The Hazards of Debt: Rollover Freezes, Incentives, and Bailouts∗

Ing-Haw Cheng,† Konstantin Milbradt‡

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Abstract

We investigate the trade-off between incentive provision and inefficient rollover freezes for a firm financed with short-term debt. First, debt maturity that is too short-term is inefficient, even with incentive provision. The optimal maturity is an interior solution that avoids excessive rollover risk while providing sufficient incentives for the manager to avoid risk-shifting when the firm is in good health. Second, allowing the manager to risk-shift during a freeze actually increases creditor confidence. Debt policy should not prevent the manager from holding what may appear to be otherwise low-mean strategies that have option value during a freeze. Third, a limited but not perfectly reliable form of emergency financing during a freeze - a “bailout” - may improve the terms of the trade-off and increase total ex-ante value by instilling confidence in the creditor markets. Our conclusions highlight the endogenous interaction between risk from the asset and liability sides of the balance sheet.

Keywords: Rollover risk; rollover freezes; risk-shifting; optimal maturity; bailouts; financial firms; liquidity

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†University of Michigan, Ross School of Business, ingcheng@umich.edu.
‡Massachusetts Institute of Technology, Sloan School of Management, milbradt@mit.edu.
Is the use of short-term debt optimal? Recent research has focused on the role of a freeze in short-term debt markets as a leading amplification mechanism that led to the worst financial crisis since the Great Depression. The basic premise is that the non-bank financial sector, which experienced rapid growth in the early and mid-2000’s, and which relied heavily on staggered short-term debt to finance risky long-term and illiquid assets, experienced a *rollover freeze* during the crisis. Short-term creditors refused to roll over their debt for fear of future deterioration in the real estate market, leading to financial distress for those firms far exceeding the level of losses (Brunnermeier, 2009). Under this view, short-term debt creates a liability-side risk, or funding risk, for firms. More formally, He and Xiong (2009a) analyze this "rollover risk* and point out that financing an illiquid long-term asset with staggered short-term debt leads to a dynamic coordination problem among creditors and possibly inefficient liquidations.

An alternative literature, dating back to Calomiris and Kahn (1991), emphasizes the role of short-term debt as a disciplining device for moral hazard, for example to prevent risk-shifting by managers (Jensen and Meckling, 1976). Kashyap, Rajan, and Stein (2008) note that "short-term debt may reflect a privately optimal response to governance problems." Under this premise, short-term debt for the non-bank sector lowers risk on the asset-side of the balance sheet and increases value, much in the way depositors are value-increasing for the banking sector: the fragility of the institution itself provides incentives for depositors to monitor management and thus mitigates agency issues (Diamond and Rajan, 2001; Diamond and Rajan, 2000). Nevertheless, the literature has yet to fully resolve the trade-off between incentives and rollover risk, the latter of which was, if not the match that lit the fire, arguably the accelerant that set the crisis ablaze.

In this paper, we attempt to reconcile these views. Our research question is to ask what the optimal structure of debt should be in the presence of both rollover freezes (the liability-side risk) and risk-shifting problems (the asset-side risk). Our contribution is three-fold. First, we show that in the presence of both a risk-shifting problem and coordination problem

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1See, for example, He and Xiong (2009a), Morris and Shin (2009), Acharya (2009), Gorton and Metrick (2010) and Brunnermeier (2009).
among creditors, debt that is excessively short-term is inefficient from the perspective of total firm value, as it leads to low creditor confidence. Intuitively, an excessively short maturity of debt destroys more \textit{ex ante} firm value through the possibility of runs than it creates through incentive provision. Second, we show that it can be inefficient for debt to contain covenants that restrict managerial choices about which assets to hold. At the optimal maturity (where the marginal value of shortening debt maturities is zero), allowing the manager to risk-shift during a rollover freeze actually alleviates the creditor coordination problem, thereby making covenants sub-optimal \textit{ex ante}. Third, our paper shows that making a moderate amount of emergency financing available in the event of a rollover freeze - what we term a “bailout” - is value-increasing, even when including losses for the credit provider.

We build a dynamic model that focuses on the interplay between the intertemporal nature of the coordination problem among short-term creditors and how it affects incentives and risk-shifting. Intuitively, one of the salient features of a rollover freeze is that creditors refuse to roll over debt because of the fear that future creditors may also refuse to rollover, and we show that this creates a role for volatility and hence risk-shifting to interact with the incentive to run. The time-varying fundamental plays a critical role, much as it does in option-pricing theory, in determining the effect of risk-shifting on creditor confidence. The dynamic framework also allows us to compute the expected costs of excessively short debt maturities and bailouts within an equilibrium that allows for the recurrence of freezes and risk-shifting in the future. This is in contrast to much of the existing literature where comparisons of efficiency are often made across various equilibria, usually “run” and “no-run” equilibria.

Our model contains the following elements. First, a financial firm acts as a simple investment vehicle in long-term, illiquid assets. The manager, who holds the equity of the firm, can shape the risk-profile of the asset side of the balance sheet by switching between one of two strategies, A or B, at any point in time, where strategy A is a high-mean, low-volatility “good” strategy and strategy B is a low-mean, high-volatility “bad” strategy. In the absence of other considerations, it is clearly inefficient for the manager to adopt the bad strategy at
any point, as it is dominated in expected value. The possibility that he may do so generates “asset-side risk” for the firm. Indeed, in the presence of debt financing, it becomes optimal for the manager to risk-shift or “gamble for resurrection” when firm fundamentals are low, which results in lower firm value \textit{ex ante}. We focus on risk-shifting since it is an important source of agency issues among financial firms, as noted in Acharya and Viswanathan (2009).

The firm finances its investments using staggered short-term debt and must continually roll over its debt. The staggered nature of debt creates “liability-side risk” in the form of debt runs (He and Xiong, 2009a).\footnote{We use the terms “debt run” and “rollover freeze” interchangeably.} Intuitively, staggered short-term debt creates an intertemporal coordination problem, or low “creditor confidence,” where creditors refuse to roll over their debt when firm fundamentals are sufficiently low. These rollover freezes, or “dynamic debt runs,” are unlike static bank runs in that the time-varying firm fundamentals may improve before the firm is liquidated during a freeze. Thus, a maturing creditor’s decision to roll over his debt depends on his anticipation of whether \textit{future} creditors will roll over their debt. Failing to survive a freeze results in distressed liquidation, so rollover freezes reduce firm value \textit{ex ante}.

These elements make the trade-off between shorter and longer maturities non-trivial. Although short-term debt can lead to freezes, it mitigates the risk-shifting problem. Longer maturities (such as those which are matched with asset maturity) may lead the manager to risk-shift even when the firm is not experiencing a run, a phenomenon we call “preemptive risk-shifting.” Firm value is optimized at the maturity where the manager never preemptively risk-shifts. Intuitively, risk-shifting when the firm is not experiencing a run is an inefficient transfer from debt to equity and destroys total value. Our first result is that debt maturity should be just short enough to eliminate this type of risk-shifting; shorter maturities are strictly inefficient.

On the other hand, risk-shifting during a freeze can actually increase value and improve creditor confidence. The manager will want to risk-shift during a freeze in order to maximize the equity’s option value. We show that the extra volatility creates creditor confidence during
a run. Intuitively, the intertemporal nature of a rollover freeze implies that the interests of future, non-maturing creditors and current, maturing creditors diverge during a freeze. Non-maturing creditors are effectively junior to maturing creditors and thus have more convex interests in that they want the firm to recover quickly (or at least survive long enough until their debt matures) in order to avoid being saddled with inefficient liquidation. Consequently, they prefer the volatility and higher option value of the bad strategy during a freeze, as it increases the chance that fundamentals will recover.

In equilibrium, when all agents anticipate that managers will hold the high-volatility asset during the freeze, a maturing creditor will be less worried about future creditors’ motives to withdraw funding, so that other creditors will be less worried about other creditors withdrawing funding, and so on. This results in a weaker ex ante incentive to run. Our second result is then that giving the manager the capability to risk-shift during a freeze improves value, relative to the case where the manager is restricted to only adopting the good strategy forever. Our point here is not to say that managers should seek wildly risky strategies during a freeze, but rather to highlight that debt policy should not be so stringent as to inadvertently prevent managers from holding assets or taking actions that would be deemed too risky or otherwise poor ideas during normal times.

The third part of our analysis considers how moral hazard and freezes vary with “bailout” policies where a third party (such as the government) provides emergency financing to the firm during a freeze by providing creditors just enough money to roll over their debt on the margin. Our thought experiment is to ask whether total value (including government losses) is worsened by such a policy in a stylized setting where we think of our firm as representing the broad financial sector. The emergency financing is provided only probabilistically in the sense that it is not limitless and is only provided for a limited (random) amount of time; if the funding disappears, the sector experiences a severe liquidation cost, e.g., large costs associated with systemic failure. We parameterize the “reliability” of a bailout as the expected amount of time a firm can expect to receive emergency financing during a freeze, and compute the optimal reliability including the endogenous effect of emergency financing
on rollover freezes, moral hazard, and expected government losses.

We find that a non-trivial bailout reliability improves total \textit{ex ante} value. Bailouts have two primary effects. On the one hand, they create value by boosting creditor confidence: a longer government lifeline means that a maturing creditor’s fear of whether future creditors will roll over is alleviated, feeding back in equilibrium into a decreased likelihood of freezes \textit{ex ante}. On the other hand, more reliable bailouts encourage the manager to adopt the bad strategy outside of a freeze. This is in and of itself bad for total value since the bad strategy has a low expected value. However, it also feeds back into high government losses: since rescuing the sector will only result in managers again risk-shifting even after a freeze has been overcome, creditors will require a large amount to incentivize them to roll over their debt.

Expected government losses are non-monotonic in bailout reliability due to these feedback effects. The optimal reliability of emergency financing is positive yet “mild” in the sense that, at the optimal reliability, the manager never adopts the bad strategy outside of a freeze. Providing emergency financing for a positive expected amount of time helps reduce the incidence of freezes \textit{ex ante} by boosting creditor confidence. By avoiding preemptive risk-shifting, a mild reliability is actually associated with relatively low government losses for a range of parameters; in particular, losses are smaller than the case where the government provides even \textit{less} reliable emergency financing.\textsuperscript{3}

A number of caveats attach to our conclusions. First, we are focusing on a specific type of “bailout,” one that incentivizes debtors to roll over their debt during a freeze, and not other types of bailouts, such as nationalization or direct equity injection. Second, our model is agnostic as to the source of the emergency financing, which, strictly speaking, could be provided by entities other than the government. In this sense, our point is that government-funded emergency financing may increase value if and when no one else is willing to provide it for reasons outside the model. We believe this is a reasonable starting point in light of the absence of credit in the recent crisis. Finally, our model does not explicitly model systemic

\textsuperscript{3}Of course, government losses are zero if no funding is provided, but in this case there is a large value loss due to excessive preemptive risk-shifting.
risk, the primary rationale advanced in favor of bailouts, so that we are essentially taking a reduced form approach: we take as given that there are large costs to liquidation and ask whether such policies aimed at avoiding liquidation may enhance value.

Nevertheless, the question of how short-term is too short-term with respect to debt maturity and how government intervention in these markets affects incentives is of vital importance in light of the billions of taxpayer dollars committed by governments around the world to saving large financial institutions from freezes in short-term debt markets. While our model is stylized, it attempts to address these questions in an integrated fashion.

Although our model is similar in spirit to static bank run models such as Diamond and Dybvig (1983) and Goldstein and Pauzner (2005), and the bank-run/incentive models of Diamond and Rajan (2001) and Diamond and Rajan (2000), its novel focus is on the interaction of volatility with dynamic runs. Rochet and Vives (2004) study bailouts in a global games framework and have a section on incentives, but do not consider the dynamic feedback mechanism between risk-shifting and runs. Diamond (1993) (and the closely related Diamond, 1991 paper) looks at how to optimally structure the seniority and maturity of debt contracts in response to an adverse selection problem. Our contribution is to highlight the important role of volatility in the interaction between runs and risk-shifting through an intertemporal coordination problem in a setting free from asymmetric information or multiple equilibria considerations. We show that, even in a setting without legally distinct seniority, the implicit seniority of maturing creditors has important implications for how debt financing influences incentives.

The paper proceeds as follows. Section 1 introduces the model, Section 2 describes our equilibrium, and Section 3 describes our results pertaining to optimal maturity and optimal risk-shifting. Section 4 examines the effect of emergency financing, and Section 5 discusses further implications of our analysis. Section 6 concludes.
1 The Model

Consider a financial firm that is a simple investment vehicle for long-term, illiquid assets. The firm is run by a manager who can switch between one of two investment strategies at any point in time, one of which is a high-mean, low-volatility strategy while the other strategy is a low-mean, high-volatility strategy. The firm finances itself using staggered short-term debt, such as asset-backed commercial paper, and the manager holds the equity of the firm. The staggered nature of short-term debt creates liability-side risk in the form of the possibility of rollover freezes, while the possibility of choosing different investment strategies creates asset-side risk in the form of inefficient risk-shifting.

The key quantities of interest are the maturity of debt and how it affects firm value, the shadow value of constraints (such as covenants) that force the manager to avoid risky strategies, and the value of emergency financing provided during a possible creditor run. The model is dynamic and set in continuous time with $t \in [0, \infty)$ as our results emphasize the unique role interaction that volatility has with incentives and runs. The continuous-time setting also allows us analyze all of the key quantities of interest within a single equilibrium.

1.1 The Firm, Manager, and Asset-Side Risk

The manager of the firm can employ one of two possible strategies, A or B. Employing either strategy yields a fixed cash-flow $r$ per unit of time, which is routed to creditors. The firm also yields a final random payoff of $y_{\tau_\phi}$ at a random time $\tau_\phi$ in the future. We can think of $\tau_\phi$ as representing the maturity of the type of assets underlying the firm’s strategies; when the final payoff realizes, the firm is dissolved. We model the asset maturity as random for tractability purposes; we assume that the realization time $\tau_\phi$ is exponentially distributed with intensity $\phi$, so one interpretation is that the firm invests in assets which are expected to mature $1/\phi$ years in the future.

The key distinction between the strategies is that the final payoff $y$ evolves with a high drift and low volatility while the manager employs strategy A, whereas it evolves with low
drift and high volatility while the manager employs strategy B. Mathematically, \( y \) evolves according to

\[
\frac{dy_t}{y_t} = \mu_i dt + \sigma_i dZ_t
\]

where \( dZ \) is a standard Brownian motion, \( \mu_i \) is the growth rate of the final payoff, \( \sigma_i \) is the instantaneous volatility, and \( i \) indexes whether the manager is following strategy A or B, and where \( \mu_A > \mu_B \) but \( \sigma_A < \sigma_B \).

The risk-neutral manager holds all the equity of the firm, and cares only about final wealth. The manager can switch between strategies costlessly at any point in time based on the current value of \( y \), which is observable to both creditors and the manager. We denote the region of values of \( y \) where the manager adopts strategy B with the set \( \bar{R} \). For example, the manager may choose to adopt strategy B whenever \( y \) is less than 1, in which case \( \bar{R} = (0, 1) \).

The possibility that the manager adopts strategy B at any point creates “asset-side risk” for the firm. Choosing strategy B is a real inefficiency: the more often the manager adopts strategy B, the lower the fundamental value of the firm (the expected discounted value of all future cash flows). Evidently, strategy A dominates strategy B – it has higher return for less risk. Hence we call strategy A the “good” strategy and strategy B the “bad” or “risk-shifting” strategy. While only having two possible investment strategies is a stark setup, we view them as proxies for more complex investment strategies, where the bad strategy substantially increases the volatility of the firm’s fundamentals at the expense of long-run performance. As we will see, the presence of debt creates an incentive for the manager to inefficiently “gamble for resurrection.”

### 1.2 Debt Financing and Liability-Side Risk

We choose the dynamic debt run model of He and Xiong (2009a) as our building block for liability-side risk. This choice is motivated by two considerations: first, the dynamic nature allows for treatment of volatility, and, second, the model provides a robust unique
equilibrium, which allows us to model incentives with runs in an integrated framework.

The firm finances itself using debt with total face value normalized to one. The debt is held by a continuum of risk-neutral creditors of unit measure, and each creditor holds debt of face value 1. We assume the creditors have discount rate \( \rho < r \) so that debt is financially attractive, as creditors receive the full cash flows \( r \) that the firm generates. The important feature about the firm’s debt financing is that the firm stagger sthematurti ty o fit s debt: rather than having all creditors’ debt contracts mature at the same time, only a fraction of debt comes due at each point in time. As noted in He and Xiong (2009a), many financial firms spread out the maturity of their debt expirations, often for liquidity reasons.

More formally, a creditor’s contract comes due upon the arrival of an independent Poisson shock with intensity \( \delta \) at each point in time. We refer to \( \delta \) as the maturity of the firm’s debt, although there is a distribution of debt described by \( \delta e^{-\delta T} \) with expected maturity \( 1/\delta \), plotted in Figure 1. When \( \delta = 0 \), asset maturities and debt maturities are matched: all creditors are locked in until the firm’s final asset payoff is realized at \( \tau_\phi \). On the other hand, when \( \delta > 0 \), not only is there a maturity mismatch, but a fraction of \( \delta \) of debt comes due at each point in time, with high values of \( \delta \) corresponding to very short-term debt. Section 3 endogenizes the choice of \( \delta \).

Currently maturing creditors may choose to withdraw the face value of their debt, equal to one, or roll over their debt at no additional cost and return to the non-maturing creditor.
pool. In contrast, a non-maturing creditor must wait for their debt to come due before they can extract the face value of their debt. In this sense, the non-maturing creditors are junior to currently maturing creditors, even though all claims are of ex-ante equal seniority.

If all currently maturing creditors refuse to roll over their debt, a situation we term a rollover freeze, the firm must find continued financing to survive and undergoes distressed liquidation if it cannot do so. Specifically, we assume that the company draws on emergency financing through pre-established credit lines or government money to fill the gap on the balance sheet during a rollover freeze. However, these credit lines or possible government bailouts are not perfect: when all maturing creditors in a $dt$ period decide to pull out, there is a probability $\theta \delta dt$ that the company cannot find financing. In this case, the company is liquidated in distress and its assets are sold at a fire sale discount. The parameter $\theta$ measures the reliability of the firm’s credit lines or possible government bailouts - the higher the value of $\theta$, the more likely the firm will be liquidated in distress. Section 4 endogenizes the choice of $\theta$.

If the firm is liquidated in distress (i.e., during a rollover freeze), the firm’s assets are sold on the outside market, where it fetches its expected discounted cash flow under strategy A. There is, however, a proportional cost of sale for distressed liquidations $(1 - \alpha)$, with $\alpha \in (0, 1)$ due to illiquidity of the underlying assets and/or bankruptcy costs. The project’s

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4We assume the maturity structure $\delta$ is unaffected by this; i.e., $\delta$ is stationary.

5The money from the credit lines is drawn under the standard debt contract described above; that is, the debt acquired via a credit line is identical to the existing debt of the company. The credit line providers immediately sell any such acquired debt on the secondary market at a value $D$, which is determined in equilibrium. In a wider sense, the parameter $\theta$ can also be interpreted as the strength of the company’s cash-holdings. However, actually introducing cash reserves would result in a second state-variable that would make the model very hard to solve. When we interpret the distress financing to be provided by the government, in Section 4, we assume the government uses a market based intervention: it bails out those creditors that want to leave by paying them just enough to roll over their debt.

6We assume that the liquidation value is always the risk-neutral value under strategy A (even if the firm is, at the moment, employing strategy B) for simplicity. The results are nearly identical if we assume that liquidation value is always the risk-neutral value under B, or some linear combination of the two values.
Figure 2: Example paths of \( y \) with various possible outcomes. Either the asset pays its terminal value (\( \tau_\phi \) realizes) or the firm is liquidated in a run (\( \tau_\theta \) realizes).

The outside value is

\[
L(y) = \alpha \mathbb{E}^A \left[ \int_t^{\tau_\phi} e^{-\rho(s-t)} rs + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] \\
= \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\phi + \rho - \mu y} y \\
= L + ly
\]

Due to equal seniority, the proceeds are split equally among all non-maturing creditors; excess proceeds are distributed to equity. Crucially, a firm can survive a freeze if fundamentals recover in time, as demonstrated in Figure 2.

The presence of a liquidation cost creates a coordination problem among creditors. Since maturing creditors are essentially senior to non-maturing creditors, there is a conflict of interest between maturing and non-maturing creditors. Today’s maturing creditors may stop rolling over if they anticipate that future creditors may stop rolling over, since they wish to extract their value today and avoid bearing any potential liquidation cost. The asset’s funding structure thus endogenously may lead to debt runs and inefficient liquidation, or so-called “liability-side” or “funding” risk (He and Xiong, 2009a).

It should be clear by now that the model of rollover freezes described here relies on a number of assumptions and modeling devices. First, as with asset maturity, we model the maturity of debt as random for tractability purposes. The random maturity itself is not the
key feature. Rather, the important feature is that the debt maturity is staggered. Second, there is a fire-sale discount if the firm is liquidated in distress, which we view as realistic in light of the financial crisis. Third, the terms of debt contracts are fixed, whereby the interest rate of the debt is fixed at the cash flow of the asset $r$, without possibility of renegotiation. Although the nature of our conclusions about how risk-shifting and debt runs interact clearly depend on the above assumptions, we believe these assumptions provide an important first building block about how runs interact with incentives. Therefore, to ease exposition, we take these assumptions as given for now and return to discuss the assumptions in Section 5.1.

### 1.3 Value Functions

All in all, there are two possible outcomes for the firm. It is either prematurely liquidated in distress, or the final payoff is realized. More formally, define the project’s *horizon time* $\tau = \min \{\tau_\phi, \tau_\theta\}$ as the minimum time of two possible events, either the final payoff realizing $(\tau_\phi)$ or the firm liquidating $(\tau_\theta)$ as the result of a prolonged freeze.

The value of the outcomes for the various agents in the model are as follows. First, consider today’s maturing creditors. They are faced with a choice of either withdrawing their face value of 1, or rolling over their debt and becoming a non-maturing creditor. If we let $D(y)$ denote the value of non-maturing debt as a function of the current fundamental, this means that maturing creditors can choose either to receive 1 or to roll over and receive $D(y)$ by becoming a non-maturing creditor. The value of this proposition is evidently $\max \{1, D(y)\}$.

Second, there are today’s non-maturing debtors. The current value of their debt can be described as the discounted expected value of three possible future outcomes. First, their contract could come due in the future, in which case they become a maturing creditor and will face a rollover choice, worth $\max \{1, D(y_{\tau_\delta})\}$ at that future moment. Second, the final payoff could realize at $\tau_\phi$, in which case they will receive the standard debt payoff of $\min \{y_{\tau_\delta}, 1\}$ and the firm closes up shop. Third, the firm could be liquidated at $\tau_\theta$ during a
freeze. In this case the proceeds from such a distressed liquidation are distributed to debt and equity, with each debtor receiving a payoff equal to \( \min \{ L + l y_{\tau_\theta}, 1 \} \). Between now and any one of those three future events, the non-maturing creditor will collect the cash flow \( r \).

Mathematically, we can write the value of non-maturing debt as the following expectation, or value function,

\[
D(y) = \mathbb{E}_t \left[ \begin{array}{c} \int_t^{\min\{\tau, \tau_\delta\}} re^{-\rho(s-t)} ds + 1_{\{\min\{\tau, \tau_\delta\} = \tau_\delta\}} e^{-\rho \tau_\delta} \max \{1, D(y_{\tau_\delta})\} \\ + 1_{\{\min\{\tau, \tau_\delta\} = \tau_\phi\}} e^{-\rho \tau_\phi} \min \{y_{\tau_\phi}, 1\} + 1_{\{\min\{\tau, \tau_\delta\} = \tau_\delta\}} e^{-\rho \tau_\delta} \min \{L + l y_{\tau_\theta}, 1\} \end{array} \right] 
\]

(2)

where the first term reflects the interest payment, the second term describes the rollover decision at maturity,\(^7\) the third term is the payoff from the project realizing, and the last term is the payoff from liquidation.

Third, there is the manager, who holds the firm’s equity. The current value of equity, which we denote by \( E(y) \), can be described as the discounted expected value of two possible future outcomes, which are mirror images of the outcomes for debt. First, the asset’s terminal value could realize at \( \tau_\phi \), in which case they will receive the standard equity payoff of \( \max \{y_r - 1, 0\} \) and then the firm shuts down. Second, the firm could be liquidated at \( \tau_\theta \) during a freeze, in which case equity receives \( \max \{L + l y_r - 1, 0\} \) at that future moment. Mathematically, we can thus write today’s value of equity as

\[
E(y) = \mathbb{E}_t \left[ 1_{\{\tau = \tau_\phi\}} e^{-\rho \tau_\phi} \max \{y_{\tau_\phi} - 1, 0\} + 1_{\{\tau = \tau_\delta\}} e^{-\rho \tau_\delta} \max \{L + l y_{\tau_\theta} - 1, 0\} \right] 
\]

(3)

where the first term is the payoff from the project realizing and the second term is the payoff from liquidation.

It is worth noting that these two expectations - the value of non-maturing debt and the value of equity - are expectations that take into account all possible paths of \( y \) and

\(^7\)As default/liquidation and an individual rollover decision coinciding at a time \( t \) is an event of bounded variation of order \( dt \), we can ignore this event. Thus, the creditor’s only decision at time \( \tau_\delta \) (the maturity time of their debt) is whether to roll over or to collect the face value of 1.
thus account for the possibility that the manager may change the drift and volatility of the fundamental at any point and also that creditors may run. If the critical threshold over which creditors roll over is \( y^* \) and the risk-shifting region is \( \mathcal{R} \), then these functions should be denoted as \( D(y|y^*, \mathcal{R}) \) and \( E(y|y^*, \mathcal{R}) \) instead of \( D(y) \) and \( E(y) \). For notational convenience we drop the latent variables.

For now, we conjecture that \( \mathcal{R} \) is the union of two open intervals, i.e., \( \mathcal{R} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3) \). We discuss why this conjecture for \( \mathcal{R} \) is intuitive when we solve the equilibrium. Under this conjecture, we can solve for the functions \( D(y) \) and \( E(y) \) in closed form up to a system of non-linear equations for any given \( (y^*, \mathcal{R}) \).

### 1.4 Parameter Restrictions & Numerical Benchmarks

Before turning our attention to equilibrium, we need to impose a few additional parameter restrictions for the model to make sense and the value functions to be well defined.

First, we assume that \( L + l \leq 1 \), so that the project at \( y = 1 \) is worth less if liquidated than if it realized immediately. This is important to rule out the manager unilaterally liquidating the project to cash in on the promised interest flow to the creditors. Because of this and the assumption that equity holders have no cash, there is no endogenous default triggered directly by the manager, unlike in Leland (1994) and Leland and Toft (1996).\(^8\) Let \( y_L \) be the point at which, if liquidated, the project yields just enough to pay off all creditors. From our previous assumptions, we have

\[
y_L \equiv \frac{1 - L}{l} \geq 1
\]

\(^8\)As the model nests a liquidation option, we need to consider the following question: Does the manager have an incentive to unilaterally liquidate the firm? His payout when the project realizes is \((y - 1)^+\), whereas his payout when the firm is liquidated is \((L + ly - 1)^+\). By liquidating today, the manager is able to raise cash that is related to interest payments \( r \) (summarized by coefficient \( L = \frac{\alpha r}{\alpha + \sigma} \)), which would otherwise go to the debt holders. Thus, for consistency, we need to check that \( E(y) \geq [L(y) - 1]^+ \). This holds for all cases treated in this paper.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Interpretation</th>
</tr>
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<tbody>
<tr>
<td>r</td>
<td>0.1</td>
<td>Cash flow from project per unit of time $dt$</td>
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<tr>
<td>(\rho)</td>
<td>0.05</td>
<td>Discount rate</td>
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<tr>
<td>(\phi)</td>
<td>0.2</td>
<td>Intensity at which terminal value of asset realizes ($1/\phi = 5$ years)</td>
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<td>(\theta)</td>
<td>1</td>
<td>Reliability of emergency financing</td>
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<tr>
<td>(\delta)</td>
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<td>Expected maturity of debt ($1/\delta = 0.1$ years)</td>
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<td>(\mu_A)</td>
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<td>Drift of strategy A</td>
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<tr>
<td>(\mu_B)</td>
<td>0</td>
<td>Drift of strategy B</td>
</tr>
<tr>
<td>(\sigma_A)</td>
<td>0.1</td>
<td>Volatility of strategy A</td>
</tr>
<tr>
<td>(\sigma_B)</td>
<td>0.3</td>
<td>Volatility of strategy B</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>0.7</td>
<td>Liquidation discount factor (Liquidation cost is $1 - \alpha = 30%$)</td>
</tr>
</tbody>
</table>

Table 1: Parameter Values.

so that \([y - 1]^+ \geq [L(y) - 1]^+\).

Second, as we will see in Section 2.1, we require \(\rho + \phi > r\) to ensure that there is an incentive to stop rolling over for the debtholders. This also results in \(L < 1\). Combined with the previous restriction \(r > \rho\), we have

\[
\rho < r < \rho + \phi
\]

Third, we require

\[
\mu_A < \rho + \phi
\]

in order for the firm to have finite value. Note that neither the maturity parameter \(\delta\), the volatility parameters \(\sigma_A\) and \(\sigma_B\), nor the liquidation intensity \(\theta\) enter these parameter restrictions so far.

Finally, we make the following technical assumption: \(\mu_A, \sigma_A, \mu_B, \sigma_B\) are such that the positive root of \(\frac{\sigma^2}{2} \eta^2 + (\mu_i - \frac{\sigma^2}{2}) \eta - (\phi + \rho + \theta \delta)\) is larger for asset A than for asset B. This constraint makes the risk-shifting problem non-trivial by ensuring that the manager will want to adopt the bad strategy for low \(y\).

For our numerical solutions, Table 1 lists our annualized benchmark parameter values. These parameters fulfill all our parameter restrictions above with \(y_L \approx 1.03\). The expected maturity of a debt contract is \(\frac{1}{\delta} = 0.1\) years or just above a month. The subjective discount
rate is $\rho = 5\%$, whereas debt has interest payments equal to $r = 10\%$. Asset A has a drift of 5% and asset B has a drift of 0%. However, the annualized instantaneous volatility is three times larger for B, 30%, than for A, 10%. Finally, there are liquidation costs of 30%.

For ease of interpretation, we translate these intensities into probabilities and expected horizon times. When all maturing debtors run (and continue running), i.e., conditional on staying in the freeze, the expected time until either the firm is liquidated or the asset’s final payoff realizes decreases from $\frac{1}{\phi} = 5$ years to $\frac{1}{\phi + \delta \theta} = .098$ years, or 1.17 months - survival during a rollover freeze is difficult. If either of these events happen during a freeze, the probability that it was liquidated is $\frac{\delta \theta}{\phi + \delta \theta} = 0.98$ and the probability that the asset’s final payoff realized is only 0.02. Thus, the liquidation event is by far the dominating event during a rollover freeze.

## 2 Rollover Risk and Risk-Shifting

We now turn our attention to equilibrium, where creditors symmetrically choose a rollover threshold $y^*$ to maximize their debt value (taking as given the manager’s risk-shifting strategy), and the manager chooses a risk-shifting region $\bar{R} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3)$ to maximize his equity value (taking as given the strategy of the creditors). We look for a dynamically consistent equilibrium $\{y^*, \bar{R}\}$ where the state variable is the observable fundamental $y$. In equilibrium, each individual creditor must be just indifferent between rolling over his debt and receiving face value of 1 at at $y = y^*$. This equilibrium condition can be written as $D \left( y^* \big| y^*, \bar{R} \right) = 1$.

Optimality can be represented as a so-called “super contact” condition (Dixit, 1993; Dumas, 1991) which we derive in the appendix, where, at each point $\bar{y}_i$, the second derivative of the manager’s value function must be equal whether or not he holds asset A or asset B.$^9$

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$^9$Intuitively, for the manager’s strategy to be optimal, not only must he be indifferent between A and B in the sense that the marginal utility of adopting strategy A must equal the marginal utility of adopting strategy B, but he must be indifferent between the change in marginal utility from either strategy. Loosely, if the manager expects the marginal utility of strategy A to increase over the next $dt$ interval more than he expects the marginal utility of holding asset B to increase, then, given that level of each marginal utility is equal now, it would be profitable (in expectation) for the manager to briefly hold asset A instead.
Combining these two conditions, an equilibrium in our model is defined as:

**Definition 1** A symmetric Markov-Perfect Nash equilibrium \((y^*, \bar{R})\) in cut-off strategies where maturing creditors maximize \(\max (1, D(y))\) and the manager maximizes \(\max_{\bar{R}=(\bar{y}_1, \bar{y}_2, \bar{y}_3)} E(y)\) must simultaneously satisfy

1. *(Creditors’ Indifference Condition)* \(D(y^*|y^*, \bar{R}) = 1\) with \(D(y|y^*, \bar{R})\) strictly increasing in \(y\).

2. *(Manager’s Optimality Condition)* The cut-off point \(\bar{y}_i\) in \(E(y|y^*, \bar{R})\) either satisfies the super contact condition \(\lim_{y \uparrow \bar{y}_i} E_{yy}(y|y^*, \bar{R}) = \lim_{y \downarrow \bar{y}_i} E_{yy}(y|y^*, \bar{R})\), where \(E_{yy}\) is the second derivative of the manager’s value function, or is a corner solution in the sense that \(\bar{y}_i \in \{1, y_L, y^*\}\) with no profitable deviation \(\bar{y}'_i\) available.

The rest of this section is devoted to understanding how various moving parts affect equilibrium outcomes.

### 2.1 Benchmark: Equilibrium with Rollover Risk Only

Why do creditors wish to stop rolling over their debt? In short, if a maturing creditor rolls over his debt today, he is exposed to the possibility that tomorrow’s maturing creditors may withdraw funding, causing the firm to be liquidated. Thus tomorrow’s maturing creditors exert an externality on today’s maturing creditors. Any creditor whose debt is maturing may wish to walk away with their face value now instead of facing this liquidation risk.

In more detail, consider a maturing creditor’s problem. If her roll over is \(y\), he is locked in for an expected time of \(1/\delta\), during which time he is a non-maturing creditor. As a non-maturing creditor, he would like a rollover freeze at \(y\) if and only if \(D(y) < \min \{L + Ly, 1\}\); that is, if his debt is worth less than the proceeds from distressed liquidation. In contrast, maturing creditors will stop rolling over for any \(y\) such that \(D(y) < 1\). Thus, there is clearly a wedge between the incentives of maturing and non-maturing debtholders, and choosing to roll over debt today and becoming a non-maturing creditor exposes the
creditor to movements in $y$, as low values of $y$ may precipitate a run by tomorrow’s maturing creditors. Of course, the creditor will be exposed to changes in the manager’s investment decision as well, since they affect the dynamics of $y$.

When maturities are short and $\delta$ is high, debt comes due quickly and many creditors act in quick succession. Thus, low firm fundamentals are unlikely to improve before many creditors have had a chance to act,\footnote{More formally, the expected number of creditors that get to act between now and the next maturity time is independent of $\delta$, as it is a product of the expected time $\frac{1}{\delta}$ and the flow of maturing creditors per unit of time $\delta$.} which means that today’s maturing creditors have an even stronger incentive to run since they expect future creditors to run, which we formally show in the appendix.\footnote{We show in the appendix that $y^* = 0$ (all maturing creditors always rolling over) is not an equilibrium nor is $y^* \to \infty$ (all maturing creditors never rolling over). Intuitively, it is not an equilibrium for all maturing creditors to always roll over because for low values of the fundamental, the debt value will be low and some creditors would take their face value and stop rolling over. Similarly, never rolling over is not an equilibrium because when fundamentals are high, it is a profitable one-shot deviation to stay in instead of taking the face value, even when all other creditors are running.} In this sense, short maturities erode what we call creditor confidence.

As a benchmark model, consider the case where the manager can only ever adopt strategy A. For example, there may be debt covenants that restrict the manager’s choice. Denote the equilibrium rollover threshold in this constrained model by $y^*_A$. He and Xiong (2009a) establish that there is a unique equilibrium in cutoff strategies where creditors refuse to roll over for $y < y^*_A$. Figure 3 plots $y^*_A$ for different values of $\delta$. The top left panel shows that the equilibrium rollover threshold increases as maturity decreases (i.e. $\delta$ increases).\footnote{As noted in He and Xiong (2009a), $y^*_A$’s monotonicity disappears for situations with low drift and very high volatility. We will focus on the case where $\sigma_i$ is low as the economically relevant one. Our results in the following sections are robust to higher volatility though, i.e. they when we double $\sigma_B$ to 0.6.}

In equilibrium, low maturity feeds back into a higher rollover threshold $y^*_A$ for everyone, a phenomenon we term a loss of creditor confidence. This results in a lower ex ante value of debt, equity and total firm value for short maturities, as plotted in the upper right panel, lower right panel, and lower left panel. In each of these panels, value is decreasing in $\delta$. In this benchmark model, shorter maturities are strictly inefficient, as creditors raise their rollover thresholds in anticipation of others raising their rollover thresholds.
Figure 3: Rollover thresholds and firm value under benchmark equilibrium with no risk-shifting as a function of expected debt maturity $\delta$. Longer debt maturities correspond to lower values of $\delta$ while shorter maturities correspond to higher values of $\delta$. The upper left-hand panel plots the rollover threshold $y^*_A$ on the vertical axis. The upper right-hand panel and lower right-hand panel plot debt and equity value, respectively, at $y_0 = 1$. The lower left-hand panel plots the value of debt plus equity.
2.2 Introducing Risk-Shifting

We now introduce the manager, who holds the equity of the firm. Why does the manager wish to engage in strategy B, which is strictly dominated? In the presence of debt, the equity of the firm is a call option on the fundamental $y$, which results in a risk-shifting problem (Jensen and Meckling, 1976). For example, when $y < 1$, the manager is “out-of-the-money” on this option. Since risk-shifting can increase the equity option value by trading off fatter tails against a lower mean, managers have an incentive to “gamble for resurrection.” If we impose the assumption that creditors never run (purely as an exercise, since this is clearly not an equilibrium), the manager risk-shifts for $y < 1.19$, well before his option is “out-of-the-money,” in order to maximize his equity value.

Clearly, debtors would like to prevent risk-shifting as strategy B is strictly dominated and it dilutes their claim. Although debtors cannot individually discipline the manager as they are each very small, the coordination problem among creditors can discipline the manager through the possibility of rollover freezes and the attendant likelihood of liquidation. Since risk-shifting for low values of $y$ actually increases the chance that asset fundamentals deteriorate, the possibility of a freeze and inefficient liquidation acts as a countervailing weight to the manager’s incentive to gamble.

Note that, given a rollover threshold $y^*$, it is entirely possible for a manager to want to play a non-monotone strategy in the following sense. For high values of $y$, the manager’s equity is safely in the money, and he will want to adopt the good strategy. For lower values of $y$, but not low enough for creditors to run, he may choose to risk-shift in order to increase the option value of his equity. However, for values of $y$ very close to $y^*$ (when a freeze is imminent), the volatility of the bad strategy is likely to move the fundamental into the freeze region, which is a very bad state of the world for the manager. Because of this, he may actually adopt the good strategy for some “buffer” region of $y$ slightly above $y^*$. Finally, if fundamentals are low enough so that creditors run ($y < y^*$), the threat of a run has already been exercised, so the manager will gamble and risk-shift for sure; we prove this formally in the appendix. For these reasons we investigate risk-shifting strategies of the
form $\tilde{R} = (0, y_1) \cup (y_2, y_3)$, but we allow $y_1, y_2, y_3$ to coincide. We call risk-shifting outside of the freeze region preemptive risk-shifting, as it is risk-shifting that “preempts” (i.e., occurs before) the freeze.

The dynamic nature of the creditor coordination problem provides a nice setting for analyzing this risk-shifting problem. First, as He and Xiong (2009a) show, there is a unique equilibrium in runs in the absence of incentive considerations. This makes analyzing incentives and runs jointly much easier since runs do not generate multiple equilibria. Even compared with global games techniques, incentives in our environment are especially tractable since there is symmetric information among all parties. Second, the dynamic model yields unique insights that are specific to the nature of the time-varying fundamental, as we will show. Finally, the tractable setting allows us to analyze the impact of incentives and runs on total value by accounting for runs today and in the future all within one equilibrium, in contrast to other models which typically compare the efficiency of a “run” and “no-run” equilibrium.

3 The Optimal Structure of Debt

We now proceed to solve the full equilibrium with both rollover risk and risk-shifting. Let $(y_{AB}^*, \tilde{R}_{AB})$ denote the equilibrium rollover threshold and risk-shifting set. Because we need to consider the interaction between the freeze and risk-shifting regions, we have to simultaneously solve for $y_{AB}^*$ and $\tilde{R}_{AB} = (0, \tilde{y}_{1AB}) \cup (\tilde{y}_{2AB}, \tilde{y}_{3AB})$. As we have closed form solutions for debt and equity for any given $(y^*, \tilde{R})$ given in the appendix, the optimality conditions reduce to numerically solving a system of non-linear equations in the variables $\tilde{y}_{1AB}, \tilde{y}_{2AB}, \tilde{y}_{3AB}, y_{AB}^*$ and checking sufficiency.  

Figure 4 plots the equilibrium thresholds as a function of $\delta$ in the top left panel. The rollover threshold $y_{AB}^*$ is presented as the thick black line, whereas the risk-shifting set $\tilde{R}_{AB}$

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13 We will focus on equilibria where the manager always gambles during a freeze, i.e. with $\tilde{y}_{1AB} = y^*$, since it is always optimal for a manager to gamble once a freeze has ensued. The other two risk-shifting thresholds $\tilde{y}_{2AB}$ and $\tilde{y}_{3AB}$ are then found via a super contact condition. Finally, we check via the $E(\cdot)$ function that for any candidate equilibria $(y_{AB}^*, \tilde{R}_{AB})$, there is no profitable deviation $\hat{y}'$ at any level of $y$. 

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is identified by the shaded grey areas. The dark shading identifies preemptive risk-shifting, while the light shading identifies risk-shifting during a freeze. The top right, bottom right, and bottom left panels show the ex ante values of debt, equity, and total firm values as a function of the maturity $\delta$, respectively. In all graphs, we suppose the initial value of the fundamental is $y_0 = 1$.\textsuperscript{14}

From the perspective of total value, there are two different regions of possible maturities. First, there are the “long” maturities, where the manager engages in preemptive risk-shifting, as indicated by $\delta \in (0, 19.25)$ on the top-left panel of Figure 4 (recall that low values of $\delta$ translate to long maturities). For the longest maturities of $\delta \in (0, 0.5)$, the manager plays a monotone strategy and risk-shifts for all $y < \bar{y}_{1AB}$. For more intermediate maturities of $\delta \in (0.5, 19.25)$, the manager plays a non-monotone strategy, where he preemptively risk-shifts over a range of $y$ above the rollover threshold, reverts back to the good strategy close to the rollover threshold, and then risk-shifts again once in the freeze. In this region, the marginal impact of shortening the maturity is to further discipline the manager’s risk-shifting.

For shorter maturities over the range $\delta \in (19.25, \infty)$, incentives have been maximized and the manager never adopts strategy B unless the firm is experiencing a freeze. Mathematically, $\tilde{y}_{jAB} = y^*_AB$ for $j = 1, 2, 3$. The marginal impact of shortening the maturity on this interval is purely to worsen the coordination problem and thus increase the rollover threshold $y^*_AB$. Since it is dominant for the manager to risk-shift once a freeze has occurred, this also raises the risk-shifting threshold $\tilde{y}_{1AB}$. Although it turns out that the increased risk-shifting associated with higher $\tilde{y}_{1AB}$ actually mitigates the negative value effect of the increased rollover threshold $y^*_AB$ (as we shall discuss in Section 3.2), the dominant effect in this region resulting from shortening the maturity is to reduce total firm value, as evidenced by the lower left-hand panel.

\textsuperscript{14}We have investigated a number of different initial starting values. For starting values of $y$ outside of the freeze region, our results are qualitatively unchanged and hence we take $y_0 = 1$ for expositional purposes. The key here is that we are only considering cases where we start the firm outside of a freeze, which we believe is reasonable. Note that the equilibrium points $(y^*_AB, \mathcal{R}_{AB})$ do not depend on any initial starting value: a Markov-Perfect Nash equilibrium is dynamically consistent for all values of $y$.  

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3.1 The Optimal Maturity

These observations highlight the fact that, in order to find the optimal structure of debt, we have to weigh the inefficiencies created by risk-shifting against the inefficiencies created by rollover freezes. It turns out that, for a wide range of parameterizations, the optimal maturity is the longest possible maturity that eliminates preemptive risk-shifting, which is \( \delta = 19.25 \) in our previous example, or an average debt maturity of 19 days. Debt with shorter expected maturity is “too short-term” in the sense that one could lengthen the maturity on the margin and maintain maximum incentives while lessening rollover risk.

Shortening the maturity of debt has, on the margin, three different effects on value. First, it reduces preemptive risk-shifting, which increases value since preemptive risk-shifting is strictly inefficient. Notably, this implies that it is not optimal to completely match the maturity of assets with the maturity of liabilities, as this leads to excessive preemptive risk-shifting. Second, shortening the maturity of debt also directly decreases the likelihood the firm will survive a freeze, since debt comes due very quickly and there is thus very little time for firm fundamentals to recover before the firm is liquidated from lack of funding. This decreases firm value, but the effect is small. Finally, shortening the maturity structure results in a higher rollover threshold and more fragile creditor confidence, leading to more freezes and much lower value ex ante, much as described in Section 2.1. This effect is very large. We summarize with the following:

**Result 1** Very short maturities impose excessive liquidation costs without leading to any improvement in disciplining the manager, while any longer maturity would lead to preemptive risk-shifting by the manager. The optimal maturity that maximizes the total value of the firm is just long enough to eliminate preemptive risk-shifting, and is neither too short-term nor too long-term.
Figure 4: Rollover thresholds and firm value under full equilibrium as a function of expected debt maturity $\delta$. Longer debt maturities correspond to lower values of $\delta$ while shorter maturities correspond to higher values of $\delta$. The upper left-hand panel has values of $y$ on the vertical axis and plots the rollover threshold $y^*_A B$ as the thick black line along with the risk-shifting region $\bar{R}_{AB}$ as the shaded region. Preemptive risk-shifting is shaded in dark gray while risk-shifting during a run is shaded in light gray. The upper right-hand panel and lower right-hand panel plot debt and equity value, respectively, at $y_0 = 1$. The lower left-hand panel plots the value of debt plus equity. In all panels, the vertical line marks the optimal maturity $\delta$. 
3.2 The Value of Risk-Shifting

A novel result from the equilibrium analysis is that risk-shifting by the manager during the freeze is value-increasing for both the equity-holding manager and the debtholders, and hence for total value. In order to see this, we compare firm value under the full equilibrium with firm value under the benchmark equilibrium, where the manager is always constrained to the good strategy.

Intuitively, risk-shifting during a freeze has two beneficial effects. The first is a direct effect: increasing the asset volatility actually increases the likelihood that the firm escapes the freeze before it is liquidated from lack of funding. When the firm is experiencing a freeze, it will not survive for long. While the lower drift of the bad strategy hurts the firm’s fundamental in expectation, this negative effect is outweighed by the higher volatility of the bad strategy, which increases the likelihood that the firm escapes the freeze before it is liquidated. Technically speaking, the extra volatility increases the likelihood that $y$ rises above $y_{AB}^*$ before $\tau_\theta$ realizes.

Second, there is a substantial indirect effect whereby risk-shifting during the freeze actually improves creditor confidence. As discussed in Section 2.1, the staggered maturity structure creates a wedge between the interests of creditors, where maturing creditors may withdraw funding because they are worried about future creditors withdrawing funding. Risk-shifting alleviates this concern, since it increases the likelihood that the firm fundamental escapes the run region quickly. This lowers a maturing creditor’s run-threshold, which lowers future creditors’ run-thresholds, and so forth, so that the equilibrium run threshold $y_{AB}^*$ is substantially lower if the manager is allowed to adopt strategy B in a run.

This is most clearly seen in Figure 5, which plots the run threshold, debt value, equity value, and total firm value from the full equilibrium with risk-shifting, and overlays the values from the benchmark equilibrium in Figure 3 as dashed lines. In the upper left-hand panel, note that, at the optimal maturity $\delta = 19.25$, we have $y_{AB}^* < y_A^*$; in words, constraining the manager to no risk-shifting increases the run threshold. This results in substantially lower firm value, as demonstrated in the bottom left-hand panel. Comparing the value of the
equilibrium where the manager is allowed to risk-shift (the thick black line) with that where the manager is always forced to adopt strategy A (the dashed line) at \( \delta = 19.25 \), we see that forcing the manager to always adopt strategy A reduces the value of debt plus equity from 1.38 to 1.24, an economically large 10% loss. Forcing the manager to always adopt strategy A reduces value for all but the longest maturities, as evidenced in the bottom left-hand panel of Figure 5, where the dashed line (where the manager is constrained to no risk-shifting) lies below the thick solid line (where the manager can risk-shift) for all but very low \( \delta \).

Thus, debt policy should avoid covenants restricting managers’ strategies when maturities are short. Actions that seem strictly value-decreasing in normal times may actually be beneficial in times of distress. This demonstrates the interaction between asset-side risk and liability-side risk in a dynamic setting. When a firm faces asset-side risk in the form of production inefficiencies, this “risk” may actually have a bright side in alleviating the liability-side risk stemming from how the firm is financed. This non-trivial interaction depends critically on the nature of the debtor intertemporal coordination problem and the volatility of the time-varying fundamental.

We summarize with the following:

**Result 2** *For debt maturities that are sufficiently short-term, allowing the manager the capability to risk-shift increases creditor confidence and is value-increasing. Thus, short-term debt should not contain covenants that restrict managerial investment decisions.*

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\(^{15}\)For very long maturities, it becomes optimal to constrain the manager to no-risk-shifting, since the benefit of eliminating rampant preemptive risk-shifting (due to lack of incentives) outweighs the lost benefit of improving creditor confidence during a freeze. This begs the question of which is better: issuing debt at the optimal maturity and allowing the manager to risk-shift during a freeze, or issuing debt at a very long maturity while constraining the manager not to risk-shift (perhaps through regulatory measures)? The results from the bottom left-hand panel indicate that firm values under these two policies are very close. Indeed, if constraining the manager not to risk-shift involves ancillary costs such as contracting costs, our analysis shows that issuing debt at the optimal maturity and allowing the manager to risk-shift during a freeze is superior.
Figure 5: Rollover thresholds and firm value under full equilibrium (Figure 4) overlaid with rollover thresholds and firm value under the benchmark equilibrium with no risk-shifting (Figure 3).
4 Market-Based Emergency Financing

The previous section left out one implicit claimant to the firm’s balance sheet, the provider of interim financing during a rollover freeze. We now consider an interpretation where we think of the firm in the model as the broad financial sector and where the government provides this emergency financing via a market-based intervention, which we label a “bailout.” One may think of incentives in our model as incentives in the broad financial sector to hold risky assets, and a freeze in our model as a freeze in short-term credit markets. During such a freeze, the financial system, which relies on these short-term credit markets to finance long-term assets, experiences severe distress with probability $\theta \delta dt$. One can think of this event as the failure of Lehman Brothers or some other systemically important institution. Our goal is to address how bailouts might be designed so as to address the specific nature of rollover freezes and credit freezes while preserving adequate incentives to avoid excessive risk-taking, which we interpret as adopting the bad strategy B.

In our version of a bailout, the government uses a market-based intervention by paying maturing creditors who wish to leave $[1 - D(y)]$, which is just enough on the margin to incentivize them to roll over.\(^\text{16}\) However, these bailouts may not be completely reliable in that emergency financing may dry up, in which case the freeze creates severe distress in the financial system. The bailout reliability is thus parameterized by $\theta$, which the government commits to at time 0. We assume that the government cannot condition on the exact state of the financial system, and can only condition on whether a freeze is occurring. For example, politically economy considerations may not allow a more fine-tuned intervention. More broadly, we can interpret $\theta$ as randomizing between bailouts (e.g. Bear Stearns) and failures (e.g. Lehman Brothers).

We focus our discussion on how reliable the government’s emergency financing should be, given an observed maturity structure $\delta$ of the sector. That is, we fix $\delta$ and look for the optimal $\theta$ at time 0. A no-bailouts policy is captured by $\theta \rightarrow \infty$, whereas policy

\(^{16}\)An alternative interpretation of this bailout strategy is the following. The government purchases debt from distressed firms at face value 1. It then immediately turns around and sells this debt on the open markets for $D(y)$, making a loss of $[1 - D(y)]$ per unit of debt bailed out.
that always bails out is equivalent to $\theta = 0$. A more intuitive interpretation in terms of probabilities instead of intensities is provided through the transform $P(\theta) = e^{-\theta}$, which gives the probability of survival for a continuous freeze of length $1/\delta$. Bailing out with probability one then corresponds to $P(0) = 1$, whereas bailing out with probability zero corresponds to $\lim_{\theta \to \infty} P(\theta) = 0$.

The optimal bailout reliability $\theta$ maximizes the total \textit{ex ante} value $F$, which we define to be the total value of the system (debt plus equity) less any expected government losses. To compute expected government losses, we need to measure how often and how much the government is called upon to contribute for a given strategy $\theta$, which we can compute for any value of $y$. Here the dynamic setup of the model is particularly advantageous since we can compute the expected cost to the government within the equilibrium, taking into account the possibility of any future risk-shifting and debt runs. Denote by $G(\cdot)$ this expected cost to the government as a function of $y$, so that total \textit{ex ante} value is $F(y) = D(y) + E(y) - G(y)$. Since we fix the initial value at $y_0 = 1$, we will be interested in $G(1)$ and $F(1)$.

The government pays maturing creditors who refuse to roll over the subsidy $[1 - D(y)]$ to roll over their debt. Thus, for $y < y^*$, the government continuously provides emergency financing to a measure $\delta dt$ of maturing creditors until either the bailout expires and the system experiences liquidation, the terminal value of the asset realizes, or the fundamentals improve sufficiently so that creditors have enough incentive to roll over their debt without a bailout. The function $G$ can then be written as

$$G(y|y^*, \bar{R}) = \mathbb{E}_t \left[ \int_t^\tau e^{-\rho s} \delta 1_{\{y<y^*\}} [1 - D(y)] ds \right]$$

(4)

where $\tau = \min\{\tau_\theta, \tau_\phi\}$ as before. In the Appendix, we derive the closed-form solution of $G(y|y^*, \bar{R})$ up to the thresholds $(y^*_{AB}, \bar{R}_{AB})$.  

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4.1 Optimal Emergency Financing

We look for the bailout reliability $\theta$ that maximizes the total value of the system $F$. Intuitively, a higher bailout reliability can create value by providing financing to the system during a rollover freeze and helping avoid distressed liquidation. However, there are a number of potential costs. Naturally, bailouts could incur high government losses. Furthermore, bailouts weaken the incentives provided by short-term debt, and thus could lower ex ante equity and debt value by encouraging the manager to adopt strategy B more frequently.

To disentangle these effects, we compute equilibrium run thresholds, risk-shifting regions, government losses, and total system value (debt plus equity less government losses) for a variety of different bailout policies, shown in Figure 6. To fix ideas, we take the maturity structure at a constant $\delta = 10$. The upper left-hand panel plots the run threshold $y_{AB}^*$ as the thick solid line and the risk-shifting region $\bar{R}_{AB}$ as the grey shaded areas. The upper right-hand panel plots the debt value, while the lower right-hand panel plots government losses, and the lower left-hand panel plots total system value $F$. The probability of survival for a continuous freeze of length $1/\delta$, or $P(\theta)$, lies on the horizontal axis, with high values of $P(\theta)$ corresponding to strong bailout policies that nearly guarantee the firm will survive a freeze.

From the perspective of total system value $F$, there are two distinct regions of $P(\theta)$. For $P(\theta) < 0.21$, the marginal effect of increasing bailout reliability is to boost total value, as seen in the lower left-hand panel. Interestingly, increasing the bailout reliability in this region actually involves a slight decrease in government losses. This is because boosting bailout reliability in this region decreases the run threshold $y_{AB}^*$ without creating preemptive risk-shifting, as seen in the upper left-hand panel. In other words, boosting bailout reliability here is value-increasing because it creates creditor confidence without worsening incentives and thus avoids substantial expected losses. In contrast, for $P(\theta) > 0.21$, increasing bailout reliability destroys total value as it creates preemptive risk-shifting and also results in higher expected government losses.

More specifically, the effect of a bailout policy on value has three effects and is highly
Figure 6: Rollover thresholds and firm value under full equilibrium as a function of the reliability of emergency financing $P(\theta)$. Less reliable emergency financing corresponds to low values of $P(\theta)$, while more reliable emergency financing corresponds to high values of $P(\theta)$. The upper left-hand panel has values of $y$ on the vertical axis and plots the rollover threshold $y^*_{AB}$ as the thick black line along with the risk-shifting region $\mathcal{R}_{AB}$ as the shaded region. The benchmark rollover threshold from the no-risk-shifting equilibrium $y^*_A$ is plotted as the dashed black line. The upper right-hand plots debt value at $y_0 = 1$, while the lower right-hand panel plots expected government losses at $y_0 = 1$. The lower left-hand panel plots total system value $F$, which is the value of debt plus equity less expected government losses. In all panels, the vertical line marks the optimal bailout reliability. Values from the full equilibrium are plotted as a solid black line and values from the no-risk-shifting equilibrium are plotted as a dashed black line.
non-monotonic. First, increasing bailout reliability saves the firm from distressed liquidation more often, an effect whose value actually turns out to be relatively small. Second, increasing bailout reliability boosts creditor confidence and lowers the run threshold $y_{AB}$ by relaxing the concerns of today’s maturing creditors in equilibrium. This effect is positive for firm value (debt plus equity) is quite large. In fact, in the absence of incentive considerations, it is a dominant strategy for the government to always bail out if government costs are equally weighted against debt and equity.

Third, bailouts also worsen managerial incentives, which act as a countervailing weight to these positive effects. As the possibility of distressed liquidation as a result of a debt run fades (i.e., the liquidation intensity $\theta \delta$ drops), the manager’s incentive to risk-shift increases. This decreases value as it encourages preemptive risk-shifting, which is strictly inefficient. Even worse, a high bailout reliability results in high expected government losses, as evidenced in the lower right-hand panel of Figure 6 for $P(\theta) > 0.21$. Here, even though creditors employ a low rollover threshold for high bailout reliabilities, the government’s bailout costs can actually be high. This is because incentives are weak and managers will again preemptively risk-shift after a freeze has been escaped. This means that creditors will require a large amount from the government to incentivize them to roll over their debt during a freeze, since the system is fundamentally more tilted towards the bad strategy.

The optimal bailout reliability maximizes total value (debt plus equity less government losses, equally-weighted) by balancing these three effects. In our parameterizations, the optimal bailout reliability is positive at $P(\theta) = 0.21$ and discourages excessive freezes while preserving maximum incentives and avoiding preemptive risk-shifting. Notably, this optimal bailout reliability is associated with a relatively low level of expected government losses: lowering the bailout reliability would actually not decrease government losses appreciably since freezes would be much more frequent. To summarize:

**Result 3** Limited probabilistic/randomized bailouts can be optimal by boosting creditor confidence, even in the presence of incentive effects.
Figure 7: Firm value $F(1) = D(1) + E(1) - G(1)$ along the diagonal $\theta \delta = 10$. No bailouts, i.e. $\theta \to \infty$, are suboptimal even for very long maturities that keep the liquidation intensity constant.

4.2 Robustness

Our result on the optimal bailout reliability is qualitatively similar to our result on debt structure, which is not surprising because bailouts essentially extend the expected survival time during a freeze, while short debt maturities decrease it. In our results on maturity, we varied $\delta$ while holding $\theta$ fixed. This mixes two channels since $\delta$ affects both the liquidation intensity, $\delta \theta$, and the flow of maturing creditors, $\delta$. Changing $\theta$ in turn only affected the liquidation intensity, $\delta \theta$, while leaving the flow of maturing creditors untouched. As a robustness check, we isolate the channel that works through the flow of maturing creditors while leaving the liquidation intensity untouched by looking at equilibria along the diagonal of $\theta \delta = 10$, on which our benchmark point lies. As Figure 7 shows, there is an optimal maturity $\delta$ (and thus an optimal bailout point $\theta = \frac{10}{\delta}$); preemptive risk-shifting vanishes at this point. We can clearly see that no bailouts, i.e. $\theta \to \infty$, are suboptimal even when we lengthen the maturity to keep the liquidation intensity constant. Thus, neither a very short maturity-highly reliable bailout combination nor a long maturity-unreliable bailout combination is optimal. Our results highlight that bailouts operate through a similar, yet distinct, channel from debt maturities in balancing the trade-off between $ex$ $ante$ incentives and $ex$ $post$ bankruptcy costs.
5 Discussion

5.1 Modeling Assumptions

As mentioned at the end of Section 1.2, there are a number of stylized assumptions in the model. We discuss the robustness of these assumptions.

First, we assume that the terms of the debt contracts are fixed, whereby the promised cash flow $r$ of the debt is fixed, without possibility of renegotiation. Although this is a stylized assumption, we motivate it using the following three insights. First, suppose the firm could attract new financing during a debt run by promising a higher cash flow $r$. This may actually lead creditors to preemptively demand higher payments and may tighten the firms’ financing ability instead of relaxing it, leading to a qualitatively similar “run” mechanism. Second, there may be frictions in the debt market that prevent the promised cash flow from fully adjusting. In reality there are situations, e.g., for Lehman Brothers, where no one was willing to lend at any interest rate. Third, since all of the future cash flows of the projects have already been securitized, raising new debt would mean giving up parts of the equity. But clearly there will be a point beyond which giving up part of the equity will not be sufficient to cover the fixed face value of debt. In other words, even when $r$ is flexible, there will a point at which there is not enough cash flow $r$ to continue to roll over debt of face value 1. Even deep-pocketed equity holders will have a point beyond which they are unwilling to inject money to prop up the debtholders (Leland, 1994; He and Xiong, 2009b). For these reasons, we also abstract away from a dynamic capital structure.

Second, we abstract away from endogenous debt maturity in this model. It should be clear that holding all other creditors fixed, each individual creditor will want a lower maturity on his debt. This is the so-called maturity rat-race that is discussed in more detail in Brunnermeier and Oehmke (2009) and also in He and Xiong (2009a). Our model addresses a very different question in that we ask whether certain maturities are optimal for firm value by investigating the outcome of coordination and risk-shifting problems. This allows us to look for the debt maturity that is optimal for the firm from an ex ante perspective, whereas
papers with endogenous debt maturities look for what debt maturities would arise naturally. We view these two questions as both important and complementary, but distinct.

Finally, we assume that the government commits to a fixed reliability parameter $\theta$ at the beginning of the world. Our model analyzes whether a particular bailout reliability is optimal \textit{ex ante}, and has little to say about whether specific bailouts are optimal \textit{ex post}. We view this as reasonable since this allows us to speak of \textit{ex ante} value. Another concern is that the government does not play a strategy that allows it to condition on $y$, i.e., to choose a $\theta (y)$. We leave this for potential future analysis and only remark here that it will be difficult from a political economy perspective to tailor the bailout strategies to fundamentals as such action requires a lot of political flexibility.

5.2 Where Does the Value of Risk-Shifting Come From?

The result that risk-shifting during a freeze has a role in increasing value seems initially counterintuitive, but can be understood more fully in the following context. Essentially, a freeze shrinks the total amount that both equity and debt can expect to receive in the event of a distressed liquidation. One can think of this liquidation cost as a “claimant” in the sense that it receives $(1 - \alpha)$ of the expected discounted cash flows in the event of a distressed liquidation. Risk-shifting transfers value away from the liquidation cost to debt and equity by alleviating the coordination concern among creditors. Even though the drift of strategy B is lower, this cost is very small relative to the benefit when liquidation is likely.\footnote{One can think of the providers of emergency financing (either the government or a third party) as a claimant to the firm as well. Here, the effect of risk-shifting on these credit providers is ambiguous, but is often small compared to the overall value gain on debt plus equity for moderate to low values of credit line reliability as measured by $P(\theta)$. The dashed line in the lower left panel of Figure 6, which represent value under a “good strategy only” equilibrium, highlight that constraining the manager to a good strategy is only optimal - from the perspective of debt plus equity plus credit providers - for high values of $P(\theta)$.} This result - that it may be optimal to choose the high risk strategy - has the flavor of some of the results in Leland and Toft (1996) and Leland (1998). Similar to those papers, in our model, there are implicit claimants beyond just debt and equity.

It is important to note that transferring value away from liquidation costs is not a zero-
sum transfer, but instead a real efficiency gain. In our model, inefficiencies are not a result of the liquidation cost receiving too large a share of a fixed set of cash flows when freezes are more likely. Instead, when freezes are more likely, the overall value of the firm, or “size of the pie,” is actually smaller, even when counting the liquidation cost as a claimant - that is, there is a deadweight loss that stems from using the dominated risk-shifting strategy. Thus, the model features real inefficiencies as opposed to only contractual inefficiencies.

These results draw a distinction between two types of risk-shifting: preemptive risk-shifting and risk-shifting during a freeze. The standard literature argues that managers “gamble for resurrection” when firm fundamentals are low and that this is value-destroying. Our model suggests that risk-shifting during a freeze can be optimal when liquidation costs are high. The results also offer new insights as to why we may not always observe gambling when firm fundamentals are low. In our model, the manager plays a non-monotone strategy in equilibrium: he only gambles outside of a freeze when there is a sufficient buffer between the rollover threshold and the current firm fundamental. Inside of that buffer region, it is optimal for the manager to play conservatively by holding the high-mean, low-volatility asset in order to minimize the chances of incurring a rollover freeze.

5.3 Policy Implications

Our analysis on bailouts have the clear policy implication that the government should avoid committing to never bailing out the system but should also avoid creating expectations that the system will always be bailed out. Instead, the government should commit to a limited bailout policy. It is worth discussing the caveats to this implication.

First, we note that we are focusing on a specific type of bailout. In our bailout, the government is not nationalizing the firm. Instead, it is a targeted bailout aimed at alleviating the panic among creditors that could lead to distressed liquidation. In other words, it is aimed at instilling confidence. We believe this has basis in programs such as the various liquidity facilities the Fed has implemented and also the asset guarantee programs the Treasury has implemented. Although these did not literally distribute money to creditors from
the government, they involved the markets as much as possible and were aimed at improving market confidence and avoiding panicked runs on systemically important institutions.

Second, our model is agnostic as to the source of emergency financing, which, strictly speaking, could be provided by entities other than the government. In other words, there is no market failure in our model that prevents private insurance. In this sense, our point is that government-funded bailouts may increase value if and when no one else is willing to provide it, for many potential reasons. For example, the sheer scale of such a potential failure could necessitate governmental involvement, as private capital and credit may be slow to move in or otherwise unavailable, particularly in crises.

Finally, we simplify the analysis by taking a reduced form approach to modeling systemic risk by assuming that there is a large cost to liquidation captured via $\alpha$. However, this reduced form is flexible in that we can vary the liquidation cost, $\alpha$, from the government’s perspective separately from the liquidation cost that debtors and equityholders perceive. For example, suppose the government views the failure of the firm as a huge value loss for the overall economy, so that they employ a specific $\hat{\alpha} < \alpha$ discount to the distressed liquidation value of the firm, while the manager and creditors employ the original $\alpha$ since they do not internalize the economy-wide impact of distressed liquidation. We find that this does not qualitatively affect our results. We can also extend the model by simply assigning a higher or lower weight to government losses relative to firm value in the total value function $F$, and we find that the optimal bailout policy is robust to increasing the weight on government losses.

Our analysis also highlights a new positive channel for bailouts, one that assures current maturing creditors that future maturing creditors are unlikely to create severe distress. This mechanism works in the context of rollover freezes rather than bank runs. While we acknowledge the importance of bank runs and runs due to lack of common knowledge (as in global games models), we view our analysis as largely complementary to the large bank run literature.\footnote{See, e.g., Diamond and Dybvig (1983), Goldstein and Pauzner (2005), and the bank-run/incentive models of Diamond and Rajan (2001) and Diamond and Rajan (2000), as well as Rochet and Vives (2004).} We believe that intertemporal coordination problems are also independently

\begin{align*}

18 & \text{See, e.g., Diamond and Dybvig (1983), Goldstein and Pauzner (2005), and the bank-run/incentive models of Diamond and Rajan (2001) and Diamond and Rajan (2000), as well as Rochet and Vives (2004).}

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\end{align*}
interesting, particularly in light of the possibly different prescriptions for policy on how to resolve them. In a run that reflects a lack of common knowledge, one prescription for intervention would be to gather everyone in a conference room, share all private signals and agree on a course of action. Once common knowledge is established, the efficient action can be taken. This has basis in history as with the Fed’s intervention with LTCM.

On the other hand, in a dynamic rollover freeze, it is not a failure of common knowledge, but an intertemporal coordination problem that generates the incentive to stop rolling over or run. The prescription in this case is very different: gathering everyone in a conference room will not work, as today’s non-maturing creditors cannot commit to rolling over tomorrow when their contracts mature. Bailouts relieve the incentive to run in our model through a very different mechanism. They assure current maturing creditors that future maturing creditors are unlikely to create severe distress. On a broader level, we also believe our modeling device provides insight into the effect of bailing out some firms but not others; although the government cannot commit to save everyone, it must save a few.

6 Conclusion

So is short-term debt ultimately optimal? In this paper, we constructed a model of a non-bank financial firm which faces rollover externalities due to the use of staggered short-term debt (as in He and Xiong, 2009a) and introduced equity and incentive problems in the form of managerial risk-shifting. We also introduced expected government losses as a function of the reliability of emergency financing - which we termed a “bailout” - and analyzed its interaction with freees and risk-shifting. Our results highlight that risk from the asset and liability sides of the balance sheet interact in a dynamic setting and have non-trivial implications for how incentives and debt should be structured. The primary conclusions are that debt can be too short-term, that covenants that constrain managerial actions should be avoided, and that bailouts can improve creditor confidence. Further research into the dynamic interaction of risk from the asset and liability sides of the balance sheet may shed
further light on how to prevent future financial crises.

References


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Rochet, J.-C., and X. Vives (2004): “Coordination failure and the lender of last resort: was Bagehot right after all?,” Journal of the European Economic Association, 2(6), 1116–1147.
A Appendix

A.1 The value functions $D(y)$, $E(y)$ given $y^*, \bar{R}$

**Proposition 1** Given a rollover threshold $y^*$ and a risk-shifting region $\bar{R}$, possibly non-optimal, equity has the value function

$$E(y) = C_+(y) y^\kappa(y)^+ + C_-(y) y^\kappa(y)^- + a(y) \cdot y + b(y)$$

where $\eta(y)_+ > 1 > 0 > \eta(y)_-$ solve $f(\eta, y) = 0$,

$$f(\eta, y) = \frac{\sigma_1^2}{2} \eta^2 + \left( \mu_i - \frac{\sigma_1^2}{2} \right) \eta - (\phi + \rho + 1_{y<y^*}) \theta \delta$$

and

$$a(y) = \frac{1_{(1<y)} \phi + 1_{(y_L<y<y^*)} \theta \delta}{\rho + \phi + 1_{(y<y^*)} \theta \delta - \left( 1_{\{y\in \bar{R}\}} \mu_B + 1_{\{y\in \bar{R}\}} \mu_A \right)}$$

$$b(y) = -\frac{1_{(1<y)} \phi + 1_{(y_L<y<y^*)} \theta \delta}{\rho + \phi + 1_{(y<y^*)} \theta \delta}$$

where different $\eta, a, b, C_\pm$ apply in each of the intervals composed of the boundary points $0, 1, y_L$ and the rollover and risk-shifting thresholds $y^*, y_1, y_2, y_3$. The coefficients $C_\pm(\cdot)$ are step functions solving a linear system stemming from value matching and smooth pasting at the transition points $1, y_1, y_2, y_3, y^*, y_L$ and the boundary conditions $C_{-}^0 = 0$ and $C_{+}^\infty = 0$.

Similarly, debt has the value function

$$D(y) = CC_+(y) y^\kappa(y)^+ + CC_-(y) y^\kappa(y)^- + aa(y) \cdot y + bb(y)$$

where $\kappa(y)_+ > 1 > 0 > \kappa(y)_-$ solve $f(f(k, y) = 0$,

$$f(f(k, y) = \frac{\sigma_1^2}{2} \kappa^2 + \left( \mu_i - \frac{\sigma_1^2}{2} \right) \kappa - (\phi + \rho + 1_{y<y^*}) (\theta + 1) \delta$$

and

$$aa(y) = \frac{1_{(y<1)} \phi - 1_{(y_L<y<y^*)} l \theta \delta + 1_{(y<y^*)} l \theta \delta}{\rho + \phi + 1_{(y<y^*)} (\theta + 1) \delta - \left( 1_{\{y\in \bar{R}\}} \mu_B + 1_{\{y\in \bar{R}\}} \mu_A \right)}$$

$$bb(y) = b(y) = \frac{r + 1_{(1<y)} \phi + 1_{(y_L<y<y^*)} \theta \delta (1-L) + 1_{(y<y^*)} \delta (\theta L + 1)}{\rho + \phi + 1_{(y<y^*)} (\theta + 1) \delta}$$

where different $\kappa, aa, bb, CC_\pm$ apply in each of the intervals composed of the boundary points $0, 1, y_L$ and the rollover and risk-shifting thresholds $y^*, y_1, y_2, y_3$. The coefficients $CC_\pm(\cdot)$ are step functions solving a linear system stemming from value matching and smooth pasting at the transition points $1, y_1, y_2, y_3, y^*, y_L$ and the boundary conditions $CC_{-}^0 = 0$ and $CC_{+}^\infty = 0$.

**Proof.**

Let us first look at the debt value function. Note first that $f(0, y) < 0$ and that $f(1, y) = \mu - (\phi + \rho + 1_{y<y^*}) \theta \delta < \mu - (\phi + \rho) < 0$ by assumption on $\mu_A$ and $\mu_B$. Thus, we have $\eta(y)_+ > 1 > 0 > \eta(y)_-$.

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19 Where $C_{-}^0$ is shorthand for $C_-(y)$ with $y \in (0, \min \{1, y_1, y_2, y_3, y^*, y_L\})$ and $C_{+}^\infty$ is shorthand for $C_+(y)$ with $y \in (\max \{1, y_1, y_2, y_3, y^*, y_L\}, \infty)$. 

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When the project realizes, the payoff to the creditor is \( \min \{ y, 1 \} \). Rewrite this as \( \min \{ y, 1 \} = 1_{\{ y < 1 \}} (y - 1) + 1 \). Similarly, when the project is liquidated, the payout to creditor is \( \min \{ L + ly, 1 \} = 1_{\{ y > ly \}} (1 - L - ly) + L + ly \). With a slight abuse of notation, let \( y^* \) be a (possibly suboptimal) candidate symmetric rollover threshold. As a proportion \( \delta \) of debt contracts matures each \( dt \), liquidation has an intensity of \( \delta \theta \) in the rollover freeze \( y < y^* \).

In equilibrium, we know that at the rollover threshold \( y^* \), debt is worth 1, so that \( \max \{ 0, 1 - D \} = 1_{\{ y < y^* \}} (1 - D) \). We substitute \( 1_{\{ y < y^* \}} (1 - D) \) for \( \max \{ 0, 1 - D \} \) even off the equilibrium path, so that the creditors (collectively) behave suboptimally for non-equilibrium \( y^* \). For a candidate symmetric equilibrium we then need to check that \( D (y | y^*, \bar{R}) \) is increasing in \( y \). As there is only one state variable \( y \), the HJB resulting from equation (2) will result in the following ODE

\[
\rho D = \frac{\mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) + 1 - D)}{1_{\{ y < y^* \}} (1 - D) + r} \quad \text{if } i = A \quad y \in \bar{R}^c
\]

\[
\rho D = \frac{\mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) + 1 - D)}{1_{\{ y < y^* \}} (1 - D) + r} \quad \text{if } i = B \quad y \in \bar{R}
\]

\[
\rho D = \frac{\mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) + 1 - D)}{1_{\{ y < y^* \}} (1 - D) + r} \quad \text{if } i = A \quad y \in \bar{R}^c
\]

\[
\rho D = \frac{\mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) + 1 - D)}{1_{\{ y < y^* \}} (1 - D) + r} \quad \text{if } i = B \quad y \in \bar{R}
\]

where we substituted out the \( \max \{ \cdot \} \) function via the appropriate indicator functions. The first two terms on the right-hand side (RHS) are simply the Ito terms from the dynamics of the state variable \( y \). The third term is the payoff when the project matures, \( \tau_\theta \), whereas the fourth term is the payoff of the project being liquidated, \( \tau_\delta \). The fifth term is the payoff from the debt maturing, \( \tau_\delta \) (i.e. either rolling over or collecting the face value) and the last term is simply the interest rate.

Similarly, for equity we have the following ODE

\[
\rho E = \frac{\mu_i y E_{y} + \frac{\sigma^2}{2} y^2 E_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) - E)}{1_{\{ y < y^* \}} (1 - L - ly) + 1_{\{ y < y^* \}} (L + ly - D)} \quad \text{if } i = A \quad y \in \bar{R}^c
\]

\[
\rho E = \frac{\mu_i y E_{y} + \frac{\sigma^2}{2} y^2 E_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) - E)}{1_{\{ y < y^* \}} (1 - D) + \phi (1_{\{ y < y^* \}} (L + ly - 1) - 1_{\{ y < y^* \}} E)} \quad \text{if } i = B \quad y \in \bar{R}
\]

where once again we substituted out the \( \max \{ \cdot \} \) function via the appropriate indicator functions. The first two terms on the RHS are once again simply the Ito terms for \( y \). The third term is the payoff of the project realizing, and the last term is the payoff from the project being liquidated.

Value matching and smooth pasting at transitional points, i.e. points that can be recrossed and are thus not absorbing, are properties of the value function that directly derive from its definition as a conditional expectation \( E_t [ \cdot ] \). The step functions \( C_{\pm} (\cdot) (CC_{\pm} (\cdot)) \) for equity (debt) thus solve a linear system stemming from value matching and smooth pasting at the transitional points and the boundary conditions \( C_0^0 = 0 \) \( \text{and} \quad CC_0^0 = 0 \) \( CC_0^\infty = 0 \). The boundary conditions follow from some basic economic observations: \( C_0^\infty = 0 \) follows from the fact that equity cannot grow faster than the frictionless total value of the firm under the project \( A \), \( \frac{\sigma}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu_A} y \), which is linear in \( y \), i.e. \( \lim_{y \to \infty} E (y) / y < \infty \). Similarly, \( C_0^0 = 0 \) follows from the value function remaining bounded as \( y \to 0 \). \( CC_0^0 = 0 \) \( \text{and} \quad CC_0^\infty = 0 \) follow from the observation that the payoff to debt is bounded for both \( y \to 0 \) \( \text{and} \quad y \to \infty \), respectively.

In He and Xiong (2009a) the project technology is fixed at \( A \) and there are no managerial incentive considerations. The only decision left in the model without project choice is for creditors to decide when to stop rolling over. The equilibrium will thus be solely determined by the debt function \( D \). We conjecture a cutoff Markov strategy with creditors refusing to roll over for \( y < y^* \). In our notation, if the manager always plays \( A \) the risk-shifting set is empty, \( \bar{R} = \emptyset \).
Corollary 1 (He, Xiong) For a given rollover threshold \( y^* \), the creditors value function will be

\[
D(y|y^*) = D(y|y^*, \Psi) = CC_+ (y) y^c(y)^+ + CC_- (y) y^c(y)^- + aa(y) \cdot y + bb(y)
\]

where \( \kappa(y)_+ > 1 > 0 > \kappa(y)_- \) solve \( ff (k, y) = 0, \)

\[
ff (k, y) = \frac{\sigma_A^2}{2} k^2 + \left( \mu_A - \frac{\sigma_A^2}{2} \right) k - \left( \phi + \rho + 1_{\{y<y^*\}} (\theta + 1) \delta \right)
\]

and

\[
aa(y) = \frac{1_{\{y<1\}} \phi - 1_{\{y_L<y<y^*\}} \theta \delta + 1_{\{y<y^*\}} \theta \delta}{\rho + \phi + 1_{\{y<y^*\}} (\theta + 1) \delta - \mu_A}
\]

\[
bb(y) = \frac{r + 1_{\{y<y^*\}} \phi + 1_{\{y_L<y<y^*\}} \theta \delta (1 - L) + 1_{\{y<y^*\}} \delta (\theta L + 1)}{\rho + \phi + 1_{\{y<y^*\}} (\theta + 1) \delta}
\]

and where different \( \kappa, aa, bb, CC_\pm \) apply in each of the intervals composed of the boundary points \( 0, 1, y_L, y^* \) and the rollover and risk-shifting thresholds \( y^* \). The coefficients \( CC_\pm (...) \) are step functions solving a linear system stemming from value matching and smooth pasting at the transitional points \( 1, y_L, y^* \) and the boundary conditions \( CC_\pm^0 = 0 \) and \( CC_\pm^\infty = 0 \).

Let \( y_A^* \) denote the equilibrium rollover threshold in this scenario, so that the equilibrium condition is \( D(y_A^*|y_A^*) = 1 \). He and Xiong (2009a)’s result actually goes further than the above corollary in that they can show existence and uniqueness of the symmetric equilibrium \( y_A^* \) analytically. First, note that for any finite, strictly positive \( y^* \), we have \( \lim_{y \to \infty} D(y|y^*) = \frac{\rho \phi}{\rho + \phi} > 1 \) and \( \lim_{y \to 0} D(y|y^*) = \frac{r+1}{\rho+\phi+1} < 1 \). Second, one can analytically show that \( W(y) = D(y|y) \) is increasing and only crosses 1 once (at \( y_A^* \)). Third, it is possible show that \( D(y|y_A^*) \) is strictly increasing and continuous in \( y \), so that individual optimality for refusing to roll over below the equilibrium threshold \( y_A^* \) is established.

A.2 Optimality: Derivation of the super-contact condition

Optimality of the manager’s strategy only depends on the equity value function. For a given value function, then, the manager chooses A over B (instantaneously) when

\[
\mu_A y E_y + \frac{\sigma_A^2}{2} y^2 E_{yy} + \theta \delta 1_{\{y<y^*\}} \max \{ L + l y - 1, 0 \} > \mu_B y E_y + \frac{\sigma_B^2}{2} y^2 E_{yy} + \theta \delta 1_{\{y<y^*\}} \max \{ L + l y - 1, 0 \}
\]

and B over A when the other way around. Note that A and B enter the max equation directly only through \( \mu_i \) and \( \sigma_i \), and indirectly through the value function \( E \). But suppose we are already at the optimum. Then we have no change in the value function for an instantaneous switching between A and B. Thus, only the direct impact matters, and we are left with the following boundary condition at \( \bar{y} \) from indifference between A and B:

\[
\mu_A \bar{y} E_y^{(A)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(A)} + \theta \delta 1_{\{\bar{y}<y^*\}} \max \{ L + l \bar{y} - 1, 0 \} = \mu_B \bar{y} E_y^{(B)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(B)} + \theta \delta 1_{\{\bar{y}<y^*\}} \max \{ L + l \bar{y} - 1, 0 \}
\]

where the functions \( E^{(i)} \) denote the value function with technology \( i \) in use, i.e. A applies to the right of \( \bar{y} \), and B to the left.

We can now derive the super-contact condition. Suppose that \( y^* < \bar{y} \). Then, we have by the optimality of \( \bar{y} \)

\[
\mu_A \bar{y} E_y^{(A)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(A)} = \mu_B \bar{y} E_y^{(B)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(B)}
\]

\[
\mu_A \bar{y} E_y^{(B)} + \frac{\sigma_A^2}{2} \bar{y}^2 E_{yy}^{(B)} = \mu_B \bar{y} E_y^{(A)} + \frac{\sigma_B^2}{2} \bar{y}^2 E_{yy}^{(A)}
\]
as the conditions have to hold approaching from the right (i.e. for \(E^{(A)}\)) and from the left (i.e. for \(E^{(B)}\)) of \(\bar{y}\) - the derivative does not change instantaneously when we switch strategies for a \(dt\) period. Write \(\Delta x = x_A - x_B\). Note that we have value matching and smooth pasting at \(y = \bar{y}\). Subtracting the bottom equation from the top one, we can derive the super-contact condition:

\[
\frac{\sigma_A^2}{2} y^2 \Delta E_{yy} = \frac{\sigma_B^2}{2} y^2 \Delta E_{yy} \iff \frac{\Delta \sigma_A^2}{2} y^2 \Delta E_{yy} = 0 \iff E_{yy}(\bar{y}) = E_{yy}(\bar{y})
\]

where the last line follows from \(\bar{y} \neq 0\).

Suppose instead that \(y^* > \bar{y}\). Then we have

\[
\mu_A \bar{y} E^{(A)}_y + \frac{\sigma_A^2}{2} y^2 E^{(A)}_{yy} + \theta \delta \mathbf{1}_{\{y_L < \bar{y} < y^*\}} (L + l_A \bar{y} - 1) = \mu_B \bar{y} E^{(A)}_y + \frac{\sigma_B^2}{2} y^2 E^{(A)}_{yy} + \theta \delta \mathbf{1}_{\{y_L < \bar{y} < y^*\}} (L + l_B \bar{y} - 1)
\]

and

\[
\mu_A \bar{y} E^{(B)}_y + \frac{\sigma_A^2}{2} y^2 E^{(B)}_{yy} + \theta \delta \mathbf{1}_{\{y_L < \bar{y} < y^*\}} (L + l_A \bar{y} - 1) = \mu_B \bar{y} E^{(B)}_y + \frac{\sigma_B^2}{2} y^2 E^{(B)}_{yy} + \theta \delta \mathbf{1}_{\{y_L < \bar{y} < y^*\}} (L + l_B \bar{y} - 1)
\]

Subtracting the bottom equation from the top one, we can again derive the super-contact condition. This of course only holds if the second derivative is continuous in \(\bar{y}\). This can cease to hold at transitional points \(1, y_L, y^*\) at which we can have asymptotes with a switch in sign. Sufficiency is checked numerically, as the sufficiency conditions are an even higher order condition that cannot be checked analytically in this model.

### A.3 Never and always rolling over cannot be equilibria

**Proposition 2** In the model, always rolling over and never rolling over cannot be equilibria.

**Proof.** First consider a scenario where every maturing creditor rolls over, i.e. \(y^* = 0\). We will consider a one shot deviation of a single maturing creditor, i.e. he can decide today if to rollover or not but will roll over in the future (if given the chance). As \(y \to 0\), debt will be worth \(D = \frac{r+\phi}{r+\phi+\delta} < 1\) by our parameter assumptions, whereas as \(y \to \infty\), debt will be worth \(D = \frac{r+\phi+\delta}{r+\phi+2\delta} > 1\). From the continuity of the value function we know that \(D\) crosses 1 at some point. It is below this point that the individual creditor will stop rolling over. But since the creditors are identical, they all have the same incentives and thus shift their rollover threshold up. We conclude that \(y^* = 0\) cannot be an equilibrium.

Now suppose every maturing creditor never rolls over, i.e., \(y^* = \infty\). Again, we will consider a one shot deviation of a single maturing creditor, i.e. he can decide today if to rollover or not but will not roll over in the future. As \(y \to 0\), the project will be worth \(D = \frac{r+\phi+\delta}{r+\phi+2\delta+\delta} < 1\), so clearly for low levels of \(y\) it is never profitable to roll over the debt. But as \(y \to \infty\), debt is worth \(D = \frac{r+\phi+\delta}{r+\phi+\delta+\delta} > 1\), so there will be a point at which the creditor will want to stay in the firm even if everyone else withdraws at the first chance they get. We conclude that \(y^* \to \infty\) cannot be an equilibrium either. ■

### A.4 Optimality of risk-shifting during rollover freeze

**Proposition 3** For any rollover threshold \(y^* < 1\), it is optimal for the manager to risk-shift on \([0, y^*]\).

**Proof.** We prove the statement for \(\bar{y} < y < y^*\) and \(0 < y < \bar{y} < y^*\). For all other \(y > y^*\), since there is a positive probability of reaching point \(y = y^*\), we can rely on the recursive formulation for optimality.

For expositional clarity, and WLOG, take \(\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = \bar{y}\) and assume \(\bar{y} < y^*\) (i.e. there is no preemptive risk-shifting; this doesn’t influence the incentives to risk-shift for other values of \(y\) as we have a recursive definition). Recall that \(y_L \geq 1\). We will look at the derivative of \(E(y|\bar{y}, y^*)\) w.r.t. \(\bar{y}\). After substituting in the appropriate constants \(C_-(y)\) and \(C_+(y)\) and some tedious algebra, we get the following results. Here \(\eta_i\) is shorthand for \(\eta_i(y)\) with \(y \in (0, \min \{\bar{y}, y^*, y_L, 1\})\) and so forth.
For \( y \in (\bar{y}, y^*) \), we have

\[
\frac{\partial E(y|\bar{y}, y^*)}{\partial \bar{y}} = \frac{\left(\eta_1 - \eta_2 \right) \left( \eta_1 - \eta_2 \right) \left( \eta_2 - \eta_2 \right) \phi \left( \phi + \rho - \mu_A \right)}{\left( \left( \eta_2 - \eta_3 \right) - \eta_2 \left( \eta_3 + \eta_2 \right) + \eta_2 \eta_3 \right) \eta_2 \eta_3} \\
\frac{\left( \phi + \rho \right) \left( \phi + \rho - \mu_A \right)}{\left( \left( \eta_2 - \eta_3 \right) + \eta_2 \eta_3 \eta_2 \eta_3 \right) \eta_2 \eta_3} \left( \eta_2 - \eta_2 \right) \left( \eta_2 - \eta_2 \right) \phi \left( \phi + \rho - \mu_A \right) \eta_2 \eta_3 \eta_2 \eta_3 \eta_2 \eta_3 \eta_2 \eta_3}
\]

Note that \( (\eta_1 - \eta_2) > 0, (\eta_3 - \eta_2) < 0 \) and \( (\eta_2 - \eta_3) < 0 \) directly from their definitions. Also by our assumptions in section 1.4, we have \( (\eta_1 - \eta_2) < 0 \) (4th assumption) and \( (\phi + \rho - \mu_A) > 0 \) (3rd assumption). We conclude that this expression is positive for all \( y \in (\bar{y}, y^*) \).

For \( y \in (0, \bar{y}) \), we have

\[
\frac{\partial E(y|\bar{y}, y^*)}{\partial \bar{y}} = \frac{\left(\eta_1 - \eta_2 \right) \left( \eta_1 - \eta_2 \right) \left( \eta_2 - \eta_2 \right) \phi \left( \phi + \rho - \mu_A \right)}{\left( \left( \eta_2 - \eta_3 \right) - \eta_2 \left( \eta_3 + \eta_2 \right) + \eta_2 \eta_3 \right) \eta_2 \eta_3} \\
\frac{\left( \phi + \rho \right) \left( \phi + \rho - \mu_A \right)}{\left( \left( \eta_2 - \eta_3 \right) + \eta_2 \eta_3 \eta_2 \eta_3 \right) \eta_2 \eta_3} \left( \eta_2 - \eta_2 \right) \left( \eta_2 - \eta_2 \right) \phi \left( \phi + \rho - \mu_A \right) \eta_2 \eta_3 \eta_2 \eta_3 \eta_2 \eta_3 \eta_2 \eta_3}
\]

By the same inequalities mentioned above, the expression is positive for all \( y \in (\bar{y}, y^*) \).

What remains to be shown is that these assumptions are truly WLOG, i.e. consider a strategy by the manager that has risk-shifting on an interval \((\bar{y}_2, \bar{y}_3)\) outside of \((0, y^*)\). Clearly risk-shifting in the rollover region gives lower liquidation values conditional on default (as the drift \(\mu_B \) is lower than \(\mu_A \)). Thus, the benefit of risk-shifting must come from increasing the probability of escaping the rollover-region and thus increasing the probability on the non-freeze value function. But if the value function outside of the freeze region can be improved by risk-shifting on an interval \((\bar{y}_2, \bar{y}_3)\), then clearly the trade off between liquidation value today and continuation value tomorrow becomes even more skewed towards the continuation value tomorrow, and thus the incentives of the manager are even stronger in favor of risk-shifting on the freeze region.

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**A.5 The value function** \( G(y) \) **given** \( y^*, \bar{Y} \)

**Proposition 4** Given a rollover threshold \( y^* \) and switching thresholds \( \bar{y}_1, \bar{y}_2, \bar{y}_3 \) the government’s bailout cost are

\[
G(y|y^*, \bar{Y}) = CCC_+ (y) y^{\eta(y)_+} + CCC_- (y) y^{\eta(y)_-} + \delta 1_{\{ y < y^* \}} \left[ \frac{-aa(y)}{\rho + \phi + \theta \delta - \mu} y + \frac{1 - bb(y)}{\rho + \phi + \theta \delta} \right]
\]

\[
+ \delta 1_{\{ y < y^* \}} \left[ \frac{-aa(y)}{\rho + \phi + \theta \delta - \mu} y + \frac{1 - bb(y)}{\rho + \phi + \theta \delta} \right] y^{\eta(y)_+}
\]

\[
+ \delta 1_{\{ y < y^* \}} \left[ \frac{-aa(y)}{\rho + \phi + \theta \delta - \mu} y + \frac{1 - bb(y)}{\rho + \phi + \theta \delta} \right] y^{\eta(y)_-}
\]

where the appropriate \( aa, bb, \kappa_+, \kappa_-, CC_+ \) from Proposition 1 apply in each of the intervals composed of the boundary points \( 0, 1, y_L \) and the rollover and risk-shifting thresholds \( y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3 \). The coefficients \( CCC_{\pm} (\cdot) \) are step functions solving a linear system stemming from value matching and smooth pasting at the boundary points. The appropriate boundary conditions are \( CCC_{\pm} (0) = 0 \) and \( CCC_{0} = 0 \).
**Proof.** We observe the following: First, in equilibrium, a freeze only occurs when debt is worth less than its face value, i.e. $D(y) < 1$, which occurs for $y < y^*$, and thus $G(y) > 0$. Second, we must have $\lim_{y \to \infty} G(y) = 0$, as the incidence of having to supply interim financing becomes negligible for large $y$ as no freezes occur.

The associated HJB for the government’s loss function that derives from equation (4) is

$$\rho G = \mu yG_y + \frac{\sigma^2}{2} y^2 G_{yy} - \phi G - 1_{(y < y^*)} \theta \delta G + \delta 1_{(y < y^*)} [1 - D(y)]$$

where the first two terms on the RHS are once again the Ito terms of $y$, the third term reflects the intensity of realization, the fourth term the intensity of default without a creditline/bailout and the last term the liquidity injection stemming from the continuous bailouts on $(0, y^*)$. The equation is straightforward to solve - it is a linear ODE with 2 independent solutions from the quadratic equation, plus a particular part tied to the expression $D(y)$ that is available in closed form as it is also of power form. Note that $G$ has the same fundamental equation as $E$, thus we use the same parameters $\eta_{\pm}$. Note that as $\kappa_{\pm} \neq \eta_{\pm}$ on $y < y^*$, we know that $\frac{\sigma^2}{2} \kappa_{\pm}^2 + \left(\mu - \frac{\sigma^2}{2}\right) \kappa_{\pm} - (\rho + \phi + \theta \delta) \neq 0$, so division by zero does not arise. ■