Differential Information and Dynamic Behavior of Stock Trading Volume

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May 1993
RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

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Differential Information
and
Dynamic Behavior of Stock Trading Volume

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May 1993
Finance Working Paper #228

The authors thank Bruce Grundy, Jon Ingersoll, Richard Kihlstrom, Pete Kyle, Richard Lindsey, Roni Michaely, Krishna Ramaswamy, Steven Ross, and Matthew Spiegel for helpful conversations and comments. Financial support from the Batterymarch Fellowship Program (for Hua He) is gratefully acknowledged.
Abstract

We develop a multi-period model of stock trading in which investors receive differential information concerning the underlying value of the stock. Investors trade competitively in the market based on their own private information and the information revealed by the market clearing prices as well as other public news. By showing that the hierarchy of expectations (i.e., forecasting the forecasts of others) is a closed system, we resolve the "infinite regress" problem that is common to intertemporal models with differential information and derive a rational expectations equilibrium. We analyze the dynamic behavior of equilibrium trading volume. In particular, we examine how trading volume is related to the information flow to the market and how investors' trading reveals their private information.
1 Introduction

Stock trading volume exhibits high serial correlation.\(^1\) In a competitive market, trading occurs when investors receive new information about the economy, either from private sources or from public sources.\(^2\) Therefore, the behavior of trading volume is closely related to the nature of information flow to the investors. In the case that all information is public, clustering in trading implies that arrivals of new information are correlated.\(^3\) In the case that there is private information, clustering in trading can be generated by independent information arrivals. This is because equilibrium prices are noisy and do not fully reveal all private information even when investors behave competitively (see, e.g., Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verrecchia (1981), and Grundy and McNicholes (1989)). Thus, investors may trade many rounds after their initial reception of private information until all information is revealed through the sequence of prices. Studying the link between volume and the nature of information flow has interesting implications. For example, by empirically examining the behavior of volume we may be able to make inferences about the information structure of the economy which is not directly observable.

In this paper, we consider a multi-period trading model in which investors receive both public and private information about the true value of an asset (the “stock”). The private information is of equal quality to all investors. They trade competitively in a stock market for many periods either to accommodate noise traders (“non-informational trading”) or to speculate on the true value of the stock (“informational trading”). The value of the stock is revealed on the terminal date. We use the model to analyze the dynamic behavior of trading volume and its relation with information flow to the economy.

The pattern of volume over time is closely related to the flow of information and the nature of the information, e.g., exogenous vs. endogenous and private vs. public. In our model, exogenous information includes new private signals and public announcements while endogenous information is simply the market clearing prices which is public. Exogenous information, private or public, always generates immediate trading. However, private information not only generates trading in the current period, but also leads to possible trading in future periods. Public


\(^2\)Kyle (1985, 1989) considers the situation when informed investors behave strategically in maximizing the gains of their information trading. They trade gradually in order to prevent private information from being quickly revealed and to extract more profits. Admati and Pfleiderer (1988) also study a trading model with strategic behavior. In this paper, we follow the competitive approach.

\(^3\)For models of dynamic trading without private information, see Huffman (1987) and Dumas (1991).
information, on the other hand, only generates trading in the current period.

Two factors affect investors' informational trading. One is the risk in taking positions on private information and the other is the trading opportunities remaining before the uncertainty is fully resolved. Investors trade on their private information more aggressively when there is less uncertainty in the true value of the stock. As they continue to trade, more private information is revealed through prices and there is less risk in speculating. Thus, investors trade more aggressively over time even though there is less disagreement among them. On the other hand, investors trade less aggressively when there are fewer trading opportunities left since it becomes more difficult to unload their positions when they want to. Consequently, investors intend to reduce their speculative positions as the terminal date approaches. The trade-off between these two factors gives rise to trading over time by investors long after their initial reception of private information. The pattern of volume over time is in general non-monotonic. Therefore, when there is private information, volume is not only related to the contemporaneous information flow but also related to the existing private information (received previously). In the current model, exogenous public information such as public announcements generates trading because it reduces the uncertainty about the true value of the stock and affects investors' informational trading. Consequently, investors will optimally time their trading position in anticipation of expected public announcement. They increase their speculative positions right prior to the public information. This allows them to bet on their private information without bearing unnecessary risk. After the public announcement, they immediately reduce their position and realize the gain or loss. Thus, high volume is observed with public announcements. Furthermore, we find that abnormal volume generated by public announcements depends on the timing of the announcements.

Our model also leads to interesting implications about the relation between trading volume and price volatility. Exogenous information leads to trading as well as changes in prices. Thus, high volume generated by the exogenous information, private or public, is accompanied by high volatility in prices. However, trading generated by the existing private information is not accompanied by abnormally high volatility in prices. In this case, the trading is mainly due to investors' need to unwind their positions against each other. It does not necessarily generate large changes in their expectations about the value of the stock.

The existing work that is closely related to this paper includes Grossman and Stiglitz (1980), Diamond and Verrecchia (1981), Pfleiderer (1984), Grundy and McNicholes (1989), Brown and Jennings (1989) and Kim and Verrecchia (1991a,b). For example, Grundy and McNicholes consider a three-date model similar to ours and analyze its rational expectations equilibrium.
The current model can be viewed as an extension of the existing models in two ways. First, it is a general multi-period model and second, it allows more general information flow to the economy. A truly multi-period model is necessary for studying the dynamic behavior of trading volume.\(^4\) The more general information flow in a dynamic setting allows us to analyze the impact of different type of information on trading volume.

Solving intertemporal trading models with differential information often faces the well-known "infinite regress" problem (see, for example, Townsend (1983)). In general, the equilibrium depends on the whole hierarchy of expectations including each investor's expectation of the true state of the economy, his expectation of other investors' expectations, \(\cdots\), etc. The set of expectations is usually not a closed system. This makes the solution intractable. For the current model, we are able to find a solution that makes the hierarchy of expectations a closed system. Thus, the equilibrium has a simple characterization.\(^5\)

The paper is organized as follows. We specify the model in Section 2 and solve the equilibrium in Section 3. In Section 4, we examine the behavior of prices and volume, especially the relation between information flow and the dynamics of volume. Section 5 concludes.

2 The Model

In this section, we consider a multi-period model of stock trading in which investors receive in each period private and public information concerning the future value of the stock. Investors trade competitively in the market based on their own private information. There also exists noise in the market which prevents the equilibrium price from fully revealing investors' private information. We use this model to study the equilibrium of the economy, especially the effect of differential information on stock trading volume.

2.1 Investment opportunities

There is a riskless asset and a risky asset ("stock") available for trading on dates \(1, \cdots, T - 1\). The riskless asset is of perfectly elastic supply with the rate of return \(r\) being a non-negative constant. For simplicity, we assume \(r = 0\). Each share of the stock pays a liquidation value of

\(^4\) In models with three dates \((0, 1, 2)\), trading occurs only in the first two dates. The volume in the first date depends on the specification of investors' initial endowments. The only truly endogenous volume is the volume in the second date. Thus, these models cannot be used to analyze the dynamics of volume.

\(^5\) A number of recent papers have studied dynamic trading models in which some investors are better informed than the others; see Wang (1993a), Geanette and Kyle (1991), Foster and Viswanathan (1992). The infinite regress problem vanishes in those models due to the assumption that the better informed investors observe everything known to the less informed investors.
$H+\delta$ on the final date $T$. Shares of the stock are infinitely divisible and are traded competitively in the stock market. Let the equilibrium share price of the stock on date $t$ be $P_t$.

The stock is of a given supply which may change over time. Let $\Theta_t$ be the total number of shares available in the market on date $t$, $t = 1, \cdots, T-1$. $\Theta_t$ follows an AR(1) process:

$$\Theta_t = a_\Theta \Theta_{t-1} + \epsilon_{\Theta,t},$$

where $0 \leq |a_\Theta| \leq 1$. $\epsilon_{\Theta,t}$ is normal, $\mathcal{N}(0, \sigma^2_\Theta)$, and i.i.d. over time. The assumption of a random supply of the stock is equivalent to the usual noise trading story, i.e., the noise traders supply a total of $\Theta_t$ shares of the stock. When $a_\Theta = 0$, the amount of noise trading is i.i.d. over time, which is likely to happen when the time length of two consecutive trading dates is large. When $a_\Theta = 1$, the incremental changes of noise trading are i.i.d. over time. This is likely to happen when the time length of two consecutive trading dates is small.

### 2.2 Investors

Let $\mathcal{I}$ be the set of investors in the economy. For investor $i \in \mathcal{I}$, he maximizes expected utility of the form.\(^6\)

$$\mathbb{E} \left[ -e^{-\lambda W^i_T} \bigg| \mathcal{F}^i_t \right].$$

(2.2)

Here, $W^i_T$ is his consumption on the final date $T$, $\mathcal{F}^i_t$ his information set on date $t$, and $\lambda$ his Arrow-Pratt risk aversion coefficient.

To simplify the problem, we assume that $\mathcal{I} = \{1, 2, \cdots\}$, i.e., the set of natural numbers.\(^7\) For the convenience of aggregation, we define a charge space ($\mathcal{I}, \varphi(\mathcal{I}), \nu$) where $\varphi(\mathcal{I})$ is the collection of all subsets of $\mathcal{I}$ and $\nu : \varphi(\mathcal{I}) \rightarrow \mathbb{R}_+$ is a finitely additive measure with the property that $\nu(A) = \lim_{N \rightarrow \infty} \frac{1}{N} \#(A \cap \{1, 2, \cdots, N\}), \forall A \subseteq \mathcal{I}$ whenever the limit exists.\(^8\) Clearly, investors are equally weighted according to the defined measure. The aggregation of the random variable $z^i (i \in \mathcal{I})$ over $\mathcal{I}$ with respect to $\nu$ is then given by

$$\int_{\mathcal{I}} z^i \equiv \int_{i \in \mathcal{I}} z^i d\nu(i) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N} z^i.$$  

(2.3)

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\(^6\)It is assumed that investors only consume on the last date. It can be shown that allowing intermediate consumption is a straightforward extension.

\(^7\)See Pfeifer (1984) for a similar approach. This assumption simplifies the equilibrium price function significantly.

2.3 Information structure

All investors have the same prior about $\Pi$, $\delta$ and $\Theta_0$. Assume that the prior distributions are: $\Pi \sim \mathcal{N}(0, \sigma_\Pi^2)$, $\delta \sim \mathcal{N}(0, \sigma_\delta^2)$, $\Theta_0 \sim \mathcal{N}(0, \sigma_{\Theta_0}^2)$, and $\Pi$, $\delta$, $\Theta_0$ are uncorrelated.

On each date $t$, investor $i \in \mathcal{I}$ receives a private signal $S_i^t$ about the first component of stock's liquidation value $\Pi$:

$$S_i^t = \Pi + \epsilon_{S_i}^t,$$  \hfill (2.4)

where $\epsilon_{S_i}^t$ is the noise in investor $i$'s signal. For simplicity, we assume that $\epsilon_{S_i}^t \sim \mathcal{N}(0, \sigma_{S_i}^2)$ and are i.i.d. across investors. In addition to private signals, investors receive a public signal $Y_t$ about $\Pi$:

$$Y_t = \Pi + \epsilon_{Y,t},$$  \hfill (2.5)

where $\epsilon_{Y,t} \sim \mathcal{N}(0, \sigma_Y^2)$ is the i.i.d. noise in the public signal. Also, $P_t$ is observable to all investors. Thus, we can write the investors' information set as follows:

$$\mathcal{F}_t^i = \{P_t, Y_t, S_i^t : 1 \leq t \leq T\}, \quad i \in \mathcal{I}. \quad (2.6)$$

Since $\epsilon_{S_i}^t$ is i.i.d. across investors, information is symmetrically distributed among investors.\(^9\)

For simplicity, we shall assume all the shocks to the economy, $\{\epsilon_{\Theta,t}, \epsilon_{Y,t}, \epsilon_{S_i}^t : i \in \mathcal{I}\}$, are jointly normal, independent of each other, independent over time, and independent from $\Pi$, $\delta$ and $\Theta_0$.\(^10\)

Note that there are two components in the value of the stock $\Pi$ and $\delta$. Investors are endowed with information about $\Pi$ but no information is available about $\delta$. Moreover, the total amount of private information is sufficient to infer the true value of $\Pi$ in the current model. As the number of trading dates increases, more private information will be compounded into the equilibrium price and the true value of $\Pi$ will be eventually revealed. However, the other component of the liquidating value $\delta$ is never revealed before the terminal date. So, the uncertainty about the value of the stock remains till the end of the economy when $\sigma_\delta \neq 0$.

---

\(^9\)The more general situation would be to allow $\epsilon_{S_i}^t$ to have different variance for different investors and to be correlated across investors. Let $i$ and $i'$ be two investors. In the case that $\text{Var}[\epsilon_{S_i}^t] < \text{Var}[\epsilon_{S_i'}^t]$ and $\mathbb{E}[\epsilon_{S_i}^t'\epsilon_{S_i'}^t] = 0$, investor $i$ has a signal independent from that of $i'$ but with better precision. However, since the two signals are independent, $\epsilon_{S_i}^t$ is still informative to $i$ given $\epsilon_{S_i'}^t$. If $\mathbb{E}[\epsilon_{S_i}^t'\epsilon_{S_i'}^t] \neq 0$, there would be a common noise in the two investors' signals. Suppose there is a common noise in all the investors' signals. Then aggregation of information across all investors would not reveal the true value of $\Pi$ even when the number of investors goes to infinity, see Pfeiderer (1984). Another case is when $\epsilon_{S_i}^t = \epsilon_{S_i'}^t + \epsilon_i$ and $\mathbb{E}[\epsilon_i|\epsilon_{S_i}^t] = 0$. Then, the signal $\epsilon_{S_i}^t$ strictly dominates the signal $\epsilon_{S_i'}^t$. Given $\epsilon_{S_i}^t$, no additional information is provided by $\epsilon_{S_i'}^t$. In this case, investor $i$ has superior information than $i'$. For this paper, we restrict ourselves to the case with $\epsilon_{S_i}^t$ i.i.d. across all investors.

\(^10\)As it will become clear, extensions to more general correlation structure (except for the correlation between the signals of different investors) are quite straightforward.
3 Equilibrium

In this section, we derive the rational expectations equilibrium of the multi-period economy specified in the previous section. We follow the standard circular approach. An equilibrium price function is first conjectured. Investors’ optimization problem is then solved under the conjectured price function. Finally, market clearing condition is imposed to verify the conjectured price function.

3.1 Equilibrium price function

For convenience, let $\mathcal{F}_i^c = \{P_\tau, Y_\tau : 1 \leq \tau \leq t\}$ and $\mathcal{F}_i^{p,i} = \{S_\tau^i : 1 \leq \tau \leq t\}$. Here, $\mathcal{F}_i^c$ is the information set that contains only the publicly available information (including public announcements, the current and historical stock prices) and $\mathcal{F}_i^{p,i}$ contains only investor $i$’s private signals received up to $t$. Clearly, $\mathcal{F}_i = \mathcal{F}_i^c \lor \mathcal{F}_i^{p,i}$. Also let $E[I|\mathcal{F}_i^c] = \hat{\Pi}_i^c$, $E[I|\mathcal{F}_i^{p,i}] = \hat{\Pi}_i^{p,i}$ and $E[I|\mathcal{F}_i] = \hat{\Pi}_i$, where $E[\cdot|\mathcal{F}_i]$ is the expectation conditional on $\mathcal{F}_i$. For a technical analyst or an econometrician who only observes prices and public signals, $\hat{\Pi}_i^c$ will be his estimate of the stock’s liquidation value. Given the well known properties of CARA preferences under normal distributions of payoffs and signals, we conjecture that the equilibrium price has the following form:

$$P_t = (1 - p_{\Pi,i})\hat{\Pi}_i^c + p_{\Pi,i}\Pi - p_{\Theta,i}\Theta_t. \quad (3.1)$$

$\Theta_t$ enters the price function since it affects the total number of stocks held by the investors, hence the total risk the economy has to bear. $\Pi$ is in no investor’s information set. A priori, one would not expect it to show up directly in the price function. Instead, the price should depend on the average of individual expectations of $\Pi$ which in turn depends on the average of private signals. In the current set-up, however, there are infinite number of investors and their private signals have i.i.d. noise. By the Law of Large Numbers, the average of all private signals is simply $\Pi$. Therefore, the equilibrium price depends on the true value of $\Pi$ and is not affected by the noise in individual signals.\textsuperscript{11}

In Eq.(3.1), we have chosen the coefficients of $\Pi$ and $\hat{\Pi}_i^c$ to add up to one. This is because a constant shift in $\Pi$ will shift both $\hat{\Pi}_i^c$ and $\hat{\Pi}_i^{p,i}$ as well as $P_t$ by the same constant (recall that the riskless rate is assumed to be zero here). It is important to note that the equilibrium stock price depends on the whole history of the economy. The price function in Eq.(3.1) states that

\textsuperscript{11}In the case that the noises in private signals are correlated, the average of the signals does not give the true value of $\Pi$. Instead, it will be $\Pi$ plus the common noise. Our analysis can be extended to this case.
this dependence can be summarized by a single variable $\hat{\Pi}^c$ which is a function of the history of stock prices and public signals as we will see below.

Define $\xi_t = \Pi - \mu_t \Theta_t$, where $\mu_t \equiv p_{\Theta_t} / p_{\Pi_t}$. We can rewrite the conjectured price function as $P_t = (1 - p_{\Pi_t}) \hat{\Pi}^c_t + p_{\Pi_t} \xi_t$. Since $\hat{\Pi}^c_t$ is a function of public information, observing the price is equivalent to observing $\xi_t$ which is a linear signal about $\Pi$ with a noise that is proportional to $\Theta_t$. Thus, $\mathcal{F}_t^c = \{ \xi_t, Y_t : 1 \leq t \leq t \}$. In what follows, we first derive the investors' optimal stock demand and then determine the coefficients of equilibrium price function.

3.2 Investors' filtering problem

In order to derive investor $i$'s optimal stock demand, we have to solve the investor's conditional expectations given his information set. Under the conjectured price function, the problem becomes a linear filtering problem since all the signals are linear in the state variables, including endogenous signals such as prices.

When investor $i$ forms his expectation about the true value of the stock conditional on his information set, he will try to infer other investors' expectations, or more precisely, the market average expectation of the value of the stock. The following lemma shows how this expectation about future stock prices is formed. (All proofs are in the Appendix).

Lemma 3.1 Define $w_t^{-1} \equiv \frac{1}{\sigma_{S,t}^2} + \cdots + \frac{1}{\sigma_{S,t}^2}$ and $\alpha_t \equiv \frac{\mu_t}{\sigma_{\xi_t}^2 + \mu_t}$ where $\alpha_t^c = \text{Cov}[\Pi, \Pi | \mathcal{F}_t^c]$. Then,

$$\hat{\Pi}^i_t = \alpha_t \hat{\Pi}^c_t + (1 - \alpha_t) \hat{\Pi}^{p,i}_t, \quad \hat{\Pi}^{p,i}_t = \frac{\mu_t}{\sigma_{\xi_t}^2} \sum_{\tau=1}^{t} \frac{S^i_{\tau}}{\sigma_{S,\tau}^2}. \quad (3.2)$$

The above lemma states that investor $i$'s forecast of $\Pi$ conditional on his information set is a weighted average of his forecast conditional only on the public information and his forecast conditional only on his private signals. Define $\hat{\Pi}_t \equiv \int_i \hat{\Pi}^i_t$, which is the market average of investors' forecasts of $\Pi$. Then,

$$\hat{\Pi}_t = \alpha_t \hat{\Pi}^c_t + (1 - \alpha_t) \Pi, \quad (3.3)$$

where we have used the fact that $\int_i S^i_t = \Pi$. In general, in a market with differential information, an investor's asset demand depends not only on his expectation of the true asset value but also on his expectation of the expectations of others, and so on. The whole hierarchy of expectations does not necessarily close on itself. This is the well-known infinite regress problem that often makes the solution of the equilibrium not tractable (see, for example, Townsend (1983)). Our model, however, exhibits the property
that the hierarchy of expectations is a closed system which enables us to solve the equilibrium in closed form. In order to see this, define \( \hat{\Pi}_t = E[\tilde{\Pi}_t|\mathcal{F}_t^i] \), which is investor \( i \)'s forecast of the market average forecast for \( \Pi \). It then follows that

\[
\hat{\Pi}_t = \alpha_t \hat{\Pi}_t^i + (1 - \alpha_t) \hat{\Pi}_t^i, \quad \int_i \hat{\Pi}_t = \alpha_t \hat{\Pi}_t^i + (1 - \alpha_t) \hat{\Pi}_t.
\]

Thus, investor \( i \)'s forecast of the market average forecast is a weighted average of the forecast using the public signals and the forecast using both the public and the private signals. In particular, it is a function of \( \hat{\Pi}_t^i \). It should not be difficult to see that individual \( i \)'s forecast of the market average forecast of the market average forecast \( \cdots \) is also a function of \( \hat{\Pi}_t^i \). Thus, the number of state variables necessary to characterize the equilibrium of this economy does not explode as it would in Townsend (1983).\(^{12}\)

We now present the solution to investor \( i \)'s inference problem in the following lemma. The basic technique here is the Kalman filtering.

**Lemma 3.2** \( (\hat{\Pi}_t^i, \hat{\Theta}_t^i) \) and \( (\hat{\Pi}_t^i, \hat{\Theta}_t^i) \) satisfy the following stochastic difference equations:

\[
\begin{pmatrix}
\hat{\Pi}_t^i \\
\hat{\Theta}_t^i
\end{pmatrix} =
\begin{pmatrix}
E_{t-1}^{\Pi}[\Pi] \\
E_{t-1}^{\Theta}[\Theta_i]
\end{pmatrix} + K_t \begin{pmatrix}
Y_t - E_{t-1}[Y_t] \\
\xi_t - E_{t-1}[\xi_t]
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\hat{\Pi}_t^i \\
\hat{\Theta}_t^i
\end{pmatrix} =
\begin{pmatrix}
E_{t-1}^{\Pi}[\Pi] \\
E_{t-1}^{\Theta}[\Theta_i]
\end{pmatrix} + K_t \begin{pmatrix}
Y_t - E_{t-1}[Y_t] \\
S_t - E_{t-1}[S_t] \\
\xi_t - E_{t-1}[\xi_t]
\end{pmatrix},
\]

where \( E_t^{\cdot}[] \equiv E[\cdot|\mathcal{F}_t^i] \), \( E_i^{\cdot}[\cdot] \equiv E[\cdot|\mathcal{F}_t^i] \), and \( K_t \) and \( K_t \) are respectively \((2 \times 2)\) and \((2 \times 3)\) constant matrices given in Appendix.

### 3.3 Expected excess share returns

Given the proposed price function, the excess return on one share of the stock is \( Q_{t+1} = P_{t+1} - P_t \). From Eq.(3.1) and Eq.(3.5-3.6), we have

\[
Q_{t+1} = \left[ (1 - \pi_{t+1} \Pi) \hat{\Pi}_{t+1}^i - (1 - \pi_{t,i}) \hat{\Pi}_t^i \right] + (\pi_{t+1} - \pi_{t,i}) \Pi - (\pi_{t+1} \Theta_{t+1} - \pi_{t,i} \Theta_t).
\]

The following lemma gives the expected excess share returns on the stock:

\(^{12}\)Note that our model has a finite time horizon and a finite dimensionality of unknown variables. Thus, the number of state variables needed to characterize the equilibrium is finite although it can be very large. This is different from the situation of infinite horizon considered by Townsend. However, the nature of the problem is the same. With heterogeneous information, the state of the economy generally depends on its whole history since investors' expectations are based on the sample path of the economy. When we try to express the history dependence through the expectations of investors, a hierarchy of expectations is needed. Our model provides an example in which a few layers of expectations are sufficient to characterize the equilibrium. In particular, two expectations (individual investors' conditional expectations and their expectations of the market average of individual expectations) span the space of expectations even though the dimensionality of the economy increases with the lifetime of the economy.
Lemma 3.3 Investor i's expected excess share return can be expressed as

\[ E[Q_{i+1} | X_i] = e_{\Theta, t+1} \hat{\Theta}_i^j + e_{\Pi, t+1}(\tilde{\Pi}^i_t - \hat{\Pi}^i_t), \]  

(3.8)

where \( e_{\Pi, t+1} \) and \( e_{\Theta, t+1} \) are constants given in the Appendix.

Individual i’s expected excess return has two components. The first component \( e_{\Theta, t+1} \hat{\Theta}_i^j \) represents the premium demanded to accommodate the supply shocks. As the total number of stock shares the investors have to hold increases, for example, the more risk they have to bear and higher return is expected in equilibrium.\(^{13}\) Note, however, that investors do not observe the supply shock directly. The premium they expect from this “market making” activity depends on their expectation of the shock instead of the true value of the shock. The second component \( e_{\Pi, t+1}(\tilde{\Pi}^i_t - \hat{\Pi}^i_t) \) represents the expected gains based on the investor’s private information. Investor i has private information that is not fully reflected in the price. The difference between his expected value of the stock and what is reflected in the price gives the expected change in future prices as the true value is gradually revealed. To simplify notation, define \( \Delta_i^j \equiv \tilde{\Pi}^i_t - \hat{\Pi}^i_t \).

3.4 Optimal stock holding

Given the expected excess return of individual investors, we can derive the optimal holding of the stock by solving the investors’ optimization problem:

\[ \max_{X_i^j} E \left[ -e^{-\lambda W_{i+1}^j} | X_i^j \right], \]  

(3.9)

\[ W_{i+1}^j = W_i^j + X_i^j Q_{t+1} \]

The solution is summarized in Lemma 3.4

Lemma 3.4 Investor i’s optimal stock holding has the linear form:

\[ X_i^j = d_{\Theta, t} \hat{\Theta}_i^j + d_{\Delta, t} \Delta_i^j. \]  

(3.10)

The intuition behind the stock demand function is simple. Note that \( \Theta_i^j \) and \( \Delta_i^j \) are the two variables that linearly determine investor i’s expected excess returns for reasons discussed above. As a result, investor i’s demand must depend on these two variables. Given the CARA preference and the normality of all the state variables, the dependence takes the linear form (see the Appendix).

\(^{13}\) See, for example, Grossman and Miller (1988) and Campbell and Kyle (1993).
3.5 Market equilibrium

In equilibrium, market must clear. Market clearing requires

\[ \int X_t^i = d_{\Theta,t} \hat{\Theta}_t + d_{\Delta,t}(\hat{\Pi}_t - \hat{\Pi}_t^c) = \Theta_t. \]  \hspace{1cm} (3.11)

Since \( \hat{\Pi}_t - \mu_t \hat{\Theta}_t = \Pi - \mu_t \Theta_t \), we have

\[ \hat{\Theta}_t = -\frac{1}{\mu_t} \left[ (\Pi - \mu_t \Theta_t) - \hat{\Pi}_t \right] = \Theta_t - \frac{1}{\mu_t}(\Pi - \hat{\Pi}_t). \]  \hspace{1cm} (3.12)

Substitute Eq. (3.3) and Eq. (3.12) into the market clearing condition, we obtain:

\[ 0 = \left[ -\frac{\alpha_t}{\mu_t} d_{\Theta,t} + (1 - \alpha_t) d_{\Delta,t} \right] \left( \Pi - \hat{\Pi}_t^c \right) + (d_{\Theta,t} - 1) \Theta_t. \]

This leads to two equations with two unknowns, \( p_{\Pi,t} \) and \( p_{\Theta,t} \):

\[ (d_{\Theta,t} - 1) = 0, \quad -\frac{\alpha_t}{\mu_t} d_{\Theta,t} + (1 - \alpha_t) d_{\Delta,t} = 0, \]  \hspace{1cm} (3.13)

for \( t = 1, 2, \cdots, T - 2, T - 1 \).

The investor's optimal demand function and the equilibrium price function at \( T - 1 \) can be solved explicitly. The excess return at \( T \) is \( Q_T = \Pi + \delta - P_{T-1} \). Thus,

\[ Q_T = \mathbb{E}[Q_T | \mathcal{F}_{T-1}^i] + (\Pi - \hat{\Pi}_{T-1}^i) + \delta. \]

Since there is only one period remaining, the investor i's demand function at \( T - 1 \) has the linear form

\[ X_{T-1}^i = \frac{1}{\lambda \text{Var}(Q_T | \mathcal{F}_{T-1}^i)} \cdot \frac{1}{\lambda \text{Var}(Q_T | \mathcal{F}_{T-1}^i)} \left[(1 - p_{\Pi,T-1}) \Delta_{T-1}^i + p_{\Theta,T-1} \hat{\Theta}_{T-1}^i \right]. \]

Let \( \sigma_{T-1} = \text{Var}(\Pi | \mathcal{F}_{T-1}^i) \). This gives

\[ d_{\Theta,T-1} = \frac{p_{\Theta,T-1}}{\lambda(\sigma_{T-1} + \sigma_\delta)}, \quad d_{\Delta,T-1} = \frac{1 - p_{\Pi,T-1}}{\lambda(\sigma_{T-1} + \sigma_\delta)}. \]

We conclude that

\[ p_{\Theta,T-1} = \lambda(\sigma_{T-1} + \sigma_\delta), \quad p_{\Pi,T-1} = 1 - \sigma_{T-1}. \]  \hspace{1cm} (3.14)

Equilibrium price functions for \( t \leq T - 2 \) can be solved recursively (see Appendix). This completes our derivation of the equilibrium.\(^{14}\)

---

\(^{14}\)We will not provide a formal proof for the existence of a solution to equation (3.13). When \( \sigma_\delta = 0 \), such a proof is possible as we are able to solve the investors' demand function in closed form. When \( \sigma_\delta \neq 0 \), an existence proof is difficult to construct. For the parameters chosen in the examples below, we are able to solve for an equilibrium numerically.
4 Dynamic Behavior of Prices and Volume

In this section we analyze the equilibrium price and volume. In particular, we examine the different patterns of trading volume that emerge under different information flows. We also consider how private information is gradually compounded into the equilibrium price through many rounds of trading.

4.1 The Benchmark Case: Homogeneous Information

Before we examine how private information affects equilibrium prices and volume, let us consider the special case when investors have homogeneous information. For simplicity, suppose that \( \Pi \) is known to all investors from the beginning (i.e., \( \sigma_{s,1} = 0 \)). The remaining risk in the stock’s payoff is \( \delta \).

In this case, \( \tilde{\Pi}_t = \Pi_t = \Pi \) and the equilibrium price of the stock is \( P_t = \Pi - p_{\Theta,t}\Theta_t \). \( \Pi \) represents the “fundamental value” of the stock and \(-p_{\Theta,t}\) represents the “risk adjustment”. The “risk adjustment” gives the discount on the price to compensate investors for bearing the risk in the future payoff of the stock.\(^{15}\) Since \( \Theta_t \) gives the total number of shares supplied to the market, it determines the total amount of risk investors have to bear. Thus, the “risk adjustment” is proportional to \( \Theta_t \) and \( p_{\Theta,t} > 0 \) is the discount on the price per share of stock supplied. \( p_{\Theta,t} \) also reflects the liquidity of the market since it determines the price adjustment the market needs to accommodate the noise trading. \( p_{\Theta,t} \) increases with \( \sigma_\delta \), the risk in future payoffs, and investors’ risk aversion \( \lambda \). Following Kyle (1985), we can interpret \( \frac{1}{p_{\Theta,t}} \) as a measure of market liquidity. It can be shown that under homogeneous information, \( p_{\Theta,t} \) increases when the terminal date approaches (i.e., as \( t \) goes to \( T \)). The intuition behind this is simple. As there are fewer trading opportunities left, investors are less willing to take positions to accommodate the supply shocks. Consequently, a higher premium is demanded by investors, \( p_{\Theta,t} \) is larger and market is less liquid.

Since \( \Pi \) is public information, the equilibrium price fully reveals the supply shock \( \Theta_t \). Thus, \( \tilde{\Theta}_t = \Theta_t \) and investors’ optimal stock holding in equilibrium is simply \( X_t = \Theta_t \). Each investor holds a fair share of the stock that is supplied to the market. This simply reflects the investors’ “market-making” activity in accommodating supply shocks. The equilibrium volume of trade in this case is:

\[
V_t^* = \int_i |\Theta_t - \Theta_{t-1}| = |\Theta_t - \Theta_{t-1}|
\]

\(^{15}\)See, for example, Grossman and Miller (1988), Campbell and Kyle (1993), Spiegel and Subrahmanyam (1992) and Wang (1993a) for more detailed discussion on the equilibrium price in settings similar to ours.
The expected volume at \( t \) is then \( \mathbb{E}[V_t] = \sqrt{\frac{2}{\pi} \text{Var} \left( \Delta \Theta_t \right)} \),\(^{16}\) where \( \Delta \Theta_t \equiv \Theta_t - \Theta_{t-1} \). Clearly, under homogeneous information the volume is completely determined by the exogenous supply shocks. This volume reflects the non-informational trading in the model.

### 4.2 Discussions of the General Case

Let us now consider the general case when investors have private information. We provide some general discussions in this subsection about the equilibrium price, investors’ trading strategies and equilibrium trading volume. More detailed results are presented later.

#### A. Equilibrium Price

Let us first consider the equilibrium price. From the results in the previous section, the equilibrium price of the stock is

\[
P_t = [p_{\Pi,t} II + (1 - p_{\Pi,t}) \tilde{\Pi}_t] - p_{\Theta,t} \Theta_t. \tag{4.1}
\]

A few comments about the equilibrium price are in order. First, it has two components: the first component represents investors’ expectations of the stock’s future payoff and the second component represents the adjustment for risk, as in the case of homogeneous information. Second, the first component is not the simple weighted average of investors’ expectations about the stock’s terminal payoff, which is \((1 - \alpha_t) II + \alpha_t \tilde{\Pi}_t\). This differs from the result in static settings (e.g., Diamond and Verrecchia (1981), Hellwig (1981), and Admati (1985)).\(^{17}\) In the multi-period setting, investors follow dynamic strategies when trading on their private information. They speculate on the changes in other investors’ expectations while in the static setting they speculate on the true value of the stock. Dynamic trading strategies generate equilibrium prices that are different from those generated by static strategies. Third, even though the equilibrium has the Markov nature that it only depends on the current expectations of the state variables, the values of these expectations do depend on the history of the economy. In particular, the current price does depend on past prices (see Brown and Jennings (1989) and Wang (1993a) for more discussions on this point).

As we discussed in the case of homogeneous information, \( p_{\Theta,t} \) characterizes the liquidity of the market. Two factors affect \( p_{\Theta,t} \). One is the uncertainty in the true value of the stock and the other is the number of trading opportunities remaining before the terminal date when

\(^{16}\)Here, the expectation is taken with respect to the unconditional distribution. One way to justify this is to assume that the prior is just the stationary distribution. This gives \( \text{Var} \left( \Delta \Theta_t \right) = \frac{2\sigma_\Theta^2}{\pi(1 + \sigma_\Theta^2)} \).

\(^{17}\)Eq.(3.14) shows that on \( T - 1 \), the date before the terminal date, \( p_{\Pi,T-1} = 1 - \alpha_t \), which is the static result.
uncertainty is fully resolved. The less uncertain the value of the stock, the lower the premium investors demand and the smaller \(p_{\Theta,t}\) is. On the other hand, the fewer the trading opportunities left, the less willing the investors are to take positions and the higher \(p_{\Theta,t}\) is. In the case of homogeneous and perfect information, the uncertainty about the stock's liquidation value (i.e., \(\sigma_s\)) stays constant throughout the trading periods. The liquidity is purely determined by the second effect as discussed above.

In the case of differential information, more information (about \(\Pi\)) is revealed over time through equilibrium prices while fewer trading opportunities are left. The revelation of information intends to decrease \(p_{\Theta,t}\) while the diminishing of remaining trading opportunities intends to increase it. The resulting pattern of \(p_{\Theta,t}\) over time depends on the trade-off between these two.

Since the number of investors is infinite and the noise in their signals is \(i.i.d\), the union of all private signals actually reveals \(\Pi\). However, the supply shocks introduce noise into the prices. Thus, equilibrium prices only partially reveal investors' private information. The informativeness of the current price depends on the noise generated by the current supply shock, which equals \(\mu_t^2 \sigma_{\Theta}^2\). As investors continue to trade, the sequence of prices reveals more information. The following proposition shows how informative the sequence of prices is about the true value of the stock.

**Proposition 4.1** Let \(o_t^2 = \text{Var}(\Pi | \mathcal{F}_t^\infty)\), \(o_t = \text{Var}(\Pi | \mathcal{F}_t^t)\), \(\mu_t = \frac{p_{\Theta,t}}{p_{\Pi,t}}\) and \(f_t = 1 - a_{\Theta} \frac{\mu_t}{\mu_{t-1}}\). Then,

\[
\frac{1}{o_t^2} = \frac{1}{o_{t-1}^2} + \frac{f_t^2}{\mu_t^2 \sigma_{\Theta}^2} + \frac{1}{\sigma_{\gamma,t}^2}, \quad \frac{1}{o_t} = \frac{1}{o_{t-1}} + \frac{f_t^2}{\mu_t^2 \sigma_{\Theta}^2} + \frac{1}{\sigma_{\gamma,t}^2} + \frac{1}{\sigma_{\Delta,t}^2}.
\]

(Hence, \(o_t^2\) and \(o_t\) decrease monotonically over time.

Clearly, the amount of private information revealed through the prices increases with the number of trading rounds.

**B. Trading Strategies**

Now, let us consider investors' trading strategies. Given the equilibrium conditions, investor \(i\)'s equilibrium stock holding is

\[
X_i^t = \tilde{\Theta}_i^t + h_i \Delta_i^t, \quad h_t = \frac{a_t}{\mu_t (1 - a_t)} > 0.
\]

It has two components. The first component is proportional to \(i\)'s estimate of the supply shock. This component reflects investor \(i\)'s position in accommodating the supply shocks. The second component is proportional to the difference between his estimation of the stock's underlying value and the estimation based purely on public information. This component reflects his speculative
position based on his private information. When \( \Delta_t^i = \hat{N}_t^i - \hat{N}_t^i > 0 \), the underlying value reflected in the price is lower than what \( i \) expects. Thus, investor \( i \) takes a long position in the stock to capture expected future gains. The coefficient of the second component, \( h_t \), characterizes how aggressive the investor is in taking the speculative positions.

As the state of the economy changes, both \( \Delta_t^i \) and \( \hat{\Theta}_t^i \) change and investors trade to revise their speculative positions and market-making positions. This gives rise to the two components of trading in the current model which we will call respectively the informational trading and non-informational trading. Since the non-informational trading is completely exogenous here, we focus on the informational trading in the following analysis.

The total amount of informational trading depends on both changes in \( h_t \) and changes in \( \Delta_t^i \). In the current model, more information is revealed to the market as time passes. Thus, \( \text{Var}[\Delta_t^i] \), which reflects the disagreement between individual expectations and public information, decreases over time. However, this does not necessarily imply that volume always declines over time. It is possible that investors speculate more aggressively when more information is revealed and fewer trading opportunities are left.

C. Equilibrium Volume of Trade

Given investors' optimal stock holdings, the equilibrium volume of trade at time \( t + 1 \) is\(^{18}\)

\[
\nu_{t+1} = \frac{1}{2} \int_t^1 |X_{t+1}^i - X_t^i| + \frac{1}{2} |\Theta_{t+1} - \Theta_t|. \tag{4.4}
\]

In order to analyze the behavior of trading volume, we express it in terms of the underlying shocks to the economy. Define \( \xi_t^i \equiv \hat{N}_t^i - \hat{N}_t^i \). \( \xi_t^i \) represents the difference between investor \( i \)'s forecast of \( \Pi \) and the market average forecast of \( \Pi \), and can therefore be treated as a measure of disagreement among investors. Simple calculations show that

\[
\xi_t^i = (1 - \alpha_t)w_t \sum_{\tau=1}^1 \frac{\xi_{S,\tau}^i}{\sigma_{S,\tau}^i}. \tag{4.5}
\]

Thus, the difference between investor \( i \)'s forecast of \( \Pi \) and the market average forecast of \( \Pi \) is linearly related to the errors in his signals weighted by their precision. Substituting \( \hat{\Theta}_t^i = \Theta_t - \frac{1}{\mu_t}(\Pi - \hat{N}_t^i) \) into (4.3), we have

\[
X_t^i = \Theta_t + \frac{1}{\mu_t(1 - \alpha_t)}\xi_t^i.
\]

\(^{18}\)See Pfleiderer (1984) for a discussion on trading volume when there are supply shocks and countable number of traders.
We can now express trading volume in terms of $x_t^{i}$ as follows:

$$
V_{t+1} = \frac{1}{2} \int_{1}^{\Theta_{t+1}} \left| (\Theta_{t+1} - \Theta_{t}) + \left( \frac{1}{1 - \alpha_t} \frac{E_{i+1}^{\mu}}{\mu_{t+1}} - \frac{1}{1 - \alpha_t} \frac{E_{i}^{\mu}}{\mu} \right) \right| + \frac{1}{2} |\Theta_{t+1} - \Theta_{t}| \right|
$$

$$
= \frac{1}{2} \int_{1}^{\Theta_{t+1}} \left| (\Theta_{t+1} - \Theta_{t}) + \left( \frac{w_{t+1}}{\mu_{t+1}} \sum_{t=1}^{t+1} \frac{E_{i,r}}{\sigma_{i,r}^2} - \frac{w_{t}}{\mu} \sum_{t=1}^{t} \frac{E_{i,r}}{\sigma_{i,r}^2} \right) \right| + \frac{1}{2} |\Theta_{t+1} - \Theta_{t}|. \tag{4.6}
$$

Define $\Delta x_{t+1}^{i} = \frac{w_{t+1}}{\mu_{t+1}} \sum_{t=1}^{t+1} \frac{E_{i,r}}{\sigma_{i,r}^2} - \frac{w_{t}}{\mu} \sum_{t=1}^{t} \frac{E_{i,r}}{\sigma_{i,r}^2}$. The equilibrium volume of trade can be written as

$$
V_{t+1} = \frac{1}{2} \int_{1}^{\Delta \Theta_{t+1} + \Delta x_{t+1}^{i}} \left| \Delta \Theta_{t+1} + \Delta x_{t+1}^{i} \right| + \frac{1}{2} |\Delta \Theta_{t+1}|. \tag{4.7}
$$

The expected equilibrium volume of trade at date $t + 1$ is

$$
E[V_{t+1}] = \frac{1}{\sqrt{2\pi}} \left( \sqrt{\text{Var}[\Delta \Theta_{t+1}] + \text{Var}[\Delta x_{t+1}^{i}] + \sqrt{\text{Var}[\Delta \Theta_{t+1}]}} \right). \tag{4.8}
$$

Subtracting the volume of non-informational trading, we have the volume of informational trading: $V_{t+1} \equiv E[V_{t+1}] - E[V_{t+1}^{*}]$.

$$
V_{t+1} = \frac{1}{\sqrt{2\pi}} \left( \sqrt{\text{Var}[\Delta \Theta_{t+1}] + \text{Var}[\Delta x_{t+1}^{i}] - \sqrt{\text{Var}[\Delta \Theta_{t+1}]}} \right). \tag{4.9}
$$

The volume of informational trading will be the focus of our analysis.

In what follows we examine the behavior of volume in more detail. We first analyze the situation with only private information (other than prices). We then examine how public announcements affect the pattern of trading volume. We also examine the relation between volume and price volatility and how it depends on the nature of information flow to the economy.

### 4.3 Volume and Private Information

Let us first consider the case when $\sigma_{0} = 0$. This is the situation considered in many existing models since it has a simple solution (e.g., Grundy and McNicholes (1989), Brown and Jennings (1989), and Kim and Verrecchia (1991a,b)). Explicit characterizations of the equilibrium are provided for this case. We then study the case with $\sigma_{0} \neq 0$, i.e., there is one component of the stock's liquidation value that is revealed only on the terminal date. In this case, equilibrium prices and volume have to be calculated numerically in a recursive fashion. Finally, we consider the effect of public news announcements on trading volume.

#### A. The Case with $\sigma_{0} = 0$
When \( \sigma_s = 0 \), there is no residual risk in the underlying value of the stock. If there are a large number of trading dates, the true value of the stock will be completely revealed eventually. Closed form solution to the equilibrium can be obtained in this case. We present the results by two propositions.

**Proposition 4.2** In equilibrium, the coefficients of the price functions satisfy the following relation:

\[
\mu_t = \frac{p_{\theta_t}}{p_{\Pi_t}} = \lambda w_t, \tag{4.10}
\]

for \( t = 1, \ldots, T - 1 \). In particular, if \( \sigma_{S_t}^2 = \sigma_{s}^2 \) (constant over time), then \( \mu_t = \frac{\lambda \sigma_{s}^2}{t} \).

Note that from the price, investors can infer \( \Pi - \mu_t \theta_t \) which serves as a signal for the unknown value of the stock, \( \Pi \). According to this proposition, the noise-signal ratio \( \mu_t \) decreases over time. Thus, prices become more and more informative about the true value of the stock.

Note that, to some extent, the structure of equilibrium exhibits certain degree of myopic behavior. Specifically, suppose we solve a one-period equilibrium model at time \( t \) with the assumptions that the risky asset is to be liquidated at date \( t + 1 \) and that investor \( i \) is endowed with the information set \( F_t^i \). Then, we would have \( p_{\Pi_t} = 1 - \alpha_t \) and \( p_{\theta_t} = \lambda \theta_t \), and the optimal stock holding for investor \( i \) would be

\[
X_i^t = \tilde{\theta}_i^t + \frac{\alpha_t}{\lambda \theta_t} \Delta_i^t
\]

It is easily checked that the coefficients of the equilibrium price in this one-period model also have the property that \( \mu_t = \lambda w_t \), as in Proposition 4.2. Moreover, since \( \frac{\partial \alpha_t}{\partial \theta_t} = \frac{\alpha_t}{\mu_t(1 - \alpha_t)} \), the optimal stock holding of investor \( i \) in the one-period model is exactly the same as that in the multi period model.

In order to examine the relation between trading and the information flow to the economy, we look at the expected volume of informational trading. The next proposition follows immediately from Proposition 4.2.

**Proposition 4.3** The expected equilibrium volume of informational trading is

\[
V_{t+1} = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\lambda^2 \sigma_{S_t}^2} + \frac{1}{\lambda^2 \sigma_{S_t}^2} - \sqrt{\text{Var}[\Delta \theta_{t+1}]} \right) \tag{4.11}
\]

where \( \text{Var}[\Delta \theta_{t+1}] = \frac{2 \sigma_{s}^2}{1 + 2 \sigma_s} \).

The proposition implies that informational trading occurs only when investors receive new private information. \( V_t \) is non-zero only if \( \sigma_{S_t}^2 \) is finite. Suppose that on date \( \tau \), there are no private
signals received by the investors. Then, $\sigma^2_{S,r} = \infty$ and $V_r = 0$. It is important to note that on date $r$ the price does reveal new information about the value of the stock through investors’ non-informational trading. But this information is common to all investors.

The above result extends the result obtained by Pfeiderer (1984) in a static setting to a multi-period setting. Even though there are many rounds of trading and investors maximize life-time utility in the current setting, the static nature of investors’ trading strategies in this special case ($\sigma_0 = 0$) gives rise to this simple result.

A special case of interest is when $\sigma^2_{S,t} = \infty$ for all $t$ except $t = 1$. This is the case when investors receive their private information only on the first date and there is no exogenous information coming to the economy on the following dates. The solution to the equilibrium price is shown in Figure 1.1-2. Investors take speculative positions on the first date based on their private information. On the following dates, they trade only to accommodate the supply shocks while maintain their original speculative positions. Thus, at $t = 1$ there is usually high volume due to the new private information. Since the actual value of the volume depends on assumptions about investors’ original positions, we will exclude this volume in our future discussions. On the following dates (i.e., $t > 1$), there will be no informational trading and $V_t = 0$ as shown in Figure 1.3. This result is somewhat surprising since as time passes, more private information is revealed through investors’ non-informational trading as illustrated in Proposition 4.1. The informativeness of the equilibrium price sequence increases over time even though there is no exogenous information coming to the market as shown in Figure 1.5.\(^\text{19}\) Hence, the difference between each investor’s expectations of the stock’s payoff and what is already incorporated in the price decreases over time. However, investors do not necessarily reduce their speculative positions. This is because as more information is compounded into the price there is less risk in holding the stock and investors are willing to take larger positions.\(^\text{20}\) Figure 1.4 illustrates how $h_t$, a measure of how aggressive investors trade on private information, changes over time. It is clear that when $\sigma_0 = 0$, $h_t$ increases monotonically with time.

Another special case is when $\sigma^2_{S,t} = \sigma^2_0$ is constant over time. This is the case when investors receive private information of the same quality every period. It is easy to show that $V_t$ ($t > 1$) remains constant over time in this case. As pointed out above, over time the equilibrium price becomes more informative about the value of the stock. Hence, the value of new private information diminishes over time. At the same time, the remaining uncertainty in the true value of the stock also diminishes. Thus, investors trade more aggressively over the difference between

\(^{19}\)A more detailed discussion on this point is provided by Grundy and McNicholes (1989) using a 3-date model.

\(^{20}\)This is not true in the general case when $\sigma_0 \neq 0$ as we will see later.
their private information and public information. In net, the volume stays constant over time. If instead, the precision of private signals increases as the final revelation date approaches, the volume will increase over time. In summary, the dynamics of the expected volume of trade is determined entirely by the dynamics of the precision of the private signals. Large trading volume occurs when the precision of private signals at that date is high.

B. The Case with $\sigma_\delta \neq 0$

When $\sigma_\delta \neq 0$, there is remaining uncertainty in the value of the stock even when all the private information is compounded into the price. In this case, the noise-signal ratios $\{\mu_t\}$ of the equilibrium prices do not follow Proposition 4.1. As a result, the equilibrium price function must be solved numerically. The details are described in the Appendix.

In this case, the simple results we have when $\sigma_\delta = 0$ do not hold. $\frac{w_t}{\mu_t}$ is no longer constant across time. The expected trading volume at each date depends on the precision of all signals received till that date. Specifically, the trading volume associated with differential information is positively related to

$$\text{Var}(\Delta c_{t+1}^i) = \frac{w_{t+1}^2}{\mu_{t+1}^2 \sigma_{\delta t+1}^2} + \left(\frac{w_{t+1}}{\mu_{t+1}} - \frac{w_t}{\mu_t}\right)^2 w_t^{-1},$$

(4.12)

which exhibits rich patterns over time. The behavior of trading volume becomes more complex now. We first consider the case where investors receive private information only on the first date, i.e., $\sigma_{\delta t} = \infty$ for $1 < t < T$. On the following dates, investors trade for both informational and non-informational reasons. But there is no new information coming to the market. The solution to $p_{II,t}$ and $p_{\Theta,t}$ are given in Figures 1.1-2. In this case, $w_t = w_1$ and $\text{Var}[\Delta c_{t+1}^i] = w_1 \left(\frac{1}{\mu_{t+1}} - \frac{1}{\mu_t}\right)^2$. Thus, the volume of informational trade is completely determined by the dynamics of $\mu_t$. We find that volume is not necessarily monotonic over time. In particular, volume may exhibit a peak in the middle of the trading horizon as shown in Figure 1.3. Also note that volume jumps up right before the terminal date.

In order to understand the volume pattern, let us examine investors’ trading strategies. As discussed earlier, $X_t^i = \tilde{\Theta}_s^i + h_t \Delta_t^i$ and $h_t \Delta_t^i$ reflects investor $i$’s speculative position. Since $\Delta_t^i$ is the difference between investor $i$’s expectation of $II$ and the expectation based only on public information, $h_t$ represents how aggressive he trades on private information. Figure 1.4 shows the pattern of $h_t$ over time. Investors speculate on private information most aggressively toward the middle of the trading horizon and then reduce their positions gradually. It is clear that trading occurs when $h_t$ changes. This pattern in trading is generated by two factors. On one hand, investors’ information about the value of the stock becomes more accurate towards
the terminal date, hence they trade more aggressively. On the other hand, there are fewer trading opportunities left as it is closer to the terminal date. Investors become less willing to take positions since it becomes more difficult to unload any positions if needed. The trade-off between these two factors gives rise to the hump pattern in $h_t$ and in volume. Investors increase their speculative positions at the beginning and then gradually unwind their position as the terminal date approaches. The position of the peak in volume depends on the residual risk $\sigma_\delta$. The peak occurs earlier as the residual risk increases.

The high volume right before the terminal date is easy to understand. Investors speculate on $\Pi$ based on their private information. Due to the residual risk $\delta$, they are reluctant to maintain a position till the end. They reduce their speculative positions right before the uncertainty about $\delta$ is resolved. This gives rise to the high volume on date $T - 1$.

It is interesting to examine the variance of price changes accompanying the volume. Figure 1.6 shows that high volume generated by the existing private information is not accompanied with high volatility in price. This is different from the volume generated by a new public information about the stock's payoff or a new exogenous shock in which case volume is accompanied by changes in prices (see, e.g., Kim and Verrecchia (1991a) and Campbell, Grossman and Wang (1993)). We will come back to this point later when we consider the trading generated by exogenous public information.

In the more general case that there is new private information after the first period, the pattern of volume becomes more complex. Every new information generates immediate trading as well as trading in future dates. The resulting pattern depends on the specific flow of information and the parameter values.

### 4.4 Volume and Public News Release

Abnormally high level of trading has been documented around public announcements such as earning announcements (see, e.g., Beaver (1968) and Bamber (1986)). Several authors have considered the link between abnormal trading in response to public announcements and information heterogeneity among investors. For example, Kim and Verrecchia (1991a) have shown in a 3-date setting that abnormal trading occurs only if there is some type of asymmetry among investors, either in their risk-aversion or private information. Public information does not generate

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21 Various volume patterns can be found in the strategic trading models by Forster and Viswanathan (1992), Holden and Subrahmanyam (1992) and Vayanos (1992). However, differential information or the infinite regress problem is not a major issue in those papers, although Vayanos's model has some flavor of differential information as the investors' initial endowment is not known to each other.

22 Grundy and McNicholes (1989), Kim and Verrecchia (1991a,b), Wang (1993b) analyze the link between volume on announcement days and heterogeneous information.
trading volume when there is perfect symmetry among investors.

Our model generalizes the model of Kim and Verrecchia (1991a) to a general dynamic setting. The case with \( \sigma_\delta = 0 \) is a direct extension of their model. With perfect symmetry among the investors in terms of their risk-aversion and signal quality, public news do not generate any abnormal trading. When \( \sigma_\delta = 0 \), the volume of informational trading is completely determined by the new private information as discussed earlier. Consider the special case when there is no new private information after the first period, i.e., \( \sigma_{\delta,t}^2 = \infty \ (1 < t < T) \). The volume of informational trading is determined by the changes in \( \mu_t \). Since \( \mu_t = \lambda w_t \) and \( \mu_t = w_1, \mu_t = \mu_1 \) and \( V_t = 0 \) for \( t > 1 \), independent of the flow of public information.

In the case when \( \sigma_\delta \neq 0 \), exogenous public information does generate trading even when there is perfect symmetry among investors. Figure 2.1 shows the pattern of volume when there is a public announcement. We have chosen \( T = 25 \) as in the previous figures and the announcement date to be \( t_A = 10 \) or 18. We want to compare the pattern of volume with that without the public announcement which is shown in Figure 1.3. Without the announcement, the volume exhibits a peak at \( t = 10 \) due to the endogenous informational trading among the investors. With the announcement, high volume is observed on the announcement date. For \( t_A = 18 \), the announcement date is after the peak of endogenous informational trading. The abnormal volume due to the announcement does not greatly affect the endogenous informational trading. Thus, a smooth peak occurs at \( t = 10 \) due to endogenous informational trading and a sharp peak right on the announcement date. For \( t_A = 10 \), the announcement date is around the peak of endogenous informational trading. The announcement not only generates high volume on the announcement date, but also affect the pattern of endogenous trading.

The expected announcement intends to induce investors to time their trade around the announcement date. This is best seen by examining the investors' speculative positions over time which is characterized by \( h_t \). Figure 2.2 shows that in the case of \( t_A = 10 \), the most aggressive position taking by the investors occurs shortly before the announcement date. Furthermore, investors cut back their positions right after the announcement. This pattern in their position taking clearly reflects the betting on the outcome of the announcement by the investors. In the case when \( t_A = 18 \), investors trade prior to the announcement based on the private information they received at the beginning \( (t = 1) \). Right before the announcement \( (at t = 17) \), they take additional speculative positions in the stock.

---

\(^{23}\)In a partial equilibrium model with no differential information, we would expect investors to trade less aggressively prior to the announcement date (due to their aversion towards risk). Here, we are in a general equilibrium model with differential information. Investors do trade more aggressively against each other before the value of their private information diminishes after the public announcement.
We can also examine the change in liquidity around public announcements. Figure 2.3 shows that $p_{\Theta,t}$ increases right before the announcement, indicating decreasing liquidity. The liquidity increases after the announcement due to the reduction in uncertainty by the public revelation. The decrease in liquidity prior to the announcement will cause the price to be more sensitive to supply shocks. Consequently, price volatility increases around the announcement as shown in Figure 2.4.

Another interesting point to notice is that the total amount of information revealed through the prices depends on the timing of the public announcement. Figure 2.5 plots $\omega_t^A$ over time when the announcement is at $t_A = 10$ and 18 respectively. Note that $\omega_t^A$ measures the remaining uncertainty in $\Pi$ given the history of prices and public announcements. Clearly, more private information is revealed through the prices in the case when $t_A = 10$ than in the case when $t_A = 18$. This is not surprising given investors’ trading behavior in these two cases as shown in Figure 2.2. When $t_A = 10$, there are more trading opportunities remaining before the terminal date and investors bet more aggressively on the outcome of the announcement. As a result, more private information will be revealed through the prices.

4.5 Volume and Price Volatility

Since volume is closely related to the flow of information to the economy, several authors have suggested that volume can be used to gauge the information flow to the economy (see, e.g., Clark (1973) and Lamoureux and Lastrapes (1990, 1992)). It is argued that periods of high trading volume should be the periods with clustering in new information which should also be the periods with persisting high volatility in prices. Thus, volume provides a measure of the economic time under which the information arrival is uncorrelated.

This type of argument can be justified when there is no private information. When there is private information in the market, the information flow can be exogenous or endogenous. Different structures of information flow give rise to different behavior of volume and prices. We now use our model to examine the relation between volume and price volatility under different types of information flow.

In the current model, exogenous information includes public announcements and new private signals about the value of the stock while endogenous information is simply the stock prices. As discussed in the previous subsections, new private information not only generates trading in the current period but also generates trading in future periods while new public information only affects the trading in the current period. This implies that when there is private information, independent arrival of new information can generate serially correlated volume.

Price changes whenever there is new information. Exogenous information about the value
of the stock will certainly generate changes in prices. Figure 2.4 shows the pattern of price volatility when there is a public announcement at \( t_A = 18 \) (investors receive private information at \( t = 1 \)). Price volatility jumps up at the announcement date. There is also abnormally high volume on the announcement date (see Figure 2.1). Similar results hold in the case of new private information. This implies that the volume generated by exogenous information must be accompanied by high price volatility. In the case of endogenous information, there can be trading in the absence of exogenous information. However, the volume is not accompanied by abnormally high volatility in prices. This can be seen by comparing Figures 1.6 and 2.4. There is no exogenous information except at \( t = 1 \) and \( t = 18 \). Abnormally high volume around \( t = 10 \) is generated by the existing private information. No abnormal price volatility is observed on the dates of high volume. This contrasts with the volume at \( t = 18 \) when there is a public announcement, which is accompanied by high volatility in prices.

Our examples show that when there is private information, trading is not only related to the new exogenous information, private or public, but also related to the existing private information. The trading generated by exogenous information is accompanied with high price volatility while the trading generated by the existing private information is not.

5 Concluding Remarks

In this paper, we have considered a multi-period model of stock trading in which investors have differential information about the true value of the stock. We show that there is a close link between the nature of information flow to the investors and the observed behavior of trading volume. Exogenous information, whether private or public, generates immediate trading as well as changes in prices. Private information not only generates trading in the current period, but also leads to possible trading in future periods. Public information can only generate trading in the current period. In summary, different information flows in the form of private or public information can give rise to very distinctive patterns in trading volume.

To facilitate our analysis, we have made various assumptions to make the model tractable. Some of these assumptions can be weakened. For example, it is possible to allow investors to have different degrees of risk aversion and different levels of precision for the signals they receive at any one time. The assumption that there are a countably infinite number of investors is important. Otherwise, the equilibrium price function has to be characterized by the history of state variables that cannot be summarized by \( \bar{R}_t \).
Appendix

A  Proof of Lemma 3.1

Let $S_t = (S_1, S_2, \ldots, S_t)$ and $\mathcal{F}_t^p = \mathcal{F}_t^p$. Applying the results on conditional normal distributions (see, e.g., Liptser and Shiryaev (1974)), we have

$$\mathbb{E}[\Pi | \mathcal{F}_t^p] = \tilde{\Pi}_t + \Gamma_t (S_t - \mathbb{E}[S_t | \mathcal{F}_t^p]), \quad (A.1)$$

where $\Gamma_t = \text{Cov} [\Pi, S_t | \mathcal{F}_t^p] \text{Cov} [S_t, S_t | \mathcal{F}_t^p]^{-1}$. Note that

$$\mathbb{E}[S_t | \mathcal{F}_t^p] = \tilde{\Pi}_t \xi_t,$$

$$\text{Cov} [\Pi, S_t | \mathcal{F}_t^p] = \text{Cov} [\Pi, \Pi | \mathcal{F}_t^p] \xi_t,$$

$$\text{Cov} [S_t, S_t | \mathcal{F}_t^p] = \text{Cov} [\Pi, \Pi | \mathcal{F}_t^p] \xi_t \xi_t^T + \text{diag}(\sigma_{S,1}^2, \ldots, \sigma_{S,t}^2),$$

where $\xi_t$ is the $(t \times 1)$ column vector of 1's and $\text{diag}$ is the diagonal matrix. Thus,

$$\Gamma_t = \xi_t \xi_t^T \left( \sigma_{S,1}^2, \ldots, \sigma_{S,t}^2 \right)^{-1}. \quad (A.2)$$

It can be shown by verification that $\Gamma_t = \frac{\xi_t}{\xi_t + w_t} \left( \frac{1}{\sigma_{S,1}^2}, \ldots, \frac{1}{\sigma_{S,t}^2} \right)$. Now, it is easy to verify that

$$\mathbb{E}[\Pi | \mathcal{F}_t^p] = \alpha_t \tilde{\Pi}_t + (1 - \alpha_t) \tilde{\Pi}_t^p, \quad \text{where } \tilde{\Pi}_t^p = w_t \sum_{\tau=1}^t \frac{S_t}{\sigma_{S,\tau}^2}. \quad (A.3)$$

B  Proof of Lemma 3.2

In deriving the filters of our interest, we will use the results in the following Lemma.

Lemma B.1 Let

$$x_t = A_t x_{t-1} + B_t \epsilon_{x,t}, \quad t = 1, 2, \ldots$$

$$y_t = H_t x_t + \epsilon_{y,t}.$$

$x_t$ is the $n$-vector state variable at $t$, $y_t$ is the $m$-vector of observations at $t$. $A_t$, $B_t$ and $H_t$ are respectively $(n \times n)$, $(n \times k)$, $(m \times n)$ matrices. $\{\epsilon_{x,t}, t = 1, \ldots\}$ and $\{\epsilon_{y,t}, t = 1, \ldots\}$ are respectively a $k$-vector and a $m$-vector white Gaussian sequence. $\epsilon_{x,t} \sim \mathcal{N}(0, Q_t)$, $\epsilon_{y,t} \sim \mathcal{N}(0, R_t)$ and $x_0 \sim \mathcal{N}(\tilde{x}_0, \Sigma_{x,0})$. $x_0$, $\{\epsilon_{x,t}\}$ and $\{\epsilon_{y,t}\}$ are independent. Let

$$\tilde{z}_t = \mathbb{E}[x_t | y_{\tau} : 1 \leq \tau \leq t], \quad O_t = \mathbb{E} \left[ (x_t - \tilde{z}_t)(x_t - \tilde{z}_t)^T | y_{\tau} : 1 \leq \tau \leq t \right].$$

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Then,
\[
\hat{x}_t = A_t \hat{x}_{t-1} + K_t(y_t - A_t \hat{x}_{t-1}),
\]
\[
O_t = (I_n - K_t H_t) \left( A_t O_{t-1} A_t^T + B_t Q_t B_t^T \right),
\]
\[
K_t = \left( A_t O_{t-1} A_t^T + B_t Q_t B_t^T \right) H_t^{-1} \left[ H_t (A_t O_{t-1} A_t^T + B_t Q_t B_t^T) H_t^T + R_t \right]^{-1}.
\]
where \( I_n \) is the \((n \times n)\) identity matrix.

**Proof.** See Liptser and Shiryayev (1974).

Before we apply this lemma to our problem, we first establish one relation that is useful for future use.

\[
E [(\Pi - \hat{\Pi}_t)^2 | F_t] = E \left[ \left\{ \alpha_t (\Pi - \hat{\Pi}_t) + (1 - \alpha_t) (\Pi - \hat{\Pi}_t^{p.i}) \right\}^2 | F_t \right]
\]
\[
= \alpha_t^2 E [(\Pi - \hat{\Pi}_t)^2 | F_t] + (1 - \alpha_t)^2 E \left[ (\Pi - \hat{\Pi}_t^{p.i})^2 | F_t \right]
\]
\[
= \alpha_t^2 c_i + (1 - \alpha_t)^2 \omega_i,
\]
(B.1)
which is independent of investors' private signals. Substituting in the definition of \( \alpha_t \), we conclude
\[
\frac{1}{\alpha_t} = \frac{1}{c_i} + \omega_i^{-1}.
\]
(B.2)

Thus, \( \alpha_t \) is independent of investor \( i \)'s private signals.

We can now solve for the common filters, \( \hat{\Pi}_t \) and \( \hat{\Theta}_t \), by applying Lemma B.1. Make the following substitution: \( z_t^T = (\Pi, \Theta_t), y_t^T = (Y_t, \xi_t), \epsilon_{x,t} = \epsilon_{\Theta,t}, \epsilon_{y,t} = (\epsilon_{Y,t}, 0), and\)
\[
A_t = \begin{pmatrix} 1 & 0 \\ 0 & a_{\Theta} \end{pmatrix}, \quad B_t = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H_t = \begin{pmatrix} 1 & 0 \\ 1 & -\mu_t \end{pmatrix}, \quad Q_t = \sigma_\Theta^2, \quad R_t = \begin{pmatrix} \sigma_{Y,t}^2 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Let \( E[(\Pi - \hat{\Pi}_t)^2 | F_t] = \omega_i \). Since \( p_{\Pi,t}(\Pi - \hat{\Pi}_t) = p_{\Theta,t}(\Theta_t - \hat{\Theta}_t) \), we have
\[
E[(\Theta_t - \hat{\Theta}_t)(\Theta_t - \hat{\Theta}_t)|F_t] = \frac{1}{\mu_t} \omega_i, \quad E[(\Pi - \hat{\Pi}_t)(\Theta_t - \hat{\Theta}_t)|F_t] = \frac{1}{\mu_t} \omega_i.
\]

Also define \( f_t = 1 - a_{\Theta} \frac{\mu_t}{\mu_{t-1}} \) and
\[
D_t^2 = f_t^2 \sigma_{Y,t}^2 \sigma_{Y,t-1} + \mu_t^2 \sigma_\Theta^2 (\sigma_{t-1} + \sigma_{Y,t}^2),
\]
\[
k_{\Pi,t}^c = \frac{1}{D_t} \mu_t^2 \sigma_\Theta^2 \sigma_{Y,t-1}, \quad k_{\Pi,t}^c = \frac{1}{D_t} f_t \sigma_{Y,t}^2 \sigma_{Y,t-1},
\]
\[
k_{\Theta,t}^c = \frac{1}{D_t} \mu_t \sigma_\Theta^2 \sigma_{Y,t-1}, \quad k_{\Theta,t}^c = -\frac{1}{D_t} \left[ \mu_t \sigma_\Theta^2 (\sigma_{t-1} + \sigma_{Y,t}^2) - a_{\Theta} \frac{f_t}{\mu_{t-1}} \sigma_{Y,t}^2 \sigma_{Y,t-1} \right].
\]

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From Lemma B.1, we obtain
\[
\begin{pmatrix}
\hat{H}_i^c \\
\theta_i^c
\end{pmatrix}
= 
\begin{pmatrix}
\hat{H}_{i-1}^c \\
\theta_{i-1}^c
\end{pmatrix}
+ 
\begin{pmatrix}
k_{H,t}^c \\
k_{\theta,t}^c
\end{pmatrix}
\begin{pmatrix}
Y_t - \hat{H}_{i-1}^c \\
\xi_t - (\hat{H}_{i-1}^c - a_\theta \mu_t \theta_{i-1}^c)
\end{pmatrix},
\]
and
\[
\frac{1}{\sigma_i^2} = \frac{1}{\sigma_{i-1}^2} + \frac{f_t^2}{\mu_t \sigma_{\theta}^2} + \frac{1}{\sigma_{\xi,t}^2}. \tag{B.4}
\]

At time 0, we have \( \frac{1}{\sigma_0^2} = \frac{1}{\sigma_0^2} \).

Now, consider \( \hat{H}_i^c \) and \( \theta_i^c \). The derivation is similar to that of the common filters. In applying Lemma B.1, make the following substitution: \( x_t^\top = (H_t, \theta_t, Y_t, S_t, \xi_t, \epsilon_{x,t} = \epsilon_{\theta,t}, \epsilon_{Y,t} = \epsilon_{Y,t}, \epsilon_{S,t}, 0) \), and
\[
A_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_t = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_t = \begin{pmatrix} \sigma_{Y,t}^2 & 0 & 0 \\ 0 & \sigma_{S,t}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Let \( o_t = E[(H - \hat{H}_i^c)^2 | F_t^i] \), we have
\[
E[(\theta_t - \theta_i^c)^2 | F_t^i] = \frac{1}{\mu_t} o_t, \quad E[(H - \hat{H}_i^c)(\theta_t - \theta_i^c) | F_t^i] = \frac{1}{\mu_t} o_t.
\]
Also define
\[
k_{H,t}^c = \frac{1}{D_t} \mu_t \sigma_{\theta}^2 \sigma_{\xi,t}^2 o_t - 1, \quad k_{\theta,t}^c = \frac{1}{D_t} \mu_t \sigma_{\theta}^2 \sigma_{\xi,t}^2 o_t - 1, \quad k_{H,t}^c = \frac{1}{D_t} f_t \sigma_{Y,t}^2 \sigma_{\xi,t}^2 o_t - 1,
\]
\[
k_{\theta,t}^c = \frac{1}{D_t} \mu_t \sigma_{\theta}^2 \sigma_{\xi,t}^2 o_t - 1, \quad k_{\theta,t}^c = \frac{1}{D_t} \mu_t \sigma_{\theta}^2 \sigma_{\xi,t}^2 o_t - 1,
\]
\[
k_{\theta,t}^c = -\frac{1}{D_t} \left\{ \mu_t \sigma_{\theta}^2 \left[ (\sigma_{\theta,t}^2 + \sigma_{\xi,t}^2) o_t - 1 + \sigma_{\theta,t}^2 \sigma_{\xi,t}^2 \right] - a_\theta \frac{f_t}{\mu_t} \sigma_{\theta}^2 \sigma_{\xi,t}^2 o_t - 1 \right\}.
\]

Lemma B.1 implies that
\[
\begin{pmatrix}
\hat{H}_i^c \\
\theta_i^c
\end{pmatrix}
= 
\begin{pmatrix}
\hat{H}_{i-1}^c \\
\theta_{i-1}^c
\end{pmatrix}
+ 
\begin{pmatrix}
k_{H,t}^c \\
k_{\theta,t}^c
\end{pmatrix}
\begin{pmatrix}
Y_t - \hat{H}_{i-1}^c \\
\xi_t - (\hat{H}_{i-1}^c - a_\theta \mu_t \theta_{i-1}^c)
\end{pmatrix}, \tag{B.5}
\]
and
\[
\frac{1}{o_t} = \frac{1}{o_{t-1}} + \frac{f_t^2}{\mu_t \sigma_{\theta}^2} + \frac{1}{\sigma_{\xi,t}^2}. \tag{B.6}
\]
C Proof of Lemma 3.3

From Lemma 3.2, we have

\[
\hat{\Pi}_{t+1}^i = \hat{\Pi}_{t}^i + k_{t+1}^e \left( Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t^i] \right) + k_{t+1}^c \left\{ \xi_{t+1} - E[\xi_{t+1}|\mathcal{F}_t^i] \right\} + E[\hat{\Pi}_{t+1}^i|\mathcal{F}_t^i] = \hat{\Pi}_{t}^i + \left( k_{t+1}^e + f_{t+1} k_{t+1}^c \right) \left( \hat{\Pi}_{t}^i - \hat{\Pi}_{t}^i \right).
\]

Note that \( \xi_t \subseteq \mathcal{F}_t^i \subseteq \mathcal{F}_t^i \). Thus

\[
\hat{\Pi}_{t}^i - \mu_t \hat{\epsilon}_{t}^i = \hat{\Pi}_{t}^i - \mu_t \hat{\epsilon}_{t}^i = \Pi - \mu_t \hat{\epsilon}_{t}.
\]  (C.1)

Define \( b_{t+1}^Y = (1-p_{t+1}^\nu)k_{t+1}^c \) and \( b_{t+1}^S = (1-p_{t+1}^\nu)f_{t+1}k_{t+1}^c \), we can then express investor i's expectation of excess share return as:

\[
E[Q_{t+1}|\mathcal{F}_t^i] = e_{\Pi,t+1}(\hat{\Pi}_{t}^i - \hat{\Pi}_{t}^i) + e_{\theta,t+1} \hat{\epsilon}_{t}^i,  \]  (C.2)

where \( e_{\Pi,t+1} = (p_{t+1}^\nu - p_{t+1}^\nu) + b_{t+1}^Y + b_{t+1}^S, e_{\theta,t+1} = -(a_{\theta} p_{\theta,t+1}^\nu - p_{\theta,t+1}^\nu) \). Define \( \Delta_t^i = \hat{\Pi}_{t}^i - \hat{\Pi}_{t}^i \). \( \Delta_t^i \) is the difference between investor-i's estimate of the stock value and the estimate based solely on market information. It is easy to show that \( E[Q_{t+1}|\mathcal{F}_t^i] = (1-k_{t+1}^e Y_{t+1} - f_{t+1} k_{t+1}^c) \Delta_t^i \).

D Proof of Lemma 3.4

For notational convenience, define \( \Psi_{t+1}^i = \begin{pmatrix} 1, \Delta_t^i, \hat{\epsilon}_{t}^i \end{pmatrix}, \xi_{t+1}^i = \begin{pmatrix} \Pi - \hat{\Pi}_{t}^i, e_{\theta,t+1}, e_{Y,t+1} \end{pmatrix} \). We have \( \xi_{t+1}^i | \mathcal{F}_t^i \sim \mathcal{N}(0, \Sigma_{t+1}) \) where \( \Sigma_{t+1} = \text{diag}(\sigma_{\Pi,t+1}^2, \sigma_{\theta,t+1}^2, \sigma_{Y,t+1}^2) \). Also define \( \delta_t^i = k_{t+1}^Y, \delta_{t+1}^i = k_{t+1}^c \), \( \delta_{t+1}^i = k_{t+1}^Y - k_{t+1}^c \), \( \delta_{t+1}^i = k_{t+1}^c \).

\[
A_{\psi,t+1} = \begin{pmatrix}
1 & 0 & 0 & 0
0 & 1 - k_{t+1}^c Y_{t+1} & f_{t+1} k_{t+1}^c \delta_{t+1}^i & 0
0 & 0 & a_{\theta}
\end{pmatrix},
\]

\[
B_{\psi,t+1} = \begin{pmatrix}
0 & 0 & 0 & 0
\delta_{t+1}^Y + k_{t+1}^c Y_{t+1} + f_{t+1} k_{t+1}^c \delta_{t+1}^i & -\delta_{t+1}^c \mu_{t+1} & \delta_{t+1}^Y + k_{t+1}^c \delta_{t+1}^i & 0 & 0
k_{\theta,t+1}^Y + k_{\theta,t+1}^c Y_{t+1} + f_{t+1} k_{\theta,t+1}^c \delta_{t+1}^i & -k_{\theta,t+1}^c \delta_{t+1}^i \mu_{t+1} & k_{\theta,t+1}^Y & k_{\theta,t+1}^c \delta_{t+1}^i & 0
\end{pmatrix}
\]

\( A_{Q,t+1} = \begin{pmatrix} 0, e_{\Pi,t+1}, e_{\theta,t+1} \end{pmatrix} \) and \( B_{Q,t+1} = (f_{t+1} p_{\Pi,t+1}^\nu + b_{t+1}^Y + b_{t+1}^S, -p_{\theta,t+1}^\nu b_{t+1}^Y + f_{t+1}^c, 0). \)

Then, we have

\[
\Psi_{t+1}^i = A_{\psi,t+1} \Psi_{t+1}^i + B_{\psi,t+1} \xi_{t+1}^i,
\]  (D.1)

and

\[
Q_{t+1} = E[Q_{t+1}|\mathcal{F}_t^i] + B_{Q,t+1} \xi_{t+1}^i, \quad E[Q_{t+1}|\mathcal{F}_t^i] = A_{Q,t+1} \Psi_{t+1}^i,
\]  (D.2)
Now consider the investors' optimization problem:

\[
\max_{X_t^i} \mathbb{E}\left[ -e^{-\lambda_t W_t^i} \left| \mathcal{F}_t^i \right. \right] \quad \text{s. t.} \quad W_{t+1}^i = W_t^i + X_t^i Q_{t+1}.
\]  
(D.3)

Let \( J(W_t^i; \Psi_t^i; t) \) be the value function, the Bellman equation for the optimization problem reads:

\[
0 = \max_{X_t^i} \left\{ \mathbb{E}[J(W_{t+1}^i; \Psi_{t+1}^i; t+1) | \mathcal{F}_t^i] - J(W_t^i; \Psi_t^i; t) \right\}
\text{ s. t. } \quad W_{t+1}^i = W_t^i + X_t^i Q_{t+1},
J(W_T^i; \Psi_T^i; T) = -e^{-\lambda_T W_T^i}.
\]

We assume that the value function has the form: \( J(W_t^i; \Psi_t^i; t) = -e^{-\lambda_t W_t^i - \frac{1}{2} \Psi_t^i \Psi_t^i \top U_t \Psi_t^i} \). It is straightforward to show that

\[
\mathbb{E}[J(W_{t+1}^i; \Psi_{t+1}^i; t+1) | \mathcal{F}_t^i] = -\rho_{t+1} e^{-\lambda_{t+1} W_t^i - \frac{1}{2} \Psi_t^i \Psi_t^i \top A_{t+1, t+1} A_{t+1, t+1} \Psi_t^i} \times
\]
\[e^{-\lambda_{t+1} X_t^i A_{t+1, t+1} \Psi_t^i + \frac{1}{2} (\lambda_{t+1} B_{t+1, t+1} X_t^i + B_{t+1, t+1} U_t A_{t+1, t+1} \Psi_t^i) \top \Xi_{t+1} (\lambda_{t+1} B_{t+1, t+1} X_t^i + B_{t+1, t+1} U_t A_{t+1, t+1} \Psi_t^i)},\]

where \( \Xi_{t+1} = (\Sigma_{t+1}^{-1} + B_{t+1, t+1} U_t A_{t+1, t+1} B_{t+1, t+1} \Sigma_{t+1})^{-1} \) and \( \rho_{t+1} = \sqrt{\Xi_{t+1}} / \sqrt{\Sigma_{t+1}} \). Define

\[
F_t = \left[ B_{t+1, t+1} \Xi_{t+1} B_{t+1, t+1} \right]^{-1} (A_{t+1, t+1} - B_{t+1, t+1} \Xi_{t+1} B_{t+1, t+1} U_t A_{t+1, t+1}),
M_t = F_t \top (B_{t+1, t+1} \Xi_{t+1} B_{t+1, t+1}) F_t - (B_{t+1, t+1} U_t A_{t+1, t+1}) \top \Xi_{t+1} (B_{t+1, t+1} U_t A_{t+1, t+1}) + A_{t+1, t+1} U_t A_{t+1, t+1}.
\]

It is easy to derive the following first order condition with respect to \( X_t \):

\[
X_t^i = \frac{1}{\lambda_{t+1}} F_t \Psi_t^i, \quad (1 \leq t < T).
\]  
(D.4)

The second order condition for optimality is: \( B_{t+1, t+1} \Xi_{t+1} B_{t+1, t+1} > 0 \). Furthermore,

\[
\mathbb{E}[J(W_{t+1}^i; \Psi_{t+1}^i; t+1) | \mathcal{F}_t^i] = -\rho_{t+1} e^{-\lambda_{t+1} W_t^i - \frac{1}{2} \Psi_t^i \Psi_t^i \top M_t \Psi_t^i}.
\]  
(D.5)

Substitute Eq. (D.4-D.5) into the Bellman equation, we obtain the following for \( t < T \):

\[
\lambda_t = \lambda_{t+1}, \quad \text{and} \quad U_t = M_t + c_t J_{11}^{4,4},
\]  
(D.6)

where \( c_t = -2 \ln \rho_{t+1} \) and \( J_{11}^{4,4} \) is a \((4 \times 4)\) index matrix which has all the elements being zero except element \( \{11\} \) being 1. From the solution for \( T-1 \), we can recursively solve for \( \lambda_t \) and \( U_t \) hence the value function and the optimal investment policy. Hence, we have completed the proof of Lemma 2.3.

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E Proof of Proposition 4.1

For ease of exposition, we present our proof only for the case where \( T = 3, a_\Theta = 0, \sigma_Y = \infty \) and \( \sigma_{S,t} = \sigma_S \). It then follows that \( f_t \equiv 1 \). To prove the proposition for \( t = 2 \), we recall from (3.14) that \( p_{o,2} = o_2 \) and \( p_{\Pi,2} = 1 - o_2 = \frac{o_2}{o_2 + \mu_2} \). Since \( o_2 = \left( \frac{1}{o_2} + \frac{1}{\mu_2} \right)^{-1} = \frac{o_2}{o_2 + \mu_2} \), we conclude that \( \mu_2 = \lambda \mu_2 \). In the rest of the proof, we will repeatedly use (B.2), (B.4), (B.6) and \( \mu_2 = \lambda \omega_2 \) to simplify terms.

We now prove the proposition for \( t = 1 \). Simple (but tedious) calculations show that

\[
A_{\Psi,2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - \frac{o_2}{\mu_2 \sigma_2^2} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad B_{\Psi,2} = \begin{pmatrix}
0 & 0 \\
-\frac{o_2}{\mu_2 \sigma_t} & \frac{1}{\sigma_t} (o_2 - o_2^2) \\
\frac{1}{\sigma_t} & \frac{1}{\sigma_t}
\end{pmatrix}
\]

Since \( X_2^i = \frac{1}{\lambda} (0, \frac{1}{o_2}, \lambda) \Psi_2^i \), we get

\[
U_2 = \frac{1}{\sigma_2} \begin{pmatrix}
0 \\
1 - \frac{o_2}{\sigma_2} \\
\lambda
\end{pmatrix}, \quad B_{\Psi,2} U_2 B_{\Psi,2} = \frac{1}{\sigma_2} \begin{pmatrix}
-\frac{1}{\sigma_2} \\
\lambda \\
\frac{1}{\sigma_2}
\end{pmatrix}
\]

Define \( \kappa = o_2 \left( \frac{\sigma_1}{\sigma_2} + \lambda^2 \sigma_2^2 + \frac{1}{\sigma_2} \right) \). The reader can verify that

\[
\Xi_2 = \Sigma_2 - \frac{o_2}{1 + \kappa} N^T N
\]

where \( \Sigma_2 = \text{diag}(o_1, \sigma_2^2, \sigma_2^2) \) and \( N = (-\frac{o_1}{\sigma_2}, \lambda \sigma_2^2, 1) \). We thus have

\[
\Xi_2 B_{\Psi,2}^T U_2 A_{\Psi,2} = \frac{o_2}{1 + \kappa} \begin{pmatrix}
-\frac{1}{\sigma_2} \\
\lambda \sigma_2^2 \\
1
\end{pmatrix} \begin{pmatrix}
0 \\
\frac{1}{o_2} \\
-\frac{1}{\sigma_2 \mu_2^2}
\end{pmatrix}
\]

Defining \( h = B_{Q,2} \Xi_2 B_{Q,2}^T \) and noting that \( B_{Q,2} = (p_{\Pi,2} + b_2)(1, -\mu_2, 0) \), we get

\[
h = \frac{(p_{\Pi,2} + b_2)^2 \mu_2 \sigma_2^2 \sigma_1}{o_2(1 + \kappa)}.
\]

We can now conclude that \( X_2^i = d_{\Theta,1} \Theta_1^i + d_{\Delta,1} \Delta_1^i \), where

\[
d_{\Theta,1} = \frac{1}{\lambda \nu} e_{\Theta,1} = \frac{1}{\lambda \nu} \nu_{\Theta,1},
\]

\[
d_{\Delta,1} = \frac{1}{\lambda \nu} \left( e_{\Pi,1} \frac{o_2(p_{\Pi,2} + b_2)(\frac{\sigma_1}{\sigma_2} + \lambda \mu_2 \sigma_2^2)(\frac{1}{o_2^2} - \frac{1}{\sigma_2 \mu_2^2})}{1 + \kappa} \right)
\]
where \(e\pi,2 = \pi_{\eta,2} + b_2 - \pi_{\eta,1}\) with \(b_2 = (1 - \pi_{\eta,2})\frac{\sigma_2^2}{\mu \sigma_2} \mu \sigma_2\). Imposing conditions on \(d_{\theta,1}\) and \(d_{\Delta,1}\) as in (3.13), we obtain

\[
p_{\theta,1} = \lambda h
\]
\[
p_{\pi,1} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} (p_{\pi,2} + b_2) \left(1 + \frac{\sigma_2^2 + \lambda \mu_2 \sigma_2^2}{\sigma_2^2 + \sigma_2^2} \frac{1}{1 + \kappa} \right),
\]

where we recall

\[
p_{\pi,2} = \frac{\sigma_2^2}{\sigma_2^2 + w_2}, \quad h = \frac{(p_{\pi,2} + b_2)^2 \mu_2 \sigma_2^2 \sigma_2^2}{o_2(1 + \kappa)}, \quad \mu_2 = \frac{\lambda \sigma_2^2}{2}
\]

The reader can verify that

\[
p_{\pi,1} = \frac{o_1}{\sigma_2^2} (p_{\pi,2} + b_2) \left(1 + \sigma_2^2 \left(\frac{1}{2} + \frac{\lambda \sigma_2^2}{2} \frac{1}{1 + \kappa} \right) \right) = \frac{o_1}{\sigma_2^2} (p_{\pi,2} + b_2) \left(1 + \frac{\lambda \sigma_2^2 \sigma_2^2}{2} \right)
\]

and

\[
(p_{\pi,2} + b_2) \mu_2 \sigma_2^2 = o_2 \left(1 + \frac{\lambda \sigma_2^2 \sigma_2^2}{2} \right).
\]

Thus,

\[
\mu_1 = \frac{\lambda (p_{\pi,2} + b_2) \mu_2 \sigma_2^2 \sigma_2^2}{o_2 \left(1 + \frac{\lambda \sigma_2^2 \sigma_2^2}{2} \right)} = \lambda \sigma_2^2 = \lambda w_1.
\]

This completes our proof.

**F Procedure for Solving Equilibrium Price Functions**

The equilibrium price functions can be solved by a recursive procedure. Starting with an initial guess of \(\sigma_{T-1}\), we solve the equilibrium price functions at \(T-1, T-2, \ldots\), recursively. Specifically, for given \(\sigma_{T-1}\), we first find the equilibrium price function at \(T-1\), i.e., \(p_{\pi,T-1}\) and \(p_{\theta,T-1}\) according to (3.14). We then calculate all the parameters needed for finding the equilibrium price function at \(T-2\). For example, we calculate \(f_{T-1}\), \(K_{T-1}\) and \(K_{T-2}^\pi\), as a function of \(p_{\pi,T-1}\) and \(p_{\theta,T-1}\). We also calculate \(a_{T-2}\) and \(a_{T-2}^\pi\) according to (B.6) and (B.4), as a function of \(\sigma_{T-1}, p_{\pi,T-1}\) and \(p_{\theta,T-1}\). A fixed point solution for \(p_{\pi,T-2}\) and \(p_{\theta,T-2}\) is obtained using (3.13). The procedure can be repeated for \(T-3, T-4, \ldots\).
Reference


LeBaron, B., “Persistence of the Dow Jones Index on Rising Volume,” unpublished manuscript (1992), University of Wisconsin.


Figure 1.1 - $p_{II,t}$

$T=25, \lambda=2.0, a_{\theta}=0.8,$

$\sigma_{II}^0=0.5, \sigma_{\theta}^2=0.1,$

$\sigma_{S,1}^2=0.6, \sigma_{S,t}^2=10^6 \ (1<t<T-1), \ \sigma_{Y,t}^2=10^6 \ (1\leq t\leq T-1).$
Figure 1.2 - $p_{\Theta, t}$

$T=25$, $\lambda=2.0$, $a_{\Theta}=0.8$, 

$\sigma_{H}^{2}=0.5$, $\sigma_{\Theta}^{2}=0.1$, 

$\sigma_{S,1}^{2}=0.6$, $\sigma_{S,t}^{2}=10^{6}$ $(1 \leq t \leq T-1)$, $\sigma_{Y,t}^{2}=10^{6}$ $(1 \leq t \leq T-1)$. 
Figure 1.3 - Volume of Informational Trading

\[ T=25, \quad \lambda=2.0, \quad a_0=0.8, \]
\[ \sigma_{H}^{2}=0.5, \quad \sigma_{D}^{2}=0.1, \]
\[ \sigma_{S,1}^{2}=0.6, \quad \sigma_{S,t}^{2}=10^{6} \quad (1 < t \leq T-1), \quad \sigma_{Y,t}^{2}=10^{6} \quad (1 \leq t \leq T-1). \]
Figure 1.4 - $h_t$

$T=25$, $\lambda=2.0$, $a_\emptyset=0.8$,

$\sigma_\delta^2=0.5$, $\sigma_\emptyset^2=0.1$,

$\sigma_{\delta,1}^2=0.6$, $\sigma_{\delta,t}^2=10^6$ ($1<t\leq T-1$), $\sigma_{\emptyset,t}^2=10^6$ ($1\leq t\leq T-1$).
Figure 1.5 - Informativeness of Equilibrium Prices

\[ T=25, \lambda=2.0, a_{\theta}=0.8, \]
\[ \sigma_{II}^2=0.5, \sigma_{\theta}^2=0.1, \]
\[ \sigma_{S,1}^2=0.6, \sigma_{S,t}^2=10^6 \quad (1<t<T-1), \quad \sigma_{Y,t}^2=10^6 \quad (1\leq t\leq T-1). \]
\[ \text{Var}[P_t - P_{t-1}] \]

\[ \sigma_\delta = 0.5 \]
\[ \sigma_\delta = 0 \]

Figure 1.6 - Volatility of Price Changes

\[ T=25, \lambda=2.0, a_\theta=0.8, \]
\[ \sigma_{\eta}^2=0.5, \sigma_{\theta}^2=0.1, \]
\[ \sigma_{\xi_t}^2=0.6, \sigma_{\xi_t}^2=10^6 \ (1<\xi\leq T-1), \sigma_{Y_t}^2=10^6 \ (1\leq t\leq T-1). \]
Figure 2.1 - Volume and Public Announcements

\( T = 25, \lambda = 2.0, a_{\Theta} = 0.8, \)
\( \sigma_{\Pi}^2 = 0.5, \sigma_{\Theta}^2 = 0.1, \)
\( \sigma_{S,1}^2 = 0.6, \sigma_{S,t}^2 = 10^6 \ (1 < t \leq T - 1), \sigma_{Y,t_A}^2 = 3.0, \sigma_{Y,t\neq t_A}^2 = 10^6. \)
Figure 2.2 - $h_t$ Public Announcements

$T=25$, $\lambda=2.0$, $a_\theta=0.8$,

$\varepsilon^2_t=0.5$, $\sigma^2=0.1$,

$\sigma^2_{S,1}=0.6$, $\sigma^2_{S,t}=10^6$ ($1 \leq t \leq T-1$), $\sigma^2_{Y,t_A}=3.0$, $\sigma^2_{Y,t \neq t_A}=10^6$. 
Figure 2.3 - $p_{\Theta,t}$

$T=25$, $\lambda=2.0$, $a_{\Theta}=0.8$,

$\sigma_H^2=0.5$, $\sigma_{\Theta}^2=0.1$,

$\sigma_{S,1}^2=0.6$, $\sigma_{S,t}^2=10^6$ ($1 < t \leq T-1$), $\sigma_{Y,t_A}^2=3.0$, $\sigma_{Y,t\neq t_A}^2=10^6$. 
Figure 2.4 - Volatility of Price Changes

\[ \text{Var}[P_t \mid P_{t-1}] \]

\[ T=25, \ \lambda=2.0, \ a_\theta=0.8, \]
\[ \sigma_H^2=0.5, \ \sigma_\theta^2=0.1, \]
\[ \sigma_{\xi_1}^2=0.6, \ \sigma_{\xi,t}^2=10^6 \ (1 \leq t \leq T-1), \ \sigma_{Y,tA}^2=3.0, \ \sigma_{Y,t \neq t_A}^2=10^6. \]
Figure 2.5 - Informativeness of Equilibrium Prices

\[ T=25, \lambda=2.0, a_\Theta=0.8, \]
\[ \sigma_{I_t}^2=0.5, \sigma_{\Theta}^2=0.1, \]
\[ \sigma_{S,1}^2=0.6, \sigma_{S,t}^2=10^6 \ (1 \leq t \leq T-1), \sigma_{Y,t}^2=3.0, \sigma_{Y,t \neq t_A}^2=10^6. \]