard form.
From Eq.2.1.3, 2.1.4 and 2.1.6
\[
\begin{bmatrix}
\nu - m \\
\dot{u} \\
\dot{g}
\end{bmatrix} = 
\begin{bmatrix}
-\epsilon & 0 & 0 \\
0 & -\varphi_u & 0 \\
0 & 0 & r
\end{bmatrix}
\begin{bmatrix}
\nu - m \\
u \\
g
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 & f & dt \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_v \\
\sigma_u \\
\sigma_u
\end{bmatrix}
\begin{bmatrix}
dv \\
du \\
dg
\end{bmatrix}
\] (A1.1.5)

Now, J' in A1.1.4 plus A1.1.5 are in standard form of Thm A2.1. Also, the solution can be simplified because:
(i) The parameter matrices A, N, B, F, G are independent of time.
(ii) The time horizon of J' is +\infty.

So, it is the steady state solution we are looking for, i.e. \dot{S} in Eq. A2.1.6 should equal zero.

Apply Thm A2.1, after some straightforward (but tedious!) calculations, we get:

\[
\dot{f}^* = \alpha_1 (\nu - m) + \alpha_2 u + \alpha_3 g \dot{a} a^T x
\] (A1.1.6)

where

\[
\alpha_1 = \frac{\epsilon}{2\theta (\alpha_3 + \epsilon)}
\] (A1.1.7)

\[
\alpha_2 = \frac{\varphi_u}{2\theta \varphi_u (\alpha_3 + \varphi_u)}
\] (A1.1.8)

\[
\alpha_3 = \frac{s}{2} + \sqrt{\left(\frac{s}{2}\right)^2 + \frac{\lambda}{\theta}}
\] (A1.1.9)

\[
J' = x^T(0) SX(0) + \frac{1}{x} \left[ S_{11} \sigma_v^2 + S_{22} \sigma_u^2 \right]
\] (A1.1.10)

where
\[
S = S^T = \begin{bmatrix}
\frac{\theta \alpha_1^2}{x + 2\varepsilon} & \frac{\theta \alpha_1 \alpha_2}{x + \phi_u + \varepsilon} & \frac{\alpha_3}{2(\alpha_3 + \varepsilon)} \\
\frac{\theta \alpha_2^2}{x + \phi_u} & \frac{\alpha_3}{2 \varphi_m (\alpha_3 + \phi_u)}
\end{bmatrix}
\] (A1.1.11)

The \( S \) here differs from Eq 2.1.14 by an amount of:
\[
S' = (S')^T = \begin{bmatrix}
0 & 0 & - \frac{1 - \varphi_m^{-1} \alpha_1}{2} \\
0 & \varphi_m^{-1} (1 - \alpha_2) & 2
\end{bmatrix}
\] (A1.1.12)

which responds to the second term of A1.1.4. To explain it:
A1.1.6 \( \rightarrow \) A1.1.2
\[
(p - m) = (1 - \varphi_m^{-1} \alpha_1) (v - m) + \varphi_m^{-1} (1 - \alpha_2) u - \varphi_m^{-1} \alpha_3 g
\] (A1.1.13)

From Eq. A1.1.13:
\[
- [p(0) - m] g(0) = x^T(0) s' x(0)
\] (A1.1.14)

The derivation of noise order trading representation is complete.

To derive the price deviation representation, Eq. 2.1.15 to 2.1.18, we need to express state variable, \( x \), in terms of \( y \).
A1.1.6 \( \rightarrow \) A1.1.2 and solve \( u \)
\[
u = \frac{\alpha_1}{1 - \alpha_2} (v - m) + \frac{\varphi_m}{1 - \alpha_2} (p - v) + \frac{\alpha_3}{1 - \alpha_2} g
\] (A1.1.15)

which gives us the relationship of \( x \) with \( y \) as
\[ x = \begin{bmatrix} v-m \\ u \\ g \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_1 & \varphi_m & \alpha_3 \\ 1-\alpha_2 & 1-\alpha_2 & 1-\alpha_2 \end{bmatrix} \begin{bmatrix} V \\ p-v \\ g \end{bmatrix} = Ty \quad (A1.1.16) \]

Finally, substitute A1.1.16 into noise trading representation to get price deviation representation. \[ \square \]

### A1.2 Proof of Prop. 2.2.2 (Filtering problem of Model 2)

This proof can be sketched as two steps:
(a) Write the filtering problem into the standard form of Thm.A2.2.
(b) Apply Thm.A2.2.

Rewrite Eq.2.2.4 into:
\[ p+\varphi^{-1}_m f = v + \varphi^{-1}_m u \& z \quad (A1.2.1) \]

so a linear combination of unknown state variables, \( v \) and \( u \), is observed exactly. This allows us to reduce the order of the filtering problem by one.

Take expectation \( E(\cdot | \mathcal{F}) \) on both side of A1.2.1:
\[ \mathcal{F} \]
\[ \tilde{v} + \varphi^{-1}_m \tilde{u} = z \quad (A1.2.2) \]

Combining A1.2.1 with A1.2.2, we have Eq.2.2.15, which can be rewritten into:
\[ v - \tilde{v} = -\varphi^{-1}_m (u - \tilde{u}) \quad (A1.2.3) \]

from which Eq.2.2.10 is implied.

Rewrite A1.2.2 into
\[ \tilde{u} = \varphi_m (z - \tilde{v}) \quad (A1.2.4) \]
Thus, we can first estimate \( \hat{v} \), then use A1.2.4 to get \( \hat{u} \). To get \( \hat{v} \), first write the filtering problem into standard form of Thm.A1.2:

state variable dynamics:

\[
dv = \sigma_v dB_v
\]  

(A1.2.5)

observation:

\[
d\begin{bmatrix} z_1 \\ w \end{bmatrix} = \begin{bmatrix} -\varphi_v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ w \end{bmatrix} dt + \begin{bmatrix} \varphi_u \\ I \end{bmatrix} \nu dt + \begin{bmatrix} \varphi^{-1}_m \sigma_u dB_u + \sigma_v dB_v \\ \Sigma_v dB_w \end{bmatrix}
\]  

(A1.2.6)

The first row of the observations above comes from substituting Eq.2.2.3 and 2.2.5 into A1.2.1:

\[
dz_1 = dp + \varphi^{-1}_m df = dv + \varphi^{-1}_m du
\]  

(A1.2.7)

\[
= [-\varphi_v (v + \varphi^{-1}_m u) + \varphi_u v] dt + \varphi^{-1}_m \sigma_u dB_u + \sigma_v dB_v
\]

The second equation of the observation comes from writing Eq 2.2.6 into vector form:

\[
dw = \nu dt + \Sigma_w dB_w
\]  

(A1.2.8)

where

\[
w = (w_1, \ldots, w_N)^T, \quad B_w = (B_1, \ldots, B_N)^T
\]  

(A1.2.9)

and
\[
\begin{bmatrix}
\sigma_1 & 0 \\
\vdots & \ddots \\
0 & \sigma_M
\end{bmatrix}
\]  \hspace{1cm} (A1.2.10)

Before applying Thm.A1.2, it is worth noting that the filtering problem is simplified because:

(i) The parameters are independent of time.

(ii) The initial condition comes from \(-\infty\).

So, it is the steady state solution we are looking for. This means that only an algebraic, rather than a differential, form of the Riccati equation needs to be solved.

Plugging A1.2.5 and A1.2.6 into Thm.A1.2, we get Eq.2.2.12, 2.2.13 and

\[
\Omega_y = \frac{\sigma_u^2}{2\varphi_y \sigma_m^2} \frac{2}{1 + \sqrt{1 + \frac{\varphi_m^{-2} \sigma_u^2}{\sigma_v^2} + \frac{\varphi_m^{-2} \sigma_u^2 (\varphi_m^{-2} \sigma_u^2 + \sigma_v^2)}{\varphi_u^2} \sum_{m=1}^{M} \frac{1}{\sigma_m^2}}}
\]  \hspace{1cm} (A1.2.11)

combine Eq.2.2.8, 2.2.9, 2.2.10 and A1.2.11 we get Eq.2.2.11.

A1.3 Continuation of the second proof of solution of Model 2

Similar to the proof of the solution of Model 1, the rest of the proof has three stages:

(a) transform the problem into standard form of Thm.A2.3.

(b) apply Thm.A2.3 to get noise order flow representation.

(c) use variable transformation to get price deviation representation.

Eq. 2.2.4 \(\rightarrow\) 2.2.27, and put it in standard form of Thm.A2.3
\[ J = \max F \int_{0}^{\infty} e^{-\sigma t} [u \ g \ f] \begin{bmatrix} 0 & 0 & \frac{1}{2\varphi_m} \\ -\lambda & 0 & g \\ -\theta & & f \end{bmatrix} \begin{bmatrix} u \\ g \\ f \end{bmatrix} dt + E[-g(0) [p(0) - v(0)]] \]  
(A1.3.1)  
\[ \Delta J' + E[-g(0) [p(0) - v(0)]] \]  
(A1.3.1')

rewrite dynamics equation Eq.2.2.2 and 2.2.3 into standard form of Thm.A2.3  
\[ d \begin{bmatrix} u \\ g \end{bmatrix} = \begin{bmatrix} -\varphi_u & 0 \\ 0 & \sigma_u \end{bmatrix} \begin{bmatrix} u \\ g \end{bmatrix} dt + \begin{bmatrix} 0 \\ -1 \end{bmatrix} f dt + \begin{bmatrix} \sigma_u \\ 0 \end{bmatrix} db_u \]  
(A1.3.2)

Now, J' in A1.3.1' and A1.3.2 are in standard form of Thm.A2.3. The solution can be simplified because:  
(i) The horizon of J' is \(+\infty\).  
(ii) The parameter matrixes A, N, B, F, G, E are independent of time.  
(iii) Conditional variance matrix in prop.2.2.2 is independent of time.

Applying Thm.A2.3, after some calculation, we get Eq.2.2.17 to 2.2.19 and  
\[ J' = x^T(0) S x(0) + \frac{1}{\lambda^2} \sigma_u^2 s_{11} (1 - \frac{\Omega_u}{V_u}) \]  
(A1.3.3)

where  
\[ S = S^T = \begin{bmatrix} \frac{\theta \alpha_{\lambda}}{\lambda + 2\varphi_u} & \frac{\alpha_{\lambda}}{2\varphi_m} & \frac{\alpha_{\lambda}}{2\varphi_m} \\ \frac{\lambda + 2\varphi_u}{\lambda} & \frac{\alpha_{\lambda}}{2\varphi_m (\alpha_{\lambda} + \varphi_u)} \\ \frac{-\theta \alpha_{\lambda}}{\lambda + 2\varphi_u} & \frac{-\theta \alpha_{\lambda}}{\lambda + 2\varphi_u} & \frac{-\theta \alpha_{\lambda}}{\lambda + 2\varphi_u} \end{bmatrix} \]  
(A1.3.4)

The S here differs from Eq 2.2.22 by an amount of:
\[ S' = (S')^T = \begin{bmatrix} 0 & -(1 - \alpha_1) \\ \vdots & \varphi_m \alpha_2 \\ \end{bmatrix} \] (A1.3.5)

which corresponds to the second term of A1.3.1. To explain it, take \( E[\cdot | \mathcal{F}] \) on both side of 2.2.4

\[ \varphi_m (p - \hat{V}) + f = \hat{U} \] (A1.3.6)

Eq 2.2.17 -> A1.3.6

\[ (p - \hat{V}) = \varphi_m^{-1}(1 - \alpha_1) \hat{U} + \varphi_m^{-1}(-\alpha_2) g \] (A1.3.6')

A1.3.6' -> the second term of A1.3.1' is:

\[ E[-g(0)[p(0) - \nu(0)]] = -g(0)[p(0) - \hat{V}(0)] \] (A1.3.7)

\[ = \hat{x}^T(0) \dot{S}' \hat{x}(0) \]

This completes the derivation of noise order flow representation.

To derive the price deviation representation, we need to express the state variable, \( \hat{x} \), in terms of \( \hat{y} \).

Rewrite A1.3.6

\[ \hat{U} = \frac{\varphi_m}{1 - \alpha_1} (p - \hat{V}) + \frac{\alpha_2}{1 - \alpha_1} g \] (A1.3.8)

From Eq. A1.3.8, we get the relationship of \( \hat{x} \) and \( \hat{y} \) as:

\[ \hat{x} = \begin{bmatrix} \hat{U} \\ g \end{bmatrix} = \begin{bmatrix} \frac{\varphi_m}{1 - \alpha_1} & \frac{\alpha_2}{1 - \alpha_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p - \hat{V} \\ g \end{bmatrix} \cdot T \hat{y} \] (A1.3.9)

Finally, substitute A1.3.9 into noise trading flow
representation to get the price deviation representation, Eq. 2.2.23 to Eq. 2.2.26.
Appendix 2 Some Theorems Used In Calculations

In this appendix, I present some theorems which are used in the calculations of this paper.

Lowercase letters, \( x \), \( u \), etc. are used to represent vectors and capital letters \( A \), \( B \), \( C \), etc. are used to represent matrix parameters.

The dimensions of vectors and matrices can be understood in context and will not be mentioned every time.

Thm A2.1 (Discount-Linear-Quadratic-Gaussian stochastic control with perfect information)

Given a system:

\[
\frac{dx}{dt} = F(t)x + G(t)u + E(t)dB(t)
\]  

(A2.1.1)

where \( B(t) \) is an vector Brownian Motion:

\[
B(t) \sim BM((Q(t))
\]  

(A2.1.2)

And an objective:

\[
J(x(t_0)) = \arg \max_{u(t)} \int_{t_0}^{T_f} e^{-r(t)} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} A(t) & N(t) \\ N^T(t) & B(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt 
+ e^{-r(T_f)}x^T(T_f)S_fX(T_f)
\]

(A2.1.3)

where \( A(t) \), \( B(t) \), \( S_f \) are symmetric, and \( B(t) \) is negative definite.

The optimal control policy exists iff the Riccati equation, Eq.A2.1.6, has a solution. In this case, the optimal control policy is:

\[
u^*(t) = -C(t)x(t)
\]

(A2.1.4)

where

\[
C = B^{-1}(SG + N)^T
\]

(A2.1.5)

where \( S \) is the solution of Riccati equation:
\[ \dot{S} = \mathbf{rS} - \mathbf{SF} - \mathbf{F}^\top \mathbf{S} + (\mathbf{SG} + \mathbf{N}) \mathbf{B}^{-1} (\mathbf{SG} + \mathbf{N})^\top - \mathbf{A} \]  
(A2.1.6)

\[ S(t_f) = S_f \]

Note \( S \) is symmetric. And

\[ J[x(t_0)] = x^\top(t_0) S(t_0) x(t_0) + \int_{t_0}^{t_f} e^{-\mathbf{r} t} \text{Tr}[\mathbf{E}^\top(t) S(t) E(t) Q(t)] dt \]

= the term related with the initial condition
+ the term related with input over the time interval \( (t_0, t_f) \)  
(A2.1.7)

If we define:

\[ V(x, t) = \max_{u(t)} \int_{t_0}^{t_f} e^{-\mathbf{r} t} \left[ x^\top(t) u^\top(t) \right] \begin{bmatrix} A(t) & N(t) \\ N^\top(t) & B(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt 
+ e^{-\mathbf{r} t_f} x^\top(t_f) S_f x(t_f) \]  
(A2.1.8)

then:

\[ V(x, t) = e^{-\mathbf{r} t} x^\top(t) S(t) x(t) + \int_{t}^{t_f} e^{-\mathbf{r} t} \text{Tr}[\mathbf{E}(t) S(t) E(t) Q(t)] dt \]  
(A2.1.9)

Furthermore, if only the stationary solution is of interest, then there is a stationary solution iff the algebraic Riccati equation (i.e. Eq.2.1.6 with \( \dot{S} = 0 \)) has a solution which is stabilizing.

**Proof of Thm A2.1:** The thrust of this proof is:

(i) conjecture the form:

\[ V(x, t) = e^{-\mathbf{r} t} x^\top(t) S(t) x(t) + s(t) \]  
(A2.1.10)

(ii) Prove that Eq.A2.1.10 satisfies Bellman's equation. So, according to the verification theorem, \( V(x,t) \) in Eq.A2.1.10 is the solution.
Substitute Eq.A2.1.10 into Bellman's equation:

\[
0 = \max_{u^{(*)}} \{ e^{-rt} \left[ x^T(t) \right. u^T(t) \left. \right] \begin{bmatrix} A(t) & N(t) \\ N^T(t) & B(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + \mathcal{L}V \}
\]

\[
= \max_{u^{(*)}} \{ e^{-rt} \left[ x^T(t) \right. u^T(t) \left. \right] \begin{bmatrix} A(t) & N(t) \\ N^T(t) & B(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \]
\[
+ V_t + V_x^T (Fx+Gu) + \frac{1}{2} \text{Tr}(E^T \chi EQ) \}
\]

\[
= \max_{u^{(*)}} \{ [u + B^{-1} (SG+N)^T x ]^T B [u + B^{-1} (SG+N)^T x ] \}
\]

\[
+ x^T [ \dot{S} - r S + SF + F^T S - (SG+N) B^{-1} (SG+N)^T A ] x \]
\[
+ s + e^{-rt} \text{Tr}(E^T SEQ) \}
\]

(A2.1.11)

(A2.1.11')

(A2.1.11'')

From A2.1.11''

\[
\dot{S} = rS - SF - F^T S + (SG+N) B^{-1} (SG+N)^T A
\]

\[
S(t_r) = S_r
\]

(A2.1.12)

(A2.1.13)

in which the initial conditions come from comparing Eq.A2.1.10 with A2.1.8. Therefore, we have verified that Eq.A2.1.10 is the solution.

Solving A2.1.13, we get:

\[
s(t) = \int_t^{t_r} e^{-rt} \text{Tr}[E^T(t)S(t)E(t)Q(t)]
\]

(A2.1.14)
Substituting A2.1.14 into A2.1.10, we get A2.1.9. Set t equals zero, we get A2.1.7. And

\[ u^* = B^{-1}(S+G)T x \]

is the result from Eq. A2.1.11' to A2.1.11".

**Thm.A2.2 (Kalman-Bucy Filter)**

Given a u controlled system:

\[
\begin{align*}
\dot{x} &= \left[ F_0(t) + F_1(t) x + F_2(t) z + F_3(t) \dot{\nu} + G_x(t) u \right] dt + H(t) dw(t) \\
x(t_0) &= N(\overline{x}_0, P_0)
\end{align*}
\]  
(A2.2.1)

with imperfect observation:

\[
\begin{align*}
\dot{z} &= \left[ M_0(t) + M_1(t) x + M_2(t) z + M_3(t) \dot{\nu} + G_z(t) u \right] dt + d\nu(t)
\end{align*}
\]  
(A2.2.2)

where \( w(t), v(t) \) are Brownian Motions:

\[
\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \sim BM \begin{bmatrix} D(t) & L(t) \\ L^T(t) & R(t) \end{bmatrix}
\]  
(A2.2.3)

which are independent of \( x(t_0) \). Define information set at \( t \)

\[
\mathcal{S}_t = \overline{x}_0, p_0 \text{ and } z(t), u(t), t_0 \leq t \leq t
\]  
(A2.2.4)

Then, the conditional distribution of \( x \) given \( \mathcal{S}_t \) is:

\[
(x(t) | \mathcal{S}_t) \sim \text{Normal} (\hat{x}(t), P(t))
\]  
(A2.2.5)

where \( P(t) \) satisfies the Riccati equation:

\[
\begin{align*}
\begin{bmatrix} \dot{P}(t) - P(t) & \dot{P}(t) - P(t) \\ \dot{P}(t) & H^T(t) \end{bmatrix}
&= \begin{bmatrix} P_0 & P_0 \\ P_0 & H(t) \end{bmatrix} \\
P(t_0) &= P_0
\end{align*}
\]  
(A2.2.6)

and \( \hat{x}(t) \) is the solution of following differential equation:
\[
\dot{x} = \left[ F_0(t) + F_1(t) \dot{x} + F_2(t) z + G_x(t) u \right] dt + K(t) \{ dz - [M_0(t) + M_1(t) \dot{x} + M_2(t) z + G_z(t) u] dt \}
\]
\[
\dot{x}(t_0) = \bar{x}_0
\]

where
\[
K(t) = (PM_1^T + HL) R^{-1}
\] (A2.2.8)

and innovation process,
\[
dy = dz - [M_0(t) + M(t) \dot{x} + M_2(t) z + G_z(t) u] dt
\] (A2.2.9)

is a Brownian Motion adapted into the information set \( \mathcal{F}_t \) with
\[
[y(t) | \mathcal{F}_t] \sim BM[R(t)]
\] (A2.2.10)

Remark:
This is a straightforward extension of the standard representation of Kalman Filter. The reason to put \( z, \dot{v}, u \) in both system and observation equations is to emphasize that all these variables are intertwined into the system due to the complexity of multi-agent control. The standard representation of Kalman Filter can be found in, for example, Jazwinski (’70).  

Thm.A2.3 (Discount-Linear-Quadratic-Gaussian stochastic control with imperfect information)

Given the same set-up as Thm.A2.1 except that instead of perfect information of \( x(t) \) only imperfect observation as the set-up of Thm.A2.2 is available. To get an optimal solution, we can separate the estimation process (as in Thm.A2.2) and control (as in Thm A2.1). And the optimal control policy is:
which is called the certainty equivalence principle. And

\[
(J(x) | \mathcal{F}_{t_0}) = \bar{x}_0^T S(t_0) \bar{x}_0 + \text{Tr}[S(t_0) P_0] \\
+ \int_{t_0}^{t'} e^{-rt} \text{Tr}[E^T(t) S(t) E(t) Q(t) + B(t) C(t) P(t) C^T(t)] dt 
\]

\[
(V(x, t) | \mathcal{F}_t) = e^{-rt} [\bar{x}(t) S(t) \bar{x}^T + \text{Tr}[S(t) P(t)]] \\
+ \int_t^{t'} e^{-rt} \text{Tr}[E^T(t) S(t) E(t) Q(t) + B(t) C(t) P(t) C^T(t)] dt 
\]

where $C(t)$, $S(t)$, $P(t)$ and $\bar{x}$ are same as in Thm.A2.1 and Thm.A2.2.

**Remark:**

The major difference between this theorem and the traditional linear-quadratic-Gussian-with-imperfect-information problem in a standard text book, for example Astrom (1970), is that:

-- there is a cross term $N(t)$.

-- there is a discount term $e^{-rt}$

Since these problems have been dealt with in Thm.2.1, the rest of the proof is similar to the one in, for example, Astrom (1970).

**Thm.A2.4 (Equivalent Representation Theorem)**

Assume

\[
dg = -fdt + rg dt \tag{A2.4.1} 
\]

Then, the following four representations are equivalent

(1) discount profit flow representation:

\[
E(\int_{t_0}^{t_f} e^{-rt} g^T dp) \tag{A2.4.2} 
\]
(2) discount cash flow representation:

\[
E\left[ \int_{t_0}^{t_f} e^{-rt} f^T t dt + [e^{-rt} g^T(t) p(t)] \bigg| \mathcal{F}_t \right] \quad (A2.4.3)
\]

(3) discount potential profit flow representation:

\[
E\left( \int_{t_0}^{t_f} e^{-rt} f^T (p - v) dt + [e^{-rt} g^T(t) [p(t) - v(t)]] \bigg| \mathcal{F}_t \right) \quad (A2.4.4)
\]

where \( v(t) \) is any martingale process.

(4) discount price deviation representation:

\[
E\left( \int_{t_0}^{t_f} e^{-rt} f^T (p - m) dt + [e^{-rt} g^T(t) [p(t) - m]] \bigg| \mathcal{F}_t \right) \quad (A2.4.5)
\]

where \( m \) is any constant.

**Proof:**

Because a constant is a martingale, case(4) and case(2) (zero is a constant) are special cases of case(3). Now I prove case(1) \( \iff \) case(3). Note if \( v \) is a martingale

\[
E\left( \int_{t_0}^{t_f} e^{-rt} g^T dv \right) = 0 \quad \text{for any } g \quad (A2.4.6)
\]

\[
E\left( \int_{t_0}^{t_f} e^{-rt} g^T dp \right) = E\left[ \int_{t_0}^{t_f} e^{-rt} g^T d(p - v) \right] = E\left( \int_{t_0}^{t_f} e^{-rt} f^T (p - v) dt + [e^{-rt} g^T(t) [p(t) - v(t)]] \bigg| \mathcal{F}_t \right) \quad (A2.4.7)
\]
the last equality comes from integration by parts and substituting Eq. A2.4.1.

Remark:
This is a very powerful theorem. As has been shown in this paper and Lu (1991), this theorem makes it possible to transform problems of interest into standard solvable forms. As far as I know, this is the first time this theorem has been introduced.

Thm.A2.5 (Solution of linear stochastic differential equations)

Given a system:

\[ dx = F(t)x dt + E(t) dB(t) \]

\[ x(t) \sim \text{Normal}(m_0, Q_0) \]  \hspace{1cm} (A2.5.1)

where B(t) is a vector Brownian Motion:

\[ B(t) \sim BM(Q(t)) \]  \hspace{1cm} (A2.5.2)

Then, the conditional distribution of x(t) given \((m_0, V_0)\) only is:

\[ (x(t) \mid (m_0, V_0)) \sim \text{Normal}(m, V) \]

where \( m \) is the solution of:

\[ \dot{m} = F(t)m \]
\[ m(0) = m_0 \]  \hspace{1cm} (A2.5.3)

and \( V \) is the solution of:

\[ \dot{V} = FV + VF^T + E E^T \]
\[ V(0) = V_0 \]  \hspace{1cm} (A2.5.4)

Remark:
This is a well-established result of stochastic systems.
theory. The reader can find a formal proof from, for example, Davis (1977).
Appendix 3

This appendix collects the AGFP networks of the Models in this paper. Please refer to Lu (1990) for the explanations of the symbols.

Model 1 Speculation with perfect information
Model 2 Speculation with imperfect information
References


Kumar, P. R., Pravin Varaiya, [1986], Stochastic Systems: Estimation, Identification, and Control, Prentice Hall.


